



A decision-making technique under interval-valued Fermatean fuzzy Hamacher interactive aggregation operators

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Accepted: 9 May 2023

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Abstract

The evolution of a novel technique to handle multi-attribute decision-making (MADM) problems under interval-valued Fermatean fuzzy numbers is the main motivation of this paper. We aim to introduce several initiative aggregation operators (AOs), including Hamacher interactive weighted averaging, Hamacher interactive ordered weighted averaging, Hamacher interactive hybrid weighted averaging operations, etc., to acquire our desired outcomes. Then, the distinguished characteristics of these AOs are investigated. Furthermore, the suggested AOs are carried out to build a technique to MADM issues using interval-valued Fermatean fuzzy information. A case study of mine emergency plan selection is then narrated to elaborate the practicality and effectiveness of the developed method. The influence of parametric values on decision-making outcomes is investigated considering the distinct values of parameter. After discussing the developed work and seeing its applications, we come across with the conclusion that the dominant privilege of adaptation of the above-mentioned AOs is situated in the fact that these operators allow a progressively complete approach on the matters to decision-makers. Hence, the method recommended in this study offers progressively wide, enhanced accuracy and actual outcomes when compared with the prevailing associated strategies. Therefore, this technique plays a vital role in actual-life MADM problems.

Keywords IVFF numbers · AOs · Hamacher interactive operators · MADM · Mine emergency decision-making

1 Introduction

The underground mining environment is perilous and prone to several mishaps including gas outbreaks, gas leakages, roof collapse, fire, etc. It not only simply puts the lifestyles of the miners in hazard, but also causes tremendous harm to the device, precious assets and the environment. Every 12 months, a lot of treasured lives and sources are lost in disasters in mines. To mitigate those screw ups, the subsequent

sections are running over there: (1) rescue and recovery, (2) mine sample testing and research.

People's safety depends on a number of interconnected elements such as vigorous data, swapping of surroundings, the propensity of perceiving and reacting to menaces, education, imparting, and incidents. Those elements may be crucial to action during an emergency. In an underground mine, seconds matter when something goes wrong, and the commencing reaction could be crucial to the consequences. Knowing the concerns and practices that arise in the early stages of a reaction to a mine emergency can help people escape, make it easier to rescue them, and could be useful to teach miners and DMs (decision-makers). The disaster management cycle in case of mine emergency is shown in Fig. 1.

Before proceeding to the review of literature, we first provide the list of the abbreviations and acronyms as given in Table 1 that would be helpful for the readers to further understand the subsequent study.

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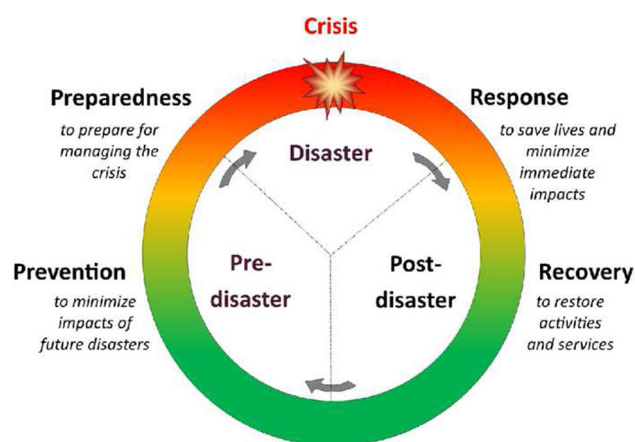


Fig. 1 Disaster management cycle

1.1 Literature review

MADM is taken into consideration as an accelerated and growing area of research that is utilized to attain the best suitable option from a group of finite alternatives similar to positive attributes. Zadeh (1965) developed FSs as a successful augmentation of crisp theory to address ambiguity and uncertainty occurring in decision experts' opinions. The main feature of FSs is its MF that lies between 0 and 1. The necessity for FSs came into being as the applications containing imprecise data cannot be modeled through crisp sets. The absence of independent NMF in FSs encouraged the researchers to propose new generalizations of FSs. Consequently, Atanassov (1986) invented IFSs having the NMF as an independent factor. Furthermore, PFSs were proposed to moderate the constraint of IFSs (the sum of squares of MF and NMF of PFSs is less than 1) (Yager 2014). The relaxed condition in case of PFSs makes these sets the most contemplated and flexible tool to deal with uncertain MADM problems. Later on, FFSs were presented by Senapati and Yager (2020) as an important augmentation of PFSs. The cube sum of MF and NMF corresponding to any object should be less than 1 in case of FFSs. As compared to IFSs and PFSs, more elements can be covered up from the square $[0, 1] \times [0, 1]$ utilizing FFSs. That's the reason that FFSs have been considered as a more effective, generalized, and powerful tool to deal with vague situations occurring in MADM. The concept of FFSs highly motivated the researchers to generalize this concept to more flexible theory, named as IVFFSs (Rani et al. 2022). The IVFFSs are considered as a useful augmentation of IVPFSs (Peng and Yang 2016).

As FFSs have more capability to handle the uncertain data and information, they contain more applications in decision-making theory. For example, Aydin (2021) created and used an entropy method in FFSs that was based on Euclidean space between FFNs. An example of 3PL-resistant assess-

Table 1 Abbreviations and acronyms

Item	Description
MADM	Multi-attribute decision-making
FSs	Fuzzy sets
MF	Membership function
NMF	Non-membership function
IFSs	Intuitionistic fuzzy sets
PFSs	Pythagorean fuzzy sets
FFSs	Fermatean fuzzy sets
IVFFSs	Interval-valued Fermatean fuzzy sets
IVPFSs	Interval-valued Pythagorean fuzzy sets
FFNs	Fermatean fuzzy numbers
3PL	Third-party logistic
SAW	Simple additive weighting
ARA	Additive ratio assessment
VIKOR	Vlse Kriterijumsa Optimizacija I Kompromisno Resenje
MCDM	Multi-criteria decision-making
CRITIC	CRiteria Importance Through Intercriteria Correlation
EDAS	Evaluation based on distance from average solution
PFH-ELECTRE I	Pythagorean fuzzy hybrid-ELimination and Choice Translating REality I
TOPSIS	Order of Preference by Similarity to an Ideal Solution
HPF	Hesitant Pythagorean fuzzy
MCGDM	Multi-criteria group decision-making
MAGDM	Multi-attribute group decision-making
MULTIMOORA	Multi-objective optimization by ratio analysis plus full MULTIplicative form
QoL	Quality of life
IMS	Intelligent manufacturing system
AOs	Aggregation operators
DM	Decision-making
OWAAO	Ordered weighted averaging aggregation operator
WAAO	Weighted averaging aggregation operator
WGAO	Weighted geometric aggregation operator
SCM	Supply chain management
IVPHFS	Interval-valued picture hesitant fuzzy set
Lq-ROFGs	Linguistic q -rung orthopair fuzzy grades
EO	Einstein operator
TSFN	T spherical fuzzy number
HFS	Hesitant fuzzy set
HFWA	Hesitant fuzzy weighted averaging
FMEA	Failure modes and effect analysis
mF	m -polar fuzzy

Table 1 continued

Item	Description
HWAO	Hamacher weighted average operator
HOWA	Hamacher ordered weighted average
HHA	Hamacher hybrid average
HOs	Hamacher operations
PAOs	Power aggregation operators
PFIAOs	Pythagorean fuzzy interaction aggregation operators
PFIHPOs	Pythagorean fuzzy interactive Hamacher power AOs
FFHAGAOs	Fermatean fuzzy Hamacher arithmetic and geometric AOs
HIWAO	Hamacher interactive weighted averaging operator
HWGO	Hamacher weighted geometric operator
HIOWO	Hamacher interactive ordered weighted operator
HIHWAO	Hamacher interactive hybrid weighted averaging operator
IVFFDMx	Interval-valued Fermatean fuzzy decision matrix

ment in case of organizing a cold chain was studied to show the implications of the developed model. Deng and Wang (2022) introduced distance measure approaches for FFSs in the fields of medical recognition and ornament identification. One distance utilized the triangular divergence and the other one is based on the Hellinger distance of FFSs. The viability and application of the suggested distance measure techniques are illustrated through various numerical examples. Xu and Shen (2021) provided additional characteristics for FFS similarity assessments. The feasibility and potency of the suggested approach have been revealed through an interpretative MADM application in the case of medical diagnosis. Moreover, SAW, ARAS, and VIKOR are three novel MCDM techniques that have been proposed by Gül (2021) to analyze the MCDM problems under FFNs. The author discussed the issue of choosing the finest examining facility to recognize COVID-19-contaminated victims using the three approaches that were suggested. Mishra et al. (2021) suggested an enhanced generalized score function for FFNs. The same authors developed the FF-CRITIC-EDAS technique to deal with MCDM issues. Being the modified and novel methods, the PFH-ELECTRE I and the PFH-TOPSIS approaches were proposed by Akram et al. (2021). Akram et al. (2022) suggested a novel ELECTRE I technique to evaluate threats in FMEA with HPF accomplishment. Shahzadi and Akram (2021) and Akram et al. (2020) presented a procedure of MCGDM to choose an antivirus mask under FF soft and FF environment. A MULTIMOORA technique under 2-

tuple linguistic FFSs was recently proposed by Akram et al. (2022); they also demonstrated how the system might be used to choose an urban QoL. Shahzadi et al. (2022) used the MOORA approach with FF data to choose an IMS. Sergi et al. (2022) presented a generalization of capital budgeting approaches under IVFFSs. Additionally, Jeevaraj (2021) introduced several new ideas for the class of IVFFNs. The score and accuracy functions for IVFFNs were given by Rani and Mishra (2022). They also addressed several axioms and presented two AOs to aggregate the information from the IVFFSs.

The idea of AOs has been studied by many researchers regarding MCDM perspectives. In passing years, fuzzy AOs fascinated many researchers to utilize these operators in the development of various DM methods. Einstein operators include Einstein sum and product and have been proved to be important possibilities toward the algebraic ones. For example, Rahman et al. (2020) gave the proposal of Einstein AOs, which include the IVPF Einstein OWAAO and the IVPF Einstein WAAO. Ali et al. (2021) proposed a number of complicated Einstein WGAOs and presented their implementations in SCM under IVPFSs. In their study of FF Einstein weighted operators, Rani and Mishra (2021) suggested a new MULTIMOORA approach to address MADM issues. In Rani and Mishra (2022) and Rani et al. (2022), other AOs for FFSs and IVFFSs are investigated. Kamacı et al. (2021) initiated the concept of IVPHFS and its operative rules on the basis of Einstein operations. The authors also developed certain IVPHF AOs to combine these sets gathered over various time frames. In the context of Lq-ROFGs, based on the EO, the authors suggested new notions such as “product-connectivity energy, extended product-connectivity energy, Laplacian energy, and signless Laplacian energy” and described a number of its desirable properties. Garg et al. (2020) created certain averaging and geometric AOs by combining the FFS’s advantageous qualities with the Yager operator. The “linguistic FF Hamy mean operator, the linguistic FF dual Hamy mean operator, the linguistic FF weighted Hamy mean operator, and the linguistic FF weighted dual Hamy mean operator” are just a few of the new areas of “linguistic FF Hamy mean operators” that were added in Akram et al. (2022). A triangular IVFFN was defined by Akram et al. (2022) and its arithmetic operations were covered. Moreover, various operational laws for TSFNs based on Archimedean t-conorm and t-norm and their crucial properties were discussed in Khan et al. (2023). Extended Z-fuzzy soft-covering-based rough matrices were presented in Sivaprakasam and Angamuthu (2023). The authors explored some algebraic characteristics of the new matrix and developed a new MAGDM model utilizing these matrices. Moreover, the HFSs were utilized to represent the uncertainty in risk evaluation (Zhou et al. 2022). Then, an improved HFWA operator was used to assess the risk evaluation for FMEA experts. Waseem et al. (2019)

introduced certain AOs, namely the $mFHWAO$, $mFHOWA$ operator, $mFHHA$ operator, etc. They also developed an algorithm to solve MADM problems in mF environment using $mFHW$ A and $mFHWG$ operators.

Wei (2019) used HOs and PAOs to design a variety of PFHPAOs. The PFHAOs were proposed by Wu and Wei (2017), and the authors talked about how to use them in MCDM. Wei (2017) proposed and talked about PFIAOs and their uses in MCDM. In Zhao and Wei (2013), new ideas for IF Einstein hybrid AOs and their application were put forth. Using Einstein AOs, Wang and Liu (2012) combined the IF data and information. In Wang et al. (2021), PFIHPAOs were used to evaluate the “express service quality using entropy weights.” Utilizing the “Hamacher T-conorm and T-norm,” Hadi et al. (2021) created unique algorithms based on FFSs and explored how they work in general. The authors proposed FFHAGAOs, which were induced by the HOs and FFS. Shahzadi et al. (2021) presented a number of AOs under FFSs, such as “Dombi, Einstein, and Hamacher.” Senapati and Chen (2021) created several unique IVPF AOs based on Hamacher triangular norms and talked about how they may be used to solve MADM problems. Hamacher interactive hybrid weighted AOs were defined by Shahzadi et al. (2021) under FFNs. Liu (2013) suggested a number of HAOs based on the IVIFNs and their uses to MCDM. Furthermore, Donyatalab et al. (2020), Akram et al. (2022), Jan et al. (2021), and Akram et al. (2022) show more significant contributions on HAOs. Many interesting operators are discussed in Deveci et al. (2021), Deveci et al. (2022), and Pamucar et al. (2022).

1.2 Motivations

We are motivated to introduce specific AOs for IVFFSs using the key characteristics of IVFFSs. These motivations are described as follows:

1. There exist more effectiveness and flexibility which are possessed by HIAOs as compared to fundamental operators and their consequences. Despite this fact, there exists no literature regarding the HIAOs under IVFFSs to deal with the IVFF data.
2. No research endeavors have been performed to study the numerous important properties of AOs for IVFFSs in the existing literature. These operators possess the characteristics to handle the interrelations between MF and NMF.
3. With the aid of numerous ambiguous and unclear pieces of information, the traditional DM approaches based on AOs are generalized, although the majority of AOs suggest constraints and restrictions when manipulating IVFF data.
4. The existing strategies present a variety of methods to analyze the challenges linked to mine emergency decision-making. However, these frameworks have limits

and drawbacks when dealing with uncertain mine emergency decision-making problems from IVFFSs.

5. The sensitivity analysis for distinct parametric values shows how we can robust the performance and enhance the consistency of decision-making methods.

1.3 Contributions

To address the above-mentioned motivations, the main contributions of this study are listed as follows:

1. A new class of AOs has been introduced under IVFF environment to overcome the deficiencies occurring in existing theories.
2. We discuss a variety of cases considering different values of parameter for which the IVFFHIWA operators reduces to IVFFIWA and IVFFEIWA for parametric values 1 and 2, respectively.
3. Then, various characteristics of these developed AOs, including idempotency, shift invariance, boundedness, monotonicity, homogeneity, etc., have been narrated and proved to enhance the applicability of these operators.
4. Moreover, a novel decision-making technique has been introduced under IVFFHIOs. The proposed technique is explained through an algorithm as well as a flowchart.
5. Then, a case study of emergencies occurring in mining is considered to apply the developed technique. The results obtained have been compared with the ranking obtained through various existing techniques, including IVFF-TOPSIS method.
6. Finally, sensitivity analysis is presented taking various values for parameter. The results show that the distinct parametric values enhance the stability of the proposed approach.

1.4 Originality and aims

The suggested work's novelty and essential qualities are described as follows: the score values for IVFFSs are calculated using a new function that allows for comparison of any number of IVFFNs. The HIWAO, HWGO, HIOWO, and HIHWAO are just a few of the AOs that are used to combine the IVFFSs. The IVFFSs are considered as a generalized environment to discuss these operators as these operators have not been studied under this more extended form of FFSs till the date. We have considered this theory as it reduces to IVPF, IVIF, and IVF theories considering square power, one power, and 0 non-membership function, respectively. This shows that we have considered the most generalized extension of FFSs as most of the other theories can be merged with the proposed technique. Thus, the applications of various extensions of FFSs can be adopted through the developed work considering distinct values of parameters or changing

membership grades. Hence, the main novelty of the proposed study can be summarized as IVFFSs have been firstly utilized to introduce and discuss HIAOs and their various types. There is a quick discussion of a few useful and beneficial qualities of potential operators. To address the MADM issues with unidentified decision-makers and unidentified weights of criteria, a composite IVFF-based framework that combines the score function and HIAOs is provided. An emergency situation at a mine is used as a case study in the IVFF environment to demonstrate the stability and durability of the proposed model. By contrasting the developed method with the existing knowledge, our developed work's sturdiness and authenticity are shown.

1.5 Problem statement

In the proposed work, we will examine the emergency DM issues of mine injuries using the proposed choice making algorithms based on IVFFHIAOs and score functions. In case of mine accidents, the mine explosion is considered as the most dangerous hazard. The mine explosion can threaten the safety of the workplace and lives of the workers as well as the safe production of mines. Since the occurrence of these explosions is often sudden and unexpected, it is difficult to predict these explosion accidents and adopt auxiliary actions and have adequate solutions. Therefore, the assessment and decision of the given emergency plans with simulations are taken into consideration essential for the disaster control of mine injuries. We will consider five emergency plans, including noxious gas concentration, reducing casualty of current events, the smoke and the dust level, the feasibility of rescue operations, and repairing facility damages caused by the emergency, as a set of alternatives. Then, the developed algorithm based on IVFFHIAOs would be applied to rank these alternatives in order to achieve the most appropriate emergency plan.

1.6 Objectives of the proposed method

In order to attain the final ranking of alternatives as mentioned in the problem statement, the main objectives of the developed technique are listed as follows:

1. To introduce IVFFHIAOs, discuss the main features of these operators and to apply these operators in order to build a novel technique of MADM.
2. To obtain an IVFF decision matrix taking into account the specifications of each alternative.
3. To determine the overall preference values of alternatives implementing the IVFFHIWA operator to the IVFF decision matrix.
4. To apply the score function in order to compare the ranking of under consideration alternatives.

5. To obtain the final ranking of alternatives to assess the most appropriate emergency plan to escape in case of mine disasters.

1.7 Structure of the study

The study is organized as follows: Sect. 2 discusses some essential FFS and IVFFS ideas. We discuss certain IVFFS operations and distance calculations. The IVFF Hamacher interactive operating laws are introduced in Sect. 3. In Sects. 4 and 5, "Hamacher weighted and ordered weighted averaging operators" for IVFFSs are introduced. The "IVFF Hamacher interactive hybrid weighted averaging operators" are covered in Sect. 6. As suggested in earlier parts, the MADM approach has been applied through IVFF AOs in Sect. 7. The technique outlined in the previous section was then applied to a case study using the mining emergency decision-making dilemma in Sect. 8. We provide a thorough comparison of the suggested strategy with several currently used strategies in Sect. 9. A delicate analysis is covered in Sect. 10. Conclusions, the proposed work's shortcomings, and potential future study directions are covered in Sect. 11.

2 Preliminaries

The definitions in this section include score functions, FFS, IVPFS, and IVFFS.

Definition 1 (Senapati and Yager 2020) "A FFS $\mathcal{T}$ on universal set $\mathcal{Q}$ is stated by

$$\mathcal{T} = \{\{x, \mathfrak{I}_{\mathcal{T}}(x), \wp_{\mathcal{T}}(x)\}\},$$

wherever $\mathfrak{I}_{\mathcal{T}} : \mathcal{Q} \rightarrow [0, 1]$, $\wp_{\mathcal{T}} : \mathcal{Q} \rightarrow [0, 1]$, and $\wp_{\mathcal{T}}(x) = \sqrt[3]{1 - (\mathfrak{I}_{\mathcal{T}}(x))^3 - (\wp_{\mathcal{T}}(x))^3}$ indicate MF, NMF, and InF, respectively."

Definition 2 (Peng and Yang 2016) "An IVPFS $\mathcal{P}$ on $\mathcal{Q}$ is stated by

$$\mathcal{P} = \{\{x, [\mathfrak{I}_{\mathcal{P}}^{lb}(x), \mathfrak{I}_{\mathcal{P}}^{ub}(x)], [\wp_{\mathcal{P}}^{lb}(x), \wp_{\mathcal{P}}^{ub}(x)]\}\},$$

wherever $0 \leq \mathfrak{I}_{\mathcal{P}}^{lb}(x) \leq \mathfrak{I}_{\mathcal{P}}^{ub}(x) \leq 1$, $0 \leq \wp_{\mathcal{P}}^{lb}(x) \leq \wp_{\mathcal{P}}^{ub}(x) \leq 1$, and $(\mathfrak{I}_{\mathcal{P}}^{ub}(x))^2 + (\wp_{\mathcal{P}}^{ub}(x))^2 \leq 1$. $\mathfrak{I}_{\mathcal{P}}(x) = [\mathfrak{I}_{\mathcal{P}}^{lb}(x), \mathfrak{I}_{\mathcal{P}}^{ub}(x)]$ and $\wp_{\mathcal{P}}(x) = [\wp_{\mathcal{P}}^{lb}(x), \wp_{\mathcal{P}}^{ub}(x)]$ symbolize the interval-valued MF and NMF, respectively. $\wp_{\mathcal{P}}(x) = [\wp_{\mathcal{P}}^{lb}(x), \wp_{\mathcal{P}}^{ub}(x)]$ is the interval-valued InF, where $\wp_{\mathcal{P}}^{lb}(x) = \sqrt{1 - (\mathfrak{I}_{\mathcal{P}}^{lb}(x))^2 - (\wp_{\mathcal{P}}^{lb}(x))^2}$ and $\wp_{\mathcal{P}}^{ub}(x) = \sqrt{1 - (\mathfrak{I}_{\mathcal{P}}^{ub}(x))^2 - (\wp_{\mathcal{P}}^{ub}(x))^2}$."

Definition 3 (Rani and Mishra 2022) “An IVFFS J on $\mathcal{Q}$ is stated by

$$J = \{ \langle x, [\mathfrak{I}_J^{lb}(x), \mathfrak{I}_J^{ub}(x)], [\wp_J^{lb}(x), \wp_J^{ub}(x)] \rangle \},$$

wherever $0 \leq \mathfrak{I}_J^{lb}(x) \leq \mathfrak{I}_J^{ub}(x) \leq 1$, $0 \leq \wp_J^{lb}(x) \leq \wp_J^{ub}(x) \leq 1$ and $(\mathfrak{I}_J^{ub}(x))^3 + (\wp_J^{ub}(x))^3 \leq 1$. $\mathfrak{I}_J(x) = [\mathfrak{I}_J^{lb}(x), \mathfrak{I}_J^{ub}(x)]$ and $\wp_J(x) = [\wp_J^{lb}(x), \wp_J^{ub}(x)]$ symbolize the interval-valued MF and NMF, respectively. $\wp_J(x) = [\wp_J^{lb}(x), \wp_J^{ub}(x)]$ is the interval-valued InF, where $\wp_J^{lb}(x) = \sqrt[3]{1 - (\mathfrak{I}_J^{ub}(x))^3 - (\wp_J^{ub}(x))^3}$ and $\wp_J^{ub}(x) = \sqrt[3]{1 - (\mathfrak{I}_J^{lb}(x))^3 - (\wp_J^{lb}(x))^3}$.”

Definition 4 (Rani and Mishra 2022) For IVFFN $J = \langle [\mathfrak{I}_J^{lb}, \mathfrak{I}_J^{ub}], [\wp_J^{lb}, \wp_J^{ub}] \rangle$, the scoring function and accuracy function are represented by

$$S(J) = \frac{1}{2} [(\mathfrak{I}_J^{lb})^3 + (\mathfrak{I}_J^{ub})^3 - (\wp_J^{lb})^3 - (\wp_J^{ub})^3],$$

where $S(J) \in [-1, 1]$,

$$\mathcal{A}(J) = \frac{1}{2} [(\mathfrak{I}_J^{lb})^3 + (\mathfrak{I}_J^{ub})^3 + (\wp_J^{lb})^3 + (\wp_J^{ub})^3],$$

where $\mathcal{A}(J) \in [0, 1]$.

Definition 5 (Rani and Mishra 2022) Consider IVFFNs $J_1 =$ (ii)

$$\langle [\mathfrak{I}_{J_1}^{lb}, \mathfrak{I}_{J_1}^{ub}], [\wp_{J_1}^{lb}, \wp_{J_1}^{ub}] \rangle \text{ and } J_2 = \langle [\mathfrak{I}_{J_2}^{lb}, \mathfrak{I}_{J_2}^{ub}], [\wp_{J_2}^{lb}, \wp_{J_2}^{ub}] \rangle.$$

1. If $S(J_1) < S(J_2)$, then $J_1 < J_2$;

2. If $S(J_1) > S(J_2)$, then $J_1 > J_2$;

3. If $S(J_1) = S(J_2)$, then

(a) If $\mathcal{A}(J_1) < \mathcal{A}(J_2)$, then $J_1 < J_2$;

(b) If $\mathcal{A}(J_1) > \mathcal{A}(J_2)$, then $J_1 > J_2$;

(c) If $\mathcal{A}(J_1) = \mathcal{A}(J_2)$, then $J_1 \sim J_2$.

3 Hamacher operations for IVFFNs

Definition 6 Let $J_1 = \langle [\mathfrak{I}_1^{lb}, \mathfrak{I}_1^{ub}], [\wp_1^{lb}, \wp_1^{ub}] \rangle$, $J_2 = \langle [\mathfrak{I}_2^{lb}, \mathfrak{I}_2^{ub}], [\wp_2^{lb}, \wp_2^{ub}] \rangle$ and $J = \langle [\mathfrak{I}^{lb}, \mathfrak{I}^{ub}], [\wp^{lb}, \wp^{ub}] \rangle$ be three IVFFNs and $\beta > 0$. The following arithmetic actions between them using Hamacher interactive norms are performed in this way:

$$J_1 \oplus J_2 = \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^2 (1 + (\mathfrak{I}_t^{lb} - 1)(\mathfrak{I}_t^{ub})^3) - \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{lb})^3)}{\prod_{t=1}^2 (1 + (\mathfrak{I}_t^{lb} - 1)(\mathfrak{I}_t^{ub})^3) + (\mathfrak{I}_t^{ub} - 1) \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{lb})^3)}}, \sqrt[3]{\frac{\prod_{t=1}^2 (1 + (\mathfrak{I}_t^{ub} - 1)(\mathfrak{I}_t^{ub})^3) - \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{ub})^3)}{\prod_{t=1}^2 (1 + (\mathfrak{I}_t^{ub} - 1)(\mathfrak{I}_t^{ub})^3) + (\mathfrak{I}_t^{ub} - 1) \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{ub})^3)}} \right], \left[\sqrt[3]{\frac{\mathfrak{I}\{ \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{lb})^3) - \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3) \}}{\prod_{t=1}^2 (1 + (\mathfrak{I}_t^{lb} - 1)(\mathfrak{I}_t^{ub})^3) + (\mathfrak{I}_t^{ub} - 1) \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{lb})^3)}}, \sqrt[3]{\frac{\mathfrak{I}\{ \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{ub})^3) - \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3) \}}{\prod_{t=1}^2 (1 + (\mathfrak{I}_t^{ub} - 1)(\mathfrak{I}_t^{ub})^3) + (\mathfrak{I}_t^{ub} - 1) \prod_{t=1}^2 (1 - (\mathfrak{I}_t^{ub})^3)}} \right] \right\rangle,$$

$$\beta \cdot J = \left\langle \left[\sqrt[3]{\frac{(1 + (\mathfrak{I}^{lb} - 1)(\mathfrak{I}^{ub})^3)^\beta - (1 - (\mathfrak{I}^{lb})^3)^\beta}{(1 + (\mathfrak{I}^{lb} - 1)(\mathfrak{I}^{ub})^3)^\beta + (\mathfrak{I}^{ub} - 1)(1 - (\mathfrak{I}^{lb})^3)^\beta}}, \sqrt[3]{\frac{(1 + (\mathfrak{I}^{ub} - 1)(\mathfrak{I}^{ub})^3)^\beta - (1 - (\mathfrak{I}^{ub})^3)^\beta}{(1 + (\mathfrak{I}^{ub} - 1)(\mathfrak{I}^{ub})^3)^\beta + (\mathfrak{I}^{ub} - 1)(1 - (\mathfrak{I}^{ub})^3)^\beta}} \right], \left[\sqrt[3]{\frac{\mathfrak{I}\{(1 - (\mathfrak{I}^{lb})^3)^\beta - (1 - (\mathfrak{I}^{lb})^3 - (\wp^{lb})^3)^\beta\}}{(1 + (\mathfrak{I}^{lb} - 1)(\mathfrak{I}^{ub})^3)^\beta + (\mathfrak{I}^{ub} - 1)(1 - (\mathfrak{I}^{lb})^3)^\beta}}, \sqrt[3]{\frac{\mathfrak{I}\{(1 - (\mathfrak{I}^{ub})^3)^\beta - (1 - (\mathfrak{I}^{ub})^3 - (\wp^{ub})^3)^\beta\}}{(1 + (\mathfrak{I}^{ub} - 1)(\mathfrak{I}^{ub})^3)^\beta + (\mathfrak{I}^{ub} - 1)(1 - (\mathfrak{I}^{ub})^3)^\beta}} \right] \right\rangle$$

4 IVFF Hamacher interactive weighted average operators

Suppose $J_t = \langle [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}], [\wp_t^{lb}, \wp_t^{ub}] \rangle$ ($t = 1, 2, \dots, q$) is a selection of IVFFNs and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_q)^T$ is its weight vector (WV) s.t $\kappa_t > 0$ and $\sum_{t=1}^q \kappa_t = 1$, then $IVFFHIWA : \Omega^q \rightarrow \Omega$ is defined as

$$IVFFHIWA(J_1, J_2, \dots, J_q) = \kappa_1 J_1 \oplus \kappa_2 J_2 \oplus \dots \kappa_q J_q.$$

Theorem 1 Let $J_t = \langle [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}], [\wp_t^{lb}, \wp_t^{ub}] \rangle$ be a collection of IVFFNs, then $IVFFHIWA(J_1, J_2, \dots, J_q) =$

$$\left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}}], \left[\sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right] \right\rangle.$$

Proof For $q = 1, \kappa = \kappa_1 = 1$,

$$\begin{aligned} IVFFHIWA(J_1) &= \kappa_1 J_1 \\ &= J_1 = \langle [\mathfrak{I}_1^{lb}, \mathfrak{I}_1^{ub}], [\wp_1^{lb}, \wp_1^{ub}] \rangle \\ &= \left\langle \left[\sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{lb})^3) - (1 - (\mathfrak{I}_1^{lb})^3)}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{lb})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_1^{lb})^3)}}, \sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{ub})^3) - (1 - (\mathfrak{I}_1^{ub})^3)}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{ub})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_1^{ub})^3)}} \right], \right. \\ &\quad \left[\sqrt[3]{\frac{\mathfrak{E} \{ (1 - (\mathfrak{I}_1^{lb})^3) - (1 - (\mathfrak{I}_1^{lb})^3 - (\wp_1^{lb})^3) \}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{lb})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_1^{lb})^3)}}, \sqrt[3]{\frac{\mathfrak{E} \{ (1 - (\mathfrak{I}_1^{ub})^3) - (1 - (\mathfrak{I}_1^{ub})^3 - (\wp_1^{ub})^3) \}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{ub})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_1^{ub})^3)}} \right] \right\rangle. \end{aligned}$$

Thus, result holds for $q = 1$. Suppose the outcome is true for $q = p$, i.e.,

$$\begin{aligned} IVFFHIWA(J_1, J_2, \dots, J_p) &= \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right], \right. \\ &\quad \left[\sqrt[3]{\frac{\mathfrak{E} \{ (1 - (\mathfrak{I}_1^{lb})^3)^{\kappa_{p+1}} - (1 - (\mathfrak{I}_1^{lb})^3 - (\wp_1^{lb})^3)^{\kappa_{p+1}} \}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{lb})^3)^{\kappa_{p+1}} + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_1^{lb})^3)^{\kappa_{p+1}}}}, \sqrt[3]{\frac{\mathfrak{E} \{ (1 - (\mathfrak{I}_1^{ub})^3)^{\kappa_{p+1}} - (1 - (\mathfrak{I}_1^{ub})^3 - (\wp_1^{ub})^3)^{\kappa_{p+1}} \}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_1^{ub})^3)^{\kappa_{p+1}} + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_1^{ub})^3)^{\kappa_{p+1}}}}} \right] \right\rangle. \end{aligned}$$

$$\left\langle \left[\sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right] \right\rangle.$$

Now, for $q = p + 1$

$$\begin{aligned} IVFFHIWA(J_1, J_2, \dots, J_{p+1}) &= \bigoplus_{t=1}^{p+1} \kappa_t J_t \\ &= \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right], \right. \\ &\quad \left[\sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^p (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^p (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right] \right\rangle \\ &\quad \oplus \left\langle \left[\sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_{p+1}^{lb})^3)^{\kappa_{p+1}} - (1 - (\mathfrak{I}_{p+1}^{lb})^3)^{\kappa_{p+1}}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_{p+1}^{lb})^3)^{\kappa_{p+1}} + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_{p+1}^{lb})^3)^{\kappa_{p+1}}}}, \sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_{p+1}^{ub})^3)^{\kappa_{p+1}} - (1 - (\mathfrak{I}_{p+1}^{ub})^3)^{\kappa_{p+1}}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_{p+1}^{ub})^3)^{\kappa_{p+1}} + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_{p+1}^{ub})^3)^{\kappa_{p+1}}}}} \right], \right. \\ &\quad \left[\sqrt[3]{\frac{\mathfrak{E} \{ (1 - (\mathfrak{I}_{p+1}^{lb})^3)^{\kappa_{p+1}} - (1 - (\mathfrak{I}_{p+1}^{lb})^3 - (\wp_{p+1}^{lb})^3)^{\kappa_{p+1}} \}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_{p+1}^{lb})^3)^{\kappa_{p+1}} + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_{p+1}^{lb})^3)^{\kappa_{p+1}}}}, \sqrt[3]{\frac{\mathfrak{E} \{ (1 - (\mathfrak{I}_{p+1}^{ub})^3)^{\kappa_{p+1}} - (1 - (\mathfrak{I}_{p+1}^{ub})^3 - (\wp_{p+1}^{ub})^3)^{\kappa_{p+1}} \}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_{p+1}^{ub})^3)^{\kappa_{p+1}} + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_{p+1}^{ub})^3)^{\kappa_{p+1}}}}} \right] \right\rangle \\ &= \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right], \right. \\ &\quad \left[\sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^{p+1} (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right] \right\rangle. \end{aligned}$$

$$\left[\sqrt[3]{\frac{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\mathfrak{E} \left\{ \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \right\}}{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\mathfrak{E} \left\{ \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^{p+1} (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^{p+1} (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}} \right] \Bigg\rangle.$$

⇒ Result is true.

Example 1 Let $J_1 = \langle [0.2, 0.4], [0.3, 0.6] \rangle$, $J_2 = \langle [0.4, 0.7], [0.1, 0.3] \rangle$, $J_3 = \langle [0.4, 0.9], [0.1, 0.3] \rangle$ and $J_4 = \langle [0.1, 0.5], [0.3, 0.7] \rangle$ be four interval-valued Fermatean fuzzy values and $\kappa = (0.2, 0.3, 0.4, 0.1)^T$ be the WV of J_t ($t = 1, 2, 3, 4$), then

$$\begin{aligned} \mathfrak{Z}_1^{lb} &= 0.2, \mathfrak{Z}_1^{ub} = 0.4, \\ \mathfrak{Z}_2^{lb} &= 0.4, \mathfrak{Z}_2^{ub} = 0.7, \\ \mathfrak{Z}_3^{lb} &= 0.4, \mathfrak{Z}_3^{ub} = 0.9, \\ \mathfrak{Z}_4^{lb} &= 0.1, \mathfrak{Z}_4^{ub} = 0.5, \\ \wp_1^{lb} &= 0.3, \wp_1^{ub} = 0.6, \\ \wp_2^{lb} &= 0.1, \wp_2^{ub} = 0.3, \\ \wp_3^{lb} &= 0.1, \wp_3^{ub} = 0.3, \\ \wp_4^{lb} &= 0.3, \wp_4^{ub} = 0.7. \end{aligned}$$

Thus, for $\mathfrak{E} = 2$

$$\begin{aligned} &IVFFHIWA(J_1, J_2, J_3, J_4) \\ &= \left\langle \sqrt[3]{\frac{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{\mathfrak{E} \left\{ \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \right\}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\mathfrak{E} \left\{ \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}} \right] \Bigg\rangle \\ &= ([0.34, 0.77], [0.24, 0.42]). \end{aligned}$$

Remark 1 We discuss different cases for IVFFHIWA operator.

- For $\mathfrak{E} = 1$, IVFFHIWA operator reduces to IVFF interactive weighted averaging (IVFFIWA) operator:

$$\begin{aligned} &IVFFIWA(J_1, J_2, \dots, J_q) \\ &= \left\langle \sqrt[3]{1 - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}, \sqrt[3]{1 - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}, \right. \\ &\quad \left. \sqrt[3]{\prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t}}, \sqrt[3]{\prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t}} \right] \Bigg\rangle. \end{aligned}$$

- For $\mathfrak{E} = 2$, IVFFHIWA operator reduces to IVFF Einstein interactive weighted averaging (IVFFEIWA) operator:

$$\begin{aligned} &IVFFEIWA(J_1, J_2, \dots, J_q) \\ &= \left\langle \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_t^{lb})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}}, \right. \\ &\quad \left. \sqrt[3]{\frac{2 \left\{ \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_t^{lb})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}}, \sqrt[3]{\frac{2 \left\{ \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}} \right] \Bigg\rangle. \end{aligned}$$

Theorem 2 Let $J_t = \langle [\mathfrak{Z}_t^{lb}, \mathfrak{Z}_t^{ub}], [\wp_t^{lb}, \wp_t^{ub}] \rangle$ be IVFFNs, then clumped value by IVFFHIWA operator is a IVFFN, i.e.,

$$IVFFHIWA(J_1, J_2, \dots, J_q) \in IVFFN.$$

Proof As J_t 's are IVFFNs, so $0 \leq \mathfrak{Z}_t^{lb}, \mathfrak{Z}_t^{ub}, \wp_t^{lb}, \wp_t^{ub} \leq 1$ and $0 \leq (\mathfrak{Z}_t^{ub})^3 + (\wp_t^{ub})^3 \leq 1$. Therefore,

$$\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}$$

$$\begin{aligned}
 &= 1 - \frac{\mathfrak{L} \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}} \\
 &\leq 1 - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} \leq 1.
 \end{aligned} \quad (1)$$

Also, $(1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3) \geq (1 - (\mathfrak{I}_t^{ub})^3) \Rightarrow \prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3) - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3) \geq 0$. Therefore,

$$\begin{aligned}
 &\frac{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}} \geq 0 \\
 &\Rightarrow \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \geq 0.
 \end{aligned}$$

Thus, $0 \leq \mathfrak{I}_{IVFFHIWA}^{ub} \leq 1$.
Moreover,

$$\begin{aligned}
 &\frac{\mathfrak{L} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}} \\
 &\leq \frac{\mathfrak{L} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}} \\
 &\leq \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} \\
 &\leq 1.
 \end{aligned}$$

Also,

$$\begin{aligned}
 &\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} \geq 0 \\
 &\frac{\mathfrak{L} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}} \geq 0 \\
 &\sqrt[3]{\frac{\mathfrak{L} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \geq 0
 \end{aligned}$$

Thus, $0 \leq \mathfrak{I}_{IVFFHIWA}^{ub} \leq 1$.

Property 1 (“Idempotency”) If $J_t = J_o = \langle [\mathfrak{I}_o^{lb}, \mathfrak{I}_o^{ub}], [\mathfrak{I}_o^{lb}, \mathfrak{I}_o^{ub}] \rangle, \forall t$, then

$$IVFFHIWA(J_1, J_2, \dots, J_q) = J_o.$$

Proof As $J_t = J_o = \langle [\mathfrak{I}_o^{lb}, \mathfrak{I}_o^{ub}], [\mathfrak{I}_o^{lb}, \mathfrak{I}_o^{ub}] \rangle (\forall t = 1, 2, \dots, q)$ and $\sum_{t=1}^q \kappa_t = 1$. According to Theorem 1,

$$IVFFHIWA(J_1, J_2, \dots, J_q)$$

$$\begin{aligned}
 &= \left\langle \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_o^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_o^{lb})^3)^{\kappa_t}}}, \right. \\
 &\quad \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_o^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_o^{ub})^3)^{\kappa_t}}}, \\
 &\quad \left. \sqrt[3]{\frac{\mathfrak{L} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_o^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_o^{lb})^3 - (\mathfrak{I}_o^{lb})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_o^{lb})^3)^{\kappa_t}}}, \right. \\
 &\quad \left. \sqrt[3]{\frac{\mathfrak{L} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_o^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_o^{ub})^3 - (\mathfrak{I}_o^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3)^{\kappa_t} + (\mathfrak{L} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_o^{ub})^3)^{\kappa_t}}} \right\rangle \\
 &= \left\langle \sqrt[3]{\frac{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t} - (1 - (\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t}}{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t} + (\mathfrak{L} - 1)(1 - (\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t}}}, \right. \\
 &\quad \sqrt[3]{\frac{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t} - (1 - (\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t}}{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t} + (\mathfrak{L} - 1)(1 - (\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t}}}, \\
 &\quad \left. \sqrt[3]{\frac{\mathfrak{L} \{(1 - (\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t} - (1 - (\mathfrak{I}_o^{lb})^3 - (\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t}\}}{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t} + (\mathfrak{L} - 1)(1 - (\mathfrak{I}_o^{lb})^3)^{\sum_{t=1}^q \kappa_t}}}, \right. \\
 &\quad \left. \sqrt[3]{\frac{\mathfrak{L} \{(1 - (\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t} - (1 - (\mathfrak{I}_o^{ub})^3 - (\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t}\}}{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t} + (\mathfrak{L} - 1)(1 - (\mathfrak{I}_o^{ub})^3)^{\sum_{t=1}^q \kappa_t}}} \right\rangle \\
 &= \left\langle \sqrt[3]{\frac{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3) - (1 - (\mathfrak{I}_o^{lb})^3)}{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3) + (\mathfrak{L} - 1)(1 - (\mathfrak{I}_o^{lb})^3)}}, \right. \\
 &\quad \sqrt[3]{\frac{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3) - (1 - (\mathfrak{I}_o^{ub})^3)}{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{ub})^3) + (\mathfrak{L} - 1)(1 - (\mathfrak{I}_o^{ub})^3)}}, \\
 &\quad \left. \sqrt[3]{\frac{\mathfrak{L} \{(1 - (\mathfrak{I}_o^{lb})^3) - (1 - (\mathfrak{I}_o^{lb})^3 - (\mathfrak{I}_o^{lb})^3)\}}{(1 + (\mathfrak{L} - 1)(\mathfrak{I}_o^{lb})^3) + (\mathfrak{L} - 1)(1 - (\mathfrak{I}_o^{lb})^3)}} \right\rangle
 \end{aligned}$$

$$\sqrt[3]{\frac{\mathfrak{E}\{(1 - (\mathfrak{Z}_o^{ub})^3) - (1 - (\mathfrak{Z}_o^{ub})^3 - (\wp_o^{ub})^3)\}}{(1 + (\mathfrak{E} - 1)(\mathfrak{Z}_o^{ub})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_o^{ub})^3)}}\Bigg] \\ = \langle [\mathfrak{Z}_o^{lb}, \mathfrak{Z}_o^{ub}], [\wp_o^{lb}, \wp_o^{ub}] \rangle.$$

Property 2 (“Boundedness”) Let $J^- = \langle [\min_t(\mathfrak{Z}_t^{lb}), \min_t(\mathfrak{Z}_t^{ub}), [\max_t(\wp_t^{lb}), \max_t(\wp_t^{ub})]] \rangle$ and $J^+ = \langle [\max_t(\mathfrak{Z}_t^{lb}), \max_t(\mathfrak{Z}_t^{ub}), [\min_t(\wp_t^{lb}), \min_t(\wp_t^{ub})]] \rangle$, then

$$J^- \leq IVFFHIWA(J_1, J_2, \dots, J_q) \leq J^+.$$

Proof Assume $f(\mathfrak{r}) = \frac{1-\mathfrak{r}}{1+(\mathfrak{E}-1)\mathfrak{r}}$, $\mathfrak{r} \in [0, 1]$, then $f'(\mathfrak{r}) = -\frac{\mathfrak{E}}{(1+(\mathfrak{E}-1)\mathfrak{r})^2} < 0$, so $f(\mathfrak{r})$ is a decreasing function (DF). As $(\mathfrak{Z}_{t,\min}^{lb})^3 \leq (\mathfrak{Z}_t^{lb})^3 \leq (\mathfrak{Z}_{t,\max}^{lb})^3$, $\forall t = 1, 2, \dots, q$, $\Rightarrow f((\mathfrak{Z}_{t,\max}^{lb})^3) \leq f((\mathfrak{Z}_t^{lb})^3) \leq f((\mathfrak{Z}_{t,\min}^{lb})^3)$, i.e., $\frac{1-(\mathfrak{Z}_{t,\max}^{lb})^3}{1+(\mathfrak{E}-1)(\mathfrak{Z}_{t,\max}^{lb})^3} \leq \frac{1-(\mathfrak{Z}_t^{lb})^3}{1+(\mathfrak{E}-1)(\mathfrak{Z}_t^{lb})^3} \leq \frac{1-(\mathfrak{Z}_{t,\min}^{lb})^3}{1+(\mathfrak{E}-1)(\mathfrak{Z}_{t,\min}^{lb})^3}$, $\forall t$. Then,

$$\begin{aligned} & \left(\frac{1 - (\mathfrak{Z}_{t,\max}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3} \right)^{\kappa_t} \\ & \leq \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t} \leq \left(\frac{1 - (\mathfrak{Z}_{t,\min}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3} \right)^{\kappa_t} \\ & \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_{t,\max}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3} \right)^{\kappa_t} \leq \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t} \\ & \leq \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_{t,\min}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3} \right)^{\kappa_t} \\ & \Leftrightarrow \left(\frac{1 - (\mathfrak{Z}_{t,\max}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \leq \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t} \\ & \leq \left(\frac{1 - (\mathfrak{Z}_{t,\min}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \Leftrightarrow \left(\frac{1 - (\mathfrak{Z}_{t,\max}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \\ & \leq \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t} \leq \left(\frac{1 - (\mathfrak{Z}_{t,\min}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \\ & \Leftrightarrow (\mathfrak{E} - 1) \left(\frac{1 - (\mathfrak{Z}_{t,\max}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \\ & \leq (\mathfrak{E} - 1) \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t} \\ & \leq (\mathfrak{E} - 1) \left(\frac{1 - (\mathfrak{Z}_{t,\min}^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \\ & \Leftrightarrow \left(\frac{\mathfrak{E}}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \leq 1 + (\mathfrak{E} - 1) \\ & \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t} \leq \left(\frac{\mathfrak{E}}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3} \right)^{\sum_{t=1}^q \kappa_t} \end{aligned}$$

$$\begin{aligned} & \Leftrightarrow \left(\frac{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3}{\mathfrak{E}} \right)^{\sum_{t=1}^q \kappa_t} \\ & \leq \frac{1}{1 + (\mathfrak{E} - 1) \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t}} \\ & \leq \left(\frac{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3}{\mathfrak{E}} \right)^{\sum_{t=1}^q \kappa_t} \\ & \Leftrightarrow \left(1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3 \right)^{\sum_{t=1}^q \kappa_t} \\ & \leq \frac{\mathfrak{E}}{1 + (\mathfrak{E} - 1) \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t}} \\ & \leq \left(1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3 \right)^{\sum_{t=1}^q \kappa_t} \Leftrightarrow (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\min}^{lb})^3 \\ & \leq \frac{\mathfrak{E}}{1 + (\mathfrak{E} - 1) \prod_{t=1}^q \left(\frac{1 - (\mathfrak{Z}_t^{lb})^3}{1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3} \right)^{\kappa_t}} - 1 \\ & \leq (\mathfrak{E} - 1)(\mathfrak{Z}_{t,\max}^{lb})^3 \Leftrightarrow (\mathfrak{Z}_{t,\min}^{lb})^3 \\ & \leq \frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}} \\ & \leq (\mathfrak{Z}_{t,\max}^{lb})^3. \end{aligned}$$

Thus,

$$\mathfrak{Z}_{t,\min}^{lb} \leq \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{lb})^3)^{\kappa_t}}} \leq \mathfrak{Z}_{t,\max}^{lb}. \quad (2)$$

Similarly,

$$\mathfrak{Z}_{t,\min}^{ub} \leq \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{Z}_t^{ub})^3)^{\kappa_t}}} \leq \mathfrak{Z}_{t,\max}^{ub}. \quad (3)$$

Consider $g(\mathfrak{s}) = \frac{\mathfrak{E} - (\mathfrak{E} - 1)\mathfrak{s}}{(\mathfrak{E} - 1)\mathfrak{s}}$, $\mathfrak{s} \in (0, 1]$, then $g'(\mathfrak{s}) = -\frac{\mathfrak{E}}{(\mathfrak{E} - 1)\mathfrak{s}^2}$, i.e., $g(\mathfrak{s})$ is a DF on $(0, 1]$. Since $1 - (\mathfrak{Z}_{t,\max}^{lb})^3 \leq 1 - (\mathfrak{Z}_t^{lb})^3 \leq 1 - (\mathfrak{Z}_{t,\min}^{lb})^3$, $\Rightarrow g(1 - (\mathfrak{Z}_{t,\max}^{lb})^3) \leq g(1 - (\mathfrak{Z}_t^{lb})^3) \leq g(1 - (\mathfrak{Z}_{t,\min}^{lb})^3)$, i.e., $\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,\min}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,\min}^{lb})^3)} \leq \frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \leq \frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,\max}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,\max}^{lb})^3)}$. Then,

$$\left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,\min}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,\min}^{lb})^3)} \right)^{\sum_{t=1}^q \kappa_t}$$

$$\begin{aligned}
&\leq \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \right)^{\kappa_t} \\
&\leq \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)} \right)^{\kappa_t} \\
&\prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)} \right)^{\kappa_t} \\
&\leq \prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \right)^{\kappa_t} \\
&\leq \prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)} \right)^{\kappa_t} \\
&\Rightarrow \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)} \right)^{\sum_{t=1}^q \kappa_t} \\
&\leq \prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \right)^{\kappa_t} \\
&\leq \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)} \right)^{\sum_{t=1}^q \kappa_t} \\
&\Rightarrow \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)} \right) \\
&\leq \prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \right)^{\kappa_t} \\
&\leq \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)} \right) \\
&\Rightarrow \left(\frac{\mathfrak{E}}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)} \right) \\
&\leq \prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \right)^{\kappa_t} + 1 \\
&\leq \left(\frac{\mathfrak{E}}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)} \right) \\
&\Rightarrow \left(\frac{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,max}^{lb})^3)}{\mathfrak{E}} \right) \\
&\leq \frac{1}{\prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \right)^{\kappa_t} + 1} \\
&\leq \left(\frac{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_{t,min}^{lb})^3)}{\mathfrak{E}} \right) \\
&\Rightarrow (1 - (\mathfrak{Z}_{t,max}^{lb})^3) \\
&\leq \frac{\mathfrak{E}}{(\mathfrak{E} - 1) \prod_{t=1}^q \left(\frac{\mathfrak{E} - (\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)}{(\mathfrak{E} - 1)(1 - (\mathfrak{Z}_t^{lb})^3)} \right)^{\kappa_t} + (\mathfrak{E} - 1)} \\
&\leq (1 - (\mathfrak{Z}_{t,min}^{lb})^3) \\
&\Rightarrow (1 - (\mathfrak{Z}_{t,max}^{lb})^3)
\end{aligned}$$

$$\begin{aligned} &\leq \frac{\mathfrak{L}}{(\mathfrak{L}-1) \prod_{t=1}^q \left(\frac{\mathfrak{L}-(\mathfrak{L}-1)(\wp_t^{lb})^3}{(\mathfrak{L}-1)(\wp_t^{lb})^3} \right)^{\kappa_t}} + (\mathfrak{L}-1) \\ &\leq (1 - (\mathfrak{S}_{t,min}^{lb})^3). \end{aligned}$$

Also,

$$\begin{aligned}
1 - (\mathfrak{I}_{t,max}^{lb})^3 - (\wp_{t,min}^{lb})^3 &\leq 1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3 \\
&\leq 1 - (\mathfrak{I}_{t,min}^{lb})^3 - (\wp_{t,max}^{lb})^3 \\
&\Leftrightarrow \frac{1 - (\mathfrak{I}_{t,max}^{lb})^3 - (\wp_{t,min}^{lb})^3}{1 - (\mathfrak{I}_{t,min}^{lb})^3} \leq \frac{1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3}{1 - (\mathfrak{I}_t^{lb})^3} \\
&\leq \frac{1 - (\mathfrak{I}_{t,min}^{lb})^3 - (\wp_{t,max}^{lb})^3}{1 - (\mathfrak{I}_{t,max}^{lb})^3} \\
&\Leftrightarrow \frac{1 - (\mathfrak{I}_{t,max}^{lb})^3 - (\wp_{t,min}^{lb})^3}{1 - (\mathfrak{I}_{t,min}^{lb})^3} \leq \prod_{t=1}^q \left(\frac{1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3}{1 - (\mathfrak{I}_t^{lb})^3} \right)^{\kappa_t} \\
&\leq \frac{1 - (\mathfrak{I}_{t,min}^{lb})^3 - (\wp_{t,max}^{lb})^3}{1 - (\mathfrak{I}_{t,max}^{lb})^3} \\
&\Leftrightarrow \frac{-(\mathfrak{I}_{t,max}^{lb})^3 + (\mathfrak{I}_{t,min}^{lb})^3 + (\wp_{t,max}^{lb})^3}{1 - (\mathfrak{I}_{t,max}^{lb})^3} \\
&\leq 1 - \prod_{t=1}^q \left(\frac{1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3}{1 - (\mathfrak{I}_t^{lb})^3} \right)^{\kappa_t} \\
&\leq \frac{-(\mathfrak{I}_{t,min}^{lb})^3 + (\mathfrak{I}_{t,max}^{lb})^3 + (\wp_{t,min}^{lb})^3}{1 - (\mathfrak{I}_{t,min}^{lb})^3} \\
&\Leftrightarrow -(\mathfrak{I}_{t,max}^{lb})^3 + (\mathfrak{I}_{t,min}^{lb})^3 + (\wp_{t,max}^{lb})^3 \\
&\quad \mathfrak{E} \left\{ 1 - \prod_{t=1}^q \left(\frac{1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3}{1 - (\mathfrak{I}_t^{lb})^3} \right)^{\kappa_t} \right\} \\
&\leq \frac{(\mathfrak{E} - 1) \prod_{t=1}^q \left(\frac{1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3}{(\mathfrak{E} - 1)(1 - (\mathfrak{I}_t^{lb})^3)} \right)^{\kappa_t} + (\mathfrak{E} - 1)}{(\mathfrak{E} - 1) \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1)} \\
&\leq -(\mathfrak{I}_{t,min}^{lb})^3 + (\mathfrak{I}_{t,max}^{lb})^3 + (\wp_{t,min}^{lb})^3 \\
&\Leftrightarrow (\wp_{t,max}^{lb})^3 \\
&\quad \mathfrak{E} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \right\} \\
&\leq \frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1)} \\
&\leq \wp_{t,min}^3 \Leftrightarrow \wp_{t,max}^{lb} \\
&\leq \sqrt[3]{\frac{\mathfrak{E} \left\{ \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}} \\
&\leq \wp_{t,min}^{lb}
\end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned} & \wp_{t,max}^{ub} \\ & \leq \sqrt[3]{\frac{\mathfrak{L}\{\prod_{t=1}^q (1-(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1-(\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t}\}}{\prod_{t=1}^q (1+(\mathfrak{L}-1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{L}-1) \prod_{t=1}^q (1-(\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \\ & \leq \wp_{t,min}^{ub} \end{aligned} \quad (5)$$

Let $IVFFHIWA(J_1, J_2, \dots, J_q) = J = [\mathfrak{I}_J^{lb}, \mathfrak{I}_J^{ub}], [\wp_J^{lb}, \wp_J^{ub}]$, then from inequalities 2, 3, 4, and 5, $\mathfrak{I}_{min}^{lb} \leq \mathfrak{I}_J^{lb} \leq \mathfrak{I}_{max}^{ub}$, $\mathfrak{I}_{min}^{ub} \leq \mathfrak{I}_J^{ub} \leq \mathfrak{I}_{max}^{ub}$, $\wp_{max}^{lb} \leq \wp_J^{lb} \leq \wp_{min}^{lb}$ and $\wp_{max}^{ub} \leq \wp_J^{ub} \leq \wp_{min}^{ub}$, where $\mathfrak{I}_{min}^{lb} = \min_t \{\mathfrak{I}_t^{lb}\}$, $\mathfrak{I}_{min}^{ub} = \min_t \{\mathfrak{I}_t^{ub}\}$, $\mathfrak{I}_{max}^{lb} = \max_t \{\mathfrak{I}_t^{lb}\}$, $\mathfrak{I}_{max}^{ub} = \max_t \{\mathfrak{I}_t^{ub}\}$, $\wp_{min}^{lb} = \min_t \{\wp_t^{lb}\}$, $\wp_{min}^{ub} = \min_t \{\wp_t^{ub}\}$, $\wp_{max}^{lb} = \max_t \{\wp_t^{lb}\}$ and $\wp_{max}^{ub} = \max_t \{\wp_t^{ub}\}$. So, $S(J) = \frac{1}{2}[(\mathfrak{I}_J^{lb})^3 + (\mathfrak{I}_J^{ub})^3 - (\wp_J^{lb})^3 - (\wp_J^{ub})^3] \leq \frac{1}{2}[(\mathfrak{I}_{max}^{lb})^3 + (\mathfrak{I}_{max}^{ub})^3 - (\wp_{max}^{lb})^3 - (\wp_{max}^{ub})^3] = S(J^+)$ and

$$\begin{aligned} S(J) &= \frac{1}{2}[(\mathfrak{I}_J^{lb})^3 + (\mathfrak{I}_J^{ub})^3 - (\wp_J^{lb})^3 - (\wp_J^{ub})^3] \geq \\ & \frac{1}{2}[(\mathfrak{I}_{min}^{lb})^3 + (\mathfrak{I}_{min}^{ub})^3 - (\wp_{min}^{lb})^3 - (\wp_{min}^{ub})^3] = S(J^-). \text{ As } \\ & S(J) < S(J^+) \text{ and } S(J) > S(J^-). \text{ So } \end{aligned}$$

$$J^- \leq IVFFHIWA(J_1, J_2, \dots, J_q) \leq J^+.$$

Property 3 (“Monotonicity”) When $J_t \leq \mathcal{T}_t, \forall t$, then

$$\begin{aligned} & IVFFHIWA(J_1, J_2, \dots, J_q) \\ & \leq IVFFHIWA(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_q). \end{aligned}$$

Proof Same as above.

Property 4 (“Shift Invariance”) Suppose $\mathcal{T} = (\mathfrak{I}_{\mathcal{T}}, \wp_{\mathcal{T}})$ is other IVFFN, then

$$\begin{aligned} & IVFFHIWA(J_1 \oplus \mathcal{T}, J_2 \oplus \mathcal{T}, \dots, J_q \oplus \mathcal{T}) \\ & = IVFFHIWA(J_1, J_2, \dots, J_q) \oplus \mathcal{T}. \end{aligned}$$

Proof As $J_t, \mathcal{T} \in IVFFNs$, so

$$\begin{aligned} J_t \oplus \mathcal{T} &= \left\langle \sqrt[3]{\frac{(1+(\mathfrak{L}-1)(\mathfrak{I}_t^{lb})^3)(1+(\mathfrak{L}-1)(\mathfrak{I}_{\mathcal{T}}^{lb})^3) - (1-(\mathfrak{I}_t^{lb})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{lb})^3)}{(1+(\mathfrak{L}-1)(\mathfrak{I}_t^{lb})^3)(1+(\mathfrak{L}-1)(\mathfrak{I}_{\mathcal{T}}^{lb})^3) + (\mathfrak{L}-1)(1-(\mathfrak{I}_t^{lb})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{lb})^3)}}, \right. \\ & \sqrt[3]{\frac{(1+(\mathfrak{L}-1)(\mathfrak{I}_t^{ub})^3)(1+(\mathfrak{L}-1)(\mathfrak{I}_{\mathcal{T}}^{ub})^3) - (1-(\mathfrak{I}_t^{ub})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{ub})^3)}{(1+(\mathfrak{L}-1)(\mathfrak{I}_t^{ub})^3)(1+(\mathfrak{L}-1)(\mathfrak{I}_{\mathcal{T}}^{ub})^3) + (\mathfrak{L}-1)(1-(\mathfrak{I}_t^{ub})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{ub})^3)}}, \\ & \left[\sqrt[3]{\frac{\mathfrak{L}\{(1-(\mathfrak{I}_t^{lb})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{lb})^3) - (1-(\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{lb})^3 - (\wp_{\mathcal{T}}^{lb})^3)\}}{(1+(\mathfrak{L}-1)(\mathfrak{I}_t^{lb})^3)(1+(\mathfrak{L}-1)(\mathfrak{I}_{\mathcal{T}}^{lb})^3) + (\mathfrak{L}-1)(1-(\mathfrak{I}_t^{lb})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{lb})^3)}}, \right. \\ & \left. \sqrt[3]{\frac{\mathfrak{L}\{(1-(\mathfrak{I}_t^{ub})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{ub})^3) - (1-(\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{ub})^3 - (\wp_{\mathcal{T}}^{ub})^3)\}}{(1+(\mathfrak{L}-1)(\mathfrak{I}_t^{ub})^3)(1+(\mathfrak{L}-1)(\mathfrak{I}_{\mathcal{T}}^{ub})^3) + (\mathfrak{L}-1)(1-(\mathfrak{I}_t^{ub})^3)(1-(\mathfrak{I}_{\mathcal{T}}^{ub})^3)}} \right] \end{aligned}$$

Therefore, $IVFFHIWA(J_1 \oplus \mathcal{T}, J_2 \oplus \mathcal{T}, \dots, J_q \oplus \mathcal{T})$

[illegible]

$$\begin{aligned}
& \left[\sqrt[3]{\frac{\mathfrak{E}\left(\left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right\}(1 - (\mathfrak{I}_T^{lb})^3) - \left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t}\right\}(1 - (\mathfrak{I}_T^{lb})^3 - (\wp_T^{lb})^3)\right)}{\left\{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right\}(1 + (\mathfrak{E} - 1)(\mathfrak{I}_T^{lb})^3) + (\mathfrak{E} - 1)\left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right\}(1 - (\mathfrak{I}_T^{lb})^3)}}}, \right. \\
& \left. \sqrt[3]{\frac{\mathfrak{E}\left(\left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right\}(1 - (\mathfrak{I}_T^{ub})^3) - \left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t}\right\}(1 - (\mathfrak{I}_T^{ub})^3 - (\wp_T^{ub})^3)\right)}{\left\{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right\}(1 + (\mathfrak{E} - 1)(\mathfrak{I}_T^{ub})^3) + (\mathfrak{E} - 1)\left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right\}(1 - (\mathfrak{I}_T^{ub})^3)}}}\right] \Bigg\rangle \\
& = IVFFHIWA(J_1, J_2, \dots, J_q) \oplus \mathcal{T}.
\end{aligned}$$

Property 5 (Homogeneity) Let $\beta > 0$, then

$$\begin{aligned}
& IVFFHIWA(\beta J_1, \beta J_2, \dots, \beta J_q) \\
& = \beta IVFFHIWA(J_1, J_2, \dots, J_q).
\end{aligned}$$

Proof As $J_t = \langle [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}], [\wp_t^{lb}, \wp_t^{ub}] \rangle$ are IVFFNs. Therefore, for $\beta > 0$

$$\begin{aligned}
\beta J_t = & \left\langle \left[\sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^\beta - (1 - (\mathfrak{I}_t^{lb})^3)^\beta}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^\beta + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_t^{lb})^3)^\beta}}, \right. \right. \\
& \sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^\beta - (1 - (\mathfrak{I}_t^{ub})^3)^\beta}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^\beta + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_t^{ub})^3)^\beta}}, \\
& \left. \left[\sqrt[3]{\frac{\mathfrak{E}\{(1 - (\mathfrak{I}_t^{lb})^3)^\beta - (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^\beta\}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^\beta + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_t^{lb})^3)^\beta}}, \right. \right. \\
& \left. \left. \sqrt[3]{\frac{\mathfrak{E}\{(1 - (\mathfrak{I}_t^{ub})^3)^\beta - (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^\beta\}}{(1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^\beta + (\mathfrak{E} - 1)(1 - (\mathfrak{I}_t^{ub})^3)^\beta}} \right] \right\rangle.
\end{aligned}$$

Therefore, $IVFFHIWA(\beta J_1, \beta J_2, \dots, \beta J_q) =$

$$\begin{aligned}
& \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^\beta)^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{lb})^3)^\beta)^{\kappa_t}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^\beta)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{lb})^3)^\beta)^{\kappa_t}}}, \right. \right. \\
& \left. \sqrt[3]{\frac{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^\beta)^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{ub})^3)^\beta)^{\kappa_t}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^\beta)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{ub})^3)^\beta)^{\kappa_t}}}, \right. \\
& \left. \left[\sqrt[3]{\frac{\mathfrak{E}\{\prod_{t=1}^q ((1 - (\mathfrak{I}_t^{lb})^3)^\beta)^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^\beta)^{\kappa_t}\}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^\beta)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{lb})^3)^\beta)^{\kappa_t}}}, \right. \right. \\
& \left. \left. \sqrt[3]{\frac{\mathfrak{E}\{\prod_{t=1}^q ((1 - (\mathfrak{I}_t^{ub})^3)^\beta)^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^\beta)^{\kappa_t}\}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^\beta)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{I}_t^{ub})^3)^\beta)^{\kappa_t}}} \right] \right\rangle
\end{aligned}$$

$$\begin{aligned}
& = \left\langle \left[\sqrt[3]{\frac{\left(\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right)^\beta - \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right)^\beta}{\left(\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right)^\beta + (\mathfrak{E} - 1) \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right)^\beta}}, \right. \right. \\
& \left. \sqrt[3]{\frac{\left(\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right)^\beta - \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right)^\beta}{\left(\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right)^\beta + (\mathfrak{E} - 1) \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right)^\beta}}, \right. \\
& \left. \left[\sqrt[3]{\frac{\mathfrak{E}\left\{\left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right)^\beta - \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t}\right)^\beta\right\}}{\left(\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right)^\beta + (\mathfrak{E} - 1) \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}\right)^\beta}}, \right. \right. \\
& \left. \left. \sqrt[3]{\frac{\mathfrak{E}\left\{\left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right)^\beta - \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t}\right)^\beta\right\}}{\left(\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right)^\beta + (\mathfrak{E} - 1) \left(\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}\right)^\beta}} \right] \right\rangle \\
& = \beta \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \right. \right. \\
& \left. \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}}, \right. \\
& \left. \left[\sqrt[3]{\frac{\mathfrak{E}\left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t}\right\}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{lb})^3)^{\kappa_t}}}, \right. \right. \\
& \left. \left. \sqrt[3]{\frac{\mathfrak{E}\left\{\prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t}\right\}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_t^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_t^{ub})^3)^{\kappa_t}}} \right] \right\rangle \\
& = \beta IVFFHIWA(J_1, J_2, \dots, J_q).
\end{aligned}$$

Property 6 Let $J_t = \langle [\mathfrak{I}_{J_t}^{lb}, \mathfrak{I}_{J_t}^{ub}], [\wp_{J_t}^{lb}, \wp_{J_t}^{ub}] \rangle$ and $\mathcal{T}_t = \langle [\mathfrak{I}_{\mathcal{T}_t}^{lb}, \mathfrak{I}_{\mathcal{T}_t}^{ub}], [\wp_{\mathcal{T}_t}^{lb}, \wp_{\mathcal{T}_t}^{ub}] \rangle$ be two collections of IVFFNs, then

$$\begin{aligned} & IVFFHIWA(J_1 \oplus \mathcal{T}_1, J_2 \oplus \mathcal{T}_2, \dots, J_q \oplus \mathcal{T}_q) \\ &= IVFFHIWA(J_1, J_2, \dots, J_q) \\ &\oplus IVFFHIWA(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_q). \end{aligned}$$

Proof As $J_t = \langle [\mathfrak{J}_{J_t}^{lb}, \mathfrak{J}_{J_t}^{ub}], [\wp_{J_t}^{lb}, \wp_{J_t}^{ub}] \rangle$ and $\mathcal{T}_t = \langle [\mathfrak{J}_{\mathcal{T}_t}^{lb}, \mathfrak{J}_{\mathcal{T}_t}^{ub}], [\wp_{\mathcal{T}_t}^{lb}, \wp_{\mathcal{T}_t}^{ub}] \rangle$ are IVFFNs, then $J_t \oplus \mathcal{T}_t =$

$$\begin{aligned} & \left\langle \left[\sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3) - (1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3)}{(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3)}}, \right. \right. \\ & \left. \sqrt[3]{\frac{(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{ub})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{ub})^3) - (1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3)}{(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{ub})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{ub})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3)}}, \right. \\ & \left. \left[\sqrt[3]{\frac{\mathfrak{E}\{(1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3) - (1 - (\mathfrak{J}_{J_t}^{lb})^3 - (\wp_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3 - (\wp_{\mathcal{T}_t}^{lb})^3)\}}{(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3)}, \right. \right. \\ & \left. \left. \sqrt[3]{\frac{\mathfrak{E}\{(1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3) - (1 - (\mathfrak{J}_{J_t}^{ub})^3 - (\wp_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3 - (\wp_{\mathcal{T}_t}^{ub})^3)\}}{(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{ub})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{ub})^3) + (\mathfrak{E} - 1)(1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3)}}, \right] \right\rangle \end{aligned}$$

Therefore,

$$IVFFHIWA(J_1 \oplus \mathcal{T}_1, J_2 \oplus \mathcal{T}_2, \dots, J_q \oplus \mathcal{T}_q)$$

$$\begin{aligned} &= \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3))^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3))^{\kappa_t}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3))^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3))^{\kappa_t}}}, \right. \right. \\ & \left. \sqrt[3]{\frac{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{ub})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{ub})^3))^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3))^{\kappa_t}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{ub})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{ub})^3))^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3))^{\kappa_t}}}, \right. \\ & \left. \left[\sqrt[3]{\frac{\mathfrak{E}\{\prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3))^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{lb})^3 - (\wp_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3 - (\wp_{\mathcal{T}_t}^{lb})^3))^{\kappa_t}\}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3))^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{lb})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3))^{\kappa_t}}}, \right. \right. \\ & \left. \left. \sqrt[3]{\frac{\mathfrak{E}\{\prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3))^{\kappa_t} - \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{ub})^3 - (\wp_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3 - (\wp_{\mathcal{T}_t}^{ub})^3))^{\kappa_t}\}}{\prod_{t=1}^q ((1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{ub})^3)(1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{ub})^3))^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q ((1 - (\mathfrak{J}_{J_t}^{ub})^3)(1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3))^{\kappa_t}}}, \right] \right\rangle \\ &= \left\langle \left[\sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{J}_{J_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{J}_{J_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{J}_{\mathcal{T}_t}^{lb})^3)^{\kappa_t}}}, \right. \right. \\ & \left. \left. \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{J}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{J}_{\mathcal{T}_t}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{J}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{J}_{\mathcal{T}_t}^{ub})^3)^{\kappa_t} \right] \right\rangle \end{aligned}$$

$$\begin{aligned}
& \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t}}} \\
& \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t}}} \\
& \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t}}} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& = \left\langle \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t}}}, \right. \\
& \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t}}}, \\
& \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t}}}, \\
& \left. \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3 - (\mathfrak{I}_{J_t}^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{J_t}^{ub})^3)^{\kappa_t}}} \right\rangle \\
& \oplus \left\langle \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t}}}, \right. \\
& \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t}}}, \\
& \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{lb})^3 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t}}}, \\
& \left. \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3 - (\mathfrak{I}_{T_t}^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{T_t}^{ub})^3)^{\kappa_t}}} \right\rangle
\end{aligned}$$

$$= IVFFHIWA(J_1, J_2, \dots, J_q) \oplus IVFFHIWA(T_1, T_2, \dots, T_q).$$

Property 7 Suppose $J_t = \langle [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}], [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}] \rangle$ and $\mathcal{T} = \langle [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}], [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}] \rangle$ are IVFFNs and $\eta > 0$, then

$$\begin{aligned}
& IVFFHIWA(\eta J_1 \oplus \mathcal{T}, \eta J_2 \oplus \mathcal{T}, \dots, \eta J_q \oplus \mathcal{T}) \\
& = \eta IVFFHIWA(J_1, J_2, \dots, J_q) \oplus \mathcal{T}.
\end{aligned}$$

Proof To prove it, use Properties 1, 5, and 6.

5 IVFF Hamacher interactive ordered weighted average operators

Definition 7 Let $J_t = \langle [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}], [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}] \rangle$ be a selection of IVFFNs, then $IVFFHIWA : \Omega^q \rightarrow \Omega$ is

$$\begin{aligned}
& IVFFHIWA(J_1, J_2, \dots, J_q) = \kappa_1 J_{\sigma(1)} \oplus \kappa_2 J_{\sigma(2)} \\
& \oplus \dots \kappa_q J_{\sigma(q)},
\end{aligned}$$

wherever $(\sigma(1), \sigma(2), \dots, \sigma(q))$ is a permutation of $(1, 2, \dots, q)$ s.t $\sigma(t-1) \geq \sigma(t)$ for any t .

Theorem 3 Suppose $J_t = \langle [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}], [\mathfrak{I}_t^{lb}, \mathfrak{I}_t^{ub}] \rangle$ is collection of IVFFNs, then

$$\begin{aligned}
& IVFFHIWA(J_1, J_2, \dots, J_q) \\
& = \left\langle \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{I}_{\sigma(t)}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{I}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{I}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \right.
\end{aligned}$$

$$\begin{aligned}
 & \left\langle \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3 - (\mathfrak{P}_{\sigma(t)}^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3 - (\mathfrak{P}_{\sigma(t)}^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}} \right\rangle. \quad (6)
 \end{aligned}$$

Proof It is same to Theorem 1.

Example 2 Let $J_1 = \langle [0.3, 0.4], [0.4, 0.5] \rangle$, $J_2 = \langle [0.4, 0.7], [0.4, 0.6] \rangle$, $J_3 = \langle [0.2, 0.7], [0.1, 0.4] \rangle$ and $J_4 = \langle [0.5, 0.7], [0.3, 0.6] \rangle$ be four interval-valued Fermatean fuzzy values and $\kappa = (0.3, 0.1, 0.2, 0.4)^T$ be the WV of J_t ($t = 1, 2, 3, 4$), then

$$\begin{aligned}
 \mathfrak{Z}_1^{lb} &= 0.3, \mathfrak{Z}_1^{ub} = 0.4, \\
 \mathfrak{Z}_2^{lb} &= 0.4, \mathfrak{Z}_2^{ub} = 0.7, \\
 \mathfrak{Z}_3^{lb} &= 0.2, \mathfrak{Z}_3^{ub} = 0.7, \\
 \mathfrak{Z}_4^{lb} &= 0.5, \mathfrak{Z}_4^{ub} = 0.7, \\
 \mathfrak{P}_1^{lb} &= 0.4, \mathfrak{P}_1^{ub} = 0.5, \\
 \mathfrak{P}_2^{lb} &= 0.4, \mathfrak{P}_2^{ub} = 0.6, \\
 \mathfrak{P}_3^{lb} &= 0.1, \mathfrak{P}_3^{ub} = 0.4, \\
 \mathfrak{P}_4^{lb} &= 0.3, \mathfrak{P}_4^{ub} = 0.6.
 \end{aligned}$$

Apply score function to calculate scores of J_t ($t = 1, 2, 3, 4$).

$$\begin{aligned}
 \mathcal{S}(J_1) &= -0.049, \\
 \mathcal{S}(J_2) &= 0.064, \\
 \mathcal{S}(J_3) &= 0.143, \\
 \mathcal{S}(J_4) &= 0.113.
 \end{aligned}$$

Since $\mathcal{S}(J_3) > \mathcal{S}(J_4) > \mathcal{S}(J_2) > \mathcal{S}(J_1)$,

$$\begin{aligned}
 J_{\sigma(1)} &= J_3 = \langle [0.2, 0.7], [0.1, 0.4] \rangle, \\
 J_{\sigma(2)} &= J_4 = \langle [0.5, 0.7], [0.3, 0.6] \rangle, \\
 J_{\sigma(3)} &= J_2 = \langle [0.4, 0.7], [0.4, 0.6] \rangle, \\
 J_{\sigma(4)} &= J_1 = \langle [0.3, 0.4], [0.4, 0.5] \rangle.
 \end{aligned}$$

Thus, for $\mathfrak{E} = 2$

$$IVFFHIOWA(J_1, J_2, J_3, J_4)$$

$$\begin{aligned}
 & = \left\langle \sqrt[3]{\frac{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3 - (\mathfrak{P}_{\sigma(t)}^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{\mathfrak{E} \{ \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3 - (\mathfrak{P}_{\sigma(t)}^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^4 (1 + (\mathfrak{E} - 1)(\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^4 (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}} \right\rangle \\
 & = \langle [0.33, 0.62], [0.34, 0.53] \rangle
 \end{aligned}$$

Remark 2 Special cases of IVFFHIOWA operator are:

- For $\mathfrak{E} = 1$, it reduces to IVFF interactive ordered weighted averaging (IVFFIOWA) operator:

$$\begin{aligned}
 & IVFFIOWA(J_1, J_2, \dots, J_q) \\
 & = \left\langle \sqrt[3]{1 - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}, \sqrt[3]{1 - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}, \right. \\
 & \left. \sqrt[3]{\prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3 - (\mathfrak{P}_{\sigma(t)}^{lb})^3)^{\kappa_t}}, \right. \\
 & \left. \sqrt[3]{\prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3 - (\mathfrak{P}_{\sigma(t)}^{ub})^3)^{\kappa_t}} \right\rangle.
 \end{aligned}$$

- For $\mathfrak{E} = 2$, it reduces to IVFF Einstein interactive ordered weighted averaging (IVFFEIOWA) operator:

$$\begin{aligned}
 & IVFFEIOWA(J_1, J_2, \dots, J_q) \\
 & = \left\langle \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{2 \{ \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3 - (\mathfrak{P}_{\sigma(t)}^{lb})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \right. \\
 & \left. \sqrt[3]{\frac{2 \{ \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3 - (\mathfrak{P}_{\sigma(t)}^{ub})^3)^{\kappa_t} \}}{\prod_{t=1}^q (1 + (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{Z}_{\sigma(t)}^{ub})^3)^{\kappa_t}}} \right\rangle,
 \end{aligned}$$

$$\begin{aligned}
& \sqrt[3]{\frac{2\left\{\prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3 - (\wp_{\sigma(t)}^{ub})^3)^{\kappa_t}\right\}}{\prod_{t=1}^q (1 + (\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t} + \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t}}}\right\}} \\
&= \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{S}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{S}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \\
& \sqrt[3]{\frac{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t}}}, \\
& \sqrt[3]{\frac{\mathfrak{E}\left\{\prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{lb})^3 - (\wp_{\sigma(t)}^{lb})^3)^{\kappa_t}\right\}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{S}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{lb})^3)^{\kappa_t}}}, \\
& \sqrt[3]{\frac{\mathfrak{E}\left\{\prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3 - (\wp_{\sigma(t)}^{ub})^3)^{\kappa_t}\right\}}{\prod_{t=1}^q (1 + (\mathfrak{E} - 1)(\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathfrak{E} - 1) \prod_{t=1}^q (1 - (\mathfrak{S}_{\sigma(t)}^{ub})^3)^{\kappa_t}}}\right\}}. \quad (7)
\end{aligned}$$

Property 8 Let $J_t = \langle [\mathfrak{S}_t^{lb}, \mathfrak{S}_t^{ub}], [\wp_t^{lb}, \wp_t^{ub}] \rangle$ be a selection of IVFFNs.

(i) Idempotency: If $J_t = J_o = \langle [\mathfrak{S}_o^{lb}, \mathfrak{S}_o^{ub}], [\wp_o^{lb}, \wp_o^{ub}] \rangle, \forall t$, then

$$IVFFHIOWA(J_1, J_2, \dots, J_q) = J_o.$$

(ii) Boundedness: Let $J^- = \langle [\min_t(\mathfrak{S}_t^{lb}), \min_t(\mathfrak{S}_t^{ub})], [\max_t(\wp_t^{lb}), \max_t(\wp_t^{ub})] \rangle$ and $J^+ = \langle [\max_t(\mathfrak{S}_t^{lb}), \max_t(\mathfrak{S}_t^{ub})], [\min_t(\wp_t^{lb}), \min_t(\wp_t^{ub})] \rangle$, then

$$J^- \leq IVFFHIOWA(J_1, J_2, \dots, J_q) \leq J^+.$$

(iii) Monotonicity: When $J_t \leq \mathcal{T}_t, \forall t$, then

$$IVFFHIOWA(J_1, J_2, \dots, J_q) \leq IVFFHIOWA(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_q).$$

(iv) Shift Invariance: If $\mathcal{T} = \langle [\mathfrak{S}_{\mathcal{T}}^{lb}, \mathfrak{S}_{\mathcal{T}}^{ub}], [\wp_{\mathcal{T}}^{lb}, \wp_{\mathcal{T}}^{ub}] \rangle$ is another IVFFN, then

$$IVFFHIOWA(J_1 \oplus \mathcal{T}, J_2 \oplus \mathcal{T}, \dots, J_q \oplus \mathcal{T}) = IVFFHIOWA(J_1, J_2, \dots, J_q) \oplus \mathcal{T}.$$

(v) Homogeneity: Let $\beta > 0$, then

$$IVFFHIOWA(\beta J_1, \beta J_2, \dots, \beta J_q) = \beta IVFFHIOWA(J_1, J_2, \dots, J_q).$$

6 IVFF Hamacher interactive hybrid weighted averaging operator

Definition 8 Suppose $J_t = \langle [\mathfrak{S}_t^{lb}, \mathfrak{S}_t^{ub}], [\wp_t^{lb}, \wp_t^{ub}] \rangle$ is a family of IVFFNs, $IVFFHIHWA : \Omega^q \rightarrow \Omega$ is defined as

$$IVFFHIHWA(J_1, J_2, \dots, J_q) = \kappa_1 \dot{J}_1 \oplus \kappa_2 \dot{J}_2 \oplus \dots \kappa_q \dot{J}_q,$$

where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_q)^T$ is the WV associated with IVFFHIHWA operator and $\phi = (\phi_1, \phi_2, \dots, \phi_q)^T$ is the WV of J_t s.t. $\sum_{t=1}^q \phi_t = 1, \phi_t \in [0, 1]$. Assume $\dot{J}$ is the t th largest of the weighted IVFFNs ($\dot{J} = q\phi_t J_t$).

Theorem 4 Suppose $J_t = \langle [\mathfrak{S}_t^{lb}, \mathfrak{S}_t^{ub}], [\wp_t^{lb}, \wp_t^{ub}] \rangle$ is a family of IVFFNs, then

$$IVFFHIHWA(J_1, J_2, \dots, J_q)$$

Proof Similar to Theorem 1.

Example 3 Let $J_1 = \langle [0.2, 0.4], [0.3, 0.6] \rangle, J_2 = \langle [0.4, 0.7], [0.1, 0.3] \rangle, J_3 = \langle [0.2, 0.7], [0.1, 0.4] \rangle$ and $J_4 = \langle [0.5, 0.7], [0.3, 0.6] \rangle$ be four interval-valued Fermatean fuzzy values, $\phi = (0.3, 0.1, 0.2, 0.4)^T$ be the WV of J_t ($t = 1, 2, 3, 4$) and $\kappa = (0.2, 0.1, 0.2, 0.5)^T$ be the WV associated with IVFFHIHWA operator, then

$$\begin{aligned}
\mathfrak{S}_1^{lb} &= 0.2, \mathfrak{S}_1^{ub} = 0.4, \\
\mathfrak{S}_2^{lb} &= 0.4, \mathfrak{S}_2^{ub} = 0.7, \\
\mathfrak{S}_3^{lb} &= 0.2, \mathfrak{S}_3^{ub} = 0.7, \\
\mathfrak{S}_4^{lb} &= 0.5, \mathfrak{S}_4^{ub} = 0.7, \\
\wp_1^{lb} &= 0.3, \wp_1^{ub} = 0.6, \\
\wp_2^{lb} &= 0.1, \wp_2^{ub} = 0.3, \\
\wp_3^{lb} &= 0.1, \wp_3^{ub} = 0.4, \\
\wp_4^{lb} &= 0.3, \wp_4^{ub} = 0.6.
\end{aligned}$$

To get the weighted interval-valued Fermatean fuzzy values, apply $\dot{J} = q\phi_t J_t$.

$$\begin{aligned}
\dot{J}_1 &= \langle [0.24, 0.48], [0.36, 0.72] \rangle, \\
\dot{J}_2 &= \langle [0.16, 0.28], [0.04, 0.12] \rangle, \\
\dot{J}_3 &= \langle [0.16, 0.56], [0.08, 0.32] \rangle, \\
\dot{J}_4 &= \langle [0.08, 0.112], [0.048, 0.096] \rangle.
\end{aligned}$$

Apply score function to calculate scores of $\dot{J}_t$ ($t = 1, 2, 3, 4$).

$$\begin{aligned}
\mathcal{S}(\dot{J}_1) &= -0.147, \\
\mathcal{S}(\dot{J}_2) &= 0.012, \\
\mathcal{S}(\dot{J}_3) &= 0.073,
\end{aligned}$$

$$\mathcal{S}(\dot{J}_4) = 4.60 \times 10^{-4}.$$

Since $\mathcal{S}(\dot{J}_3) > \mathcal{S}(\dot{J}_2) > \mathcal{S}(\dot{J}_4) > \mathcal{S}(\dot{J}_1)$.

$$\begin{aligned} \dot{J}_{\sigma(1)} &= \dot{J}_3 = \langle [0.16, 0.56], [0.08, 0.32] \rangle, \\ \dot{J}_{\sigma(2)} &= \dot{J}_2 = \langle [0.16, 0.28], [0.04, 0.12] \rangle, \\ \dot{J}_{\sigma(3)} &= \dot{J}_4 = \langle [0.08, 0.112], [0.048, 0.096] \rangle, \\ \dot{J}_{\sigma(4)} &= \dot{J}_1 = \langle [0.24, 0.48], [0.36, 0.72] \rangle. \end{aligned}$$

Thus, for $\mathcal{E} = 2$

IVFFHIWA(J_1, J_2, J_3, J_4)

$$= \left\langle \left[\frac{\prod_{t=1}^4 (1 + (\mathcal{E} - 1)(\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3)^{\kappa_t}}{\prod_{t=1}^4 (1 + (\mathcal{E} - 1)(\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3)^{\kappa_t}} \right], \right.$$

$$\tilde{\mathcal{E}} = \begin{bmatrix} ([\mathcal{S}_{11}^{lb}, \mathcal{S}_{11}^{ub}], [\wp_{11}^{lb}, \wp_{11}^{ub}]) & ([\mathcal{S}_{12}^{lb}, \mathcal{S}_{12}^{ub}], [\wp_{12}^{lb}, \wp_{12}^{ub}]) & \cdots & ([\mathcal{S}_{1q}^{lb}, \mathcal{S}_{1q}^{ub}], [\wp_{1q}^{lb}, \wp_{1q}^{ub}]) \\ ([\mathcal{S}_{21}^{lb}, \mathcal{S}_{21}^{ub}], [\wp_{21}^{lb}, \wp_{21}^{ub}]) & ([\mathcal{S}_{22}^{lb}, \mathcal{S}_{22}^{ub}], [\wp_{22}^{lb}, \wp_{22}^{ub}]) & \cdots & ([\mathcal{S}_{2q}^{lb}, \mathcal{S}_{2q}^{ub}], [\wp_{2q}^{lb}, \wp_{2q}^{ub}]) \\ \vdots & \vdots & \ddots & \vdots \\ ([\mathcal{S}_{m1}^{lb}, \mathcal{S}_{m1}^{ub}], [\wp_{m1}^{lb}, \wp_{m1}^{ub}]) & ([\mathcal{S}_{m2}^{lb}, \mathcal{S}_{m2}^{ub}], [\wp_{m2}^{lb}, \wp_{m2}^{ub}]) & \cdots & ([\mathcal{S}_{mq}^{lb}, \mathcal{S}_{mq}^{ub}], [\wp_{mq}^{lb}, \wp_{mq}^{ub}]) \end{bmatrix}$$

$$\begin{aligned} & \left. \left[\frac{\prod_{t=1}^4 (1 + (\mathcal{E} - 1)(\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3)^{\kappa_t}}{\prod_{t=1}^4 (1 + (\mathcal{E} - 1)(\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3)^{\kappa_t}} \right], \right. \\ & \left. \left[\frac{\mathcal{E} \left\{ \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3 - (\wp_{\sigma(t)}^{lb})^3)^{\kappa_t} \right\}}{\prod_{t=1}^4 (1 + (\mathcal{E} - 1)(\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{lb})^3)^{\kappa_t}} \right], \right. \\ & \left. \left[\frac{\mathcal{E} \left\{ \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3)^{\kappa_t} - \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3 - (\wp_{\sigma(t)}^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^4 (1 + (\mathcal{E} - 1)(\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^4 (1 - (\dot{\mathcal{S}}_{\sigma(t)}^{ub})^3)^{\kappa_t}} \right] \right] \rangle \\ & = \langle [0.22, 0.45], [0.27, 0.61] \rangle. \end{aligned}$$

Remark 3 The IVFFHIWA operator fulfills the set of requirements in Property 8.

7 Decision-making analysis under Hamacher interactive operators

This section will provide a method to resolve MADM issues with IVFFNs using the IVFFHIWA or IVFFHIWG operator.

To address the MADM issue in an IVFF setting, suppose $X = \{X_1, X_2, \dots, X_m\}$ and $V = \{V_1, V_2, \dots, V_q\}$ are the set of alternatives and attributes, respectively, decided by

the decision-maker and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_q)^T$ is the WV with “ $\sum_{t=1}^q \kappa_t = 1$.” IVFF decision matrix (“IVFFDM $_x$ ”) is $\tilde{\mathcal{E}} = ([\mathcal{S}_{lt}^{lb}, \mathcal{S}_{lt}^{ub}], [\wp_{lt}^{lb}, \wp_{lt}^{ub}])_{m \times q}$, wherever $[\mathcal{S}_{lt}^{lb}, \mathcal{S}_{lt}^{ub}]$ is the positive MF by which alternative X_l satisfies the attribute V_t that is defined by decision-makers, and $[\wp_{lt}^{lb}, \wp_{lt}^{ub}]$ gave the degree that “the alternative does not satisfy the attribute” and $0 \leq (\mathcal{S}_{lt}^{ub})^3 + (\wp_{lt}^{ub})^3 \leq 1$.

Utilizing the IVFFHIWA operator, “Algorithm 1 is used to solve the MADM problem.”

Algorithm 1

Step 1. Build the database of expert assessments initially, taking into account the specifications for each possibility and form a IVFFDM $_x$ $\tilde{\mathcal{E}} = ([\mathcal{S}_{lt}^{lb}, \mathcal{S}_{lt}^{ub}], [\wp_{lt}^{lb}, \wp_{lt}^{ub}])_{m \times q}$ in the accordance:

Step 2. For normalizing, IVFFDM $_x$ $\tilde{\mathcal{E}} = ([\mathcal{S}_{lt}^{lb}, \mathcal{S}_{lt}^{ub}], [\wp_{lt}^{lb}, \wp_{lt}^{ub}])_{m \times q}$ is replaced by $\bar{\tilde{\mathcal{E}}} = ([\mathcal{S}_{lt}^{lb}, \wp_{lt}^{ub}], [\wp_{lt}^{lb}, \mathcal{S}_{lt}^{ub}])_{m \times q}$

$$\bar{\tilde{\mathcal{E}}} = \begin{cases} ([\mathcal{S}_{lt}^{lb}, \mathcal{S}_{lt}^{ub}], [\wp_{lt}^{lb}, \wp_{lt}^{ub}]); & \text{for benefit parameter,} \\ ([\wp_{lt}^{lb}, \wp_{lt}^{ub}], [\mathcal{S}_{lt}^{lb}, \mathcal{S}_{lt}^{ub}]); & \text{for cost parameter.} \end{cases}$$

If the attributes are of same type, then there is no need to normalize the values of the attributes, although if there are benefit attributes and cost attributes in decision-making, then we change the rating values of the cost parameter into the benefit parameter.

Step 3. On determining the overall preference values $\mathcal{B}_l$ of X_l , implement the IVFFHIWA operator to the IVFFDM.

$$\mathcal{B}_l = IVFFHIWA(X_{l1}, X_{l2}, \dots, X_{lq})$$

$$\begin{aligned} & = \left\langle \left[\frac{\prod_{t=1}^q (1 + (\mathcal{E} - 1)(\mathcal{S}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathcal{S}_t^{lb})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathcal{E} - 1)(\mathcal{S}_t^{lb})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^q (1 - (\mathcal{S}_t^{lb})^3)^{\kappa_t}} \right], \right. \\ & \left. \left[\frac{\prod_{t=1}^q (1 + (\mathcal{E} - 1)(\mathcal{S}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathcal{S}_t^{ub})^3)^{\kappa_t}}{\prod_{t=1}^q (1 + (\mathcal{E} - 1)(\mathcal{S}_t^{ub})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^q (1 - (\mathcal{S}_t^{ub})^3)^{\kappa_t}} \right], \right. \\ & \left. \left[\frac{\mathcal{E} \left\{ \prod_{t=1}^q (1 - (\mathcal{S}_t^{lb})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathcal{S}_t^{lb})^3 - (\wp_t^{lb})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathcal{E} - 1)(\mathcal{S}_t^{lb})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^q (1 - (\mathcal{S}_t^{lb})^3)^{\kappa_t}} \right], \right. \\ & \left. \left[\frac{\mathcal{E} \left\{ \prod_{t=1}^q (1 - (\mathcal{S}_t^{ub})^3)^{\kappa_t} - \prod_{t=1}^q (1 - (\mathcal{S}_t^{ub})^3 - (\wp_t^{ub})^3)^{\kappa_t} \right\}}{\prod_{t=1}^q (1 + (\mathcal{E} - 1)(\mathcal{S}_t^{ub})^3)^{\kappa_t} + (\mathcal{E} - 1) \prod_{t=1}^q (1 - (\mathcal{S}_t^{ub})^3)^{\kappa_t}} \right] \right] \rangle \end{aligned}$$

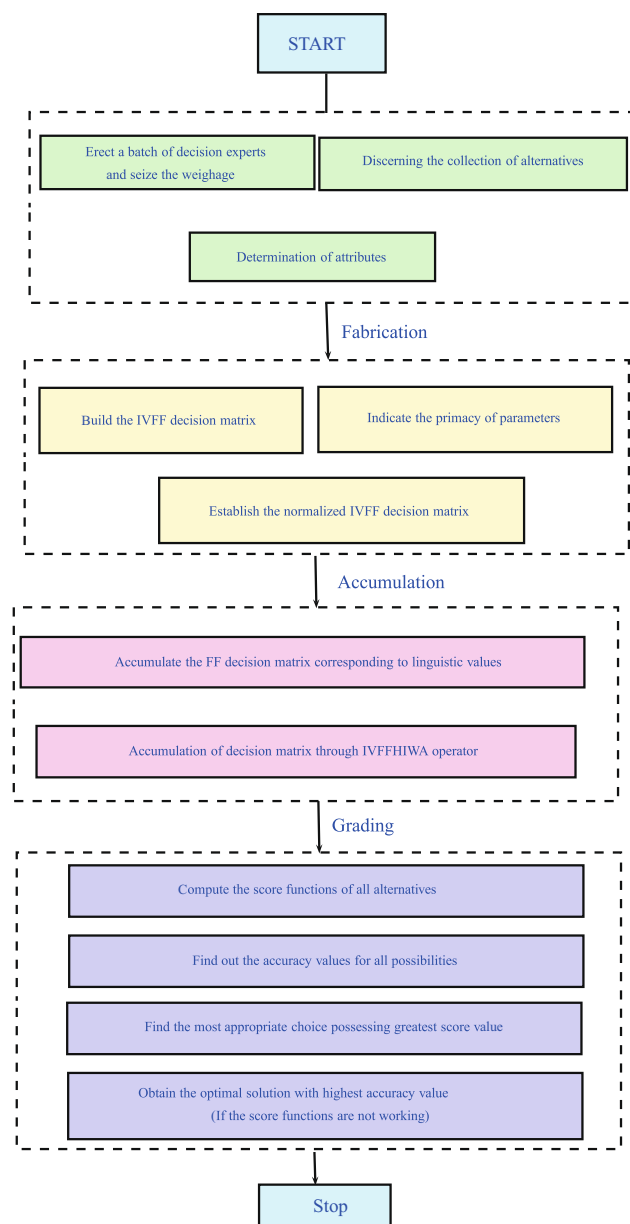


Fig. 2 Flowchart of the proposed model

$$\sqrt[3]{\frac{\mathfrak{L}\left\{\prod_{t=1}^q(1-(\mathfrak{A}_t^{ub})^3)^{\kappa_t}-\prod_{t=1}^q(1-(\mathfrak{A}_t^{ub})^3-(\mathfrak{g}_t^{ub})^3)^{\kappa_t}\right\}}{\prod_{t=1}^q(1+(\mathfrak{L}-1)(\mathfrak{A}_t^{ub})^3)^{\kappa_t}+(\mathfrak{L}-1)\prod_{t=1}^q(1-(\mathfrak{A}_t^{ub})^3)^{\kappa_t}}}\right\}.$$

Step 4. For finding the scoring values of alternatives utilize the score function $S(\mathcal{B}_l)(l = 1, 2, \dots, m)$. If two or more alternatives have same score values, then compute the accuracy functions $\mathcal{A}(\mathcal{B}_l)$ for ranking of alternatives.

Step 5. The selection will be made based on the alternative with the highest score value.

Figure 2 provides the proposed model's flowchart.

8 A case study in mine emergency decision-making

Here, we study about the emergency decision-making issues of mine accidents by using the proposed decision-making access. In mine disasters, the mine blast is one of the most dangerous hazards and it threatens the life extremely and safety of work and endangers the safety production of mine. As explosion accidents mostly happen unexpectedly, that's why it is difficult to forecast precisely the accident and make complete arrangements and urgently actions before time. Therefore, disaster simulations and emergency response plans are a significant strategy in calamity planning and appropriate reactions. The effectiveness and quality of the emergency plans will have a direct impact on the future emergency responses, which in turn will have an impact on how disasters develop and the consequent losses and damages. Therefore, it is believed that using simulations to evaluate and make decisions on the provided emergency plans is essential for the management of mine accident disasters.

8.1 Description of the problem

"Consider the five emergency plans $X = \{X_1, X_2, X_3, X_4, X_5\}$ for an explosion accident in the coal mine. The set of parameters under consideration is $\mathfrak{C} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4\}$, where noxious gas concentration $\mathfrak{C}_1$ (marked as gas), reducing casualty of current events $\mathfrak{C}_2$ (marked as casualty), the smoke and the dust level $\mathfrak{C}_3$ (marked as smoke), the feasibility of rescue operations $\mathfrak{C}_4$ (marked as feasibility) and repairing facility damages caused by the emergency $\mathfrak{C}_5$ (marked as facility). All parameters are of benefit types according to general evolving principle and the characteristics of the mine accidents. Suppose that the expert has the following prior weight set given by his/her prior experience: $\kappa = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\} = (0.2, 0.2, 0.1, 0.3, 0.2)$. The assessments for emergency plans arising from questionnaire investigation to the expert and constructing an interval-valued Fermatean fuzzy decision matrix with its tabular form given in Table 2."

Step 1. Table 2 provides the IVFFDM.

Step 2. The weights for the attributes decided by decision-maker are

$$\kappa_1 = 0.2, \kappa_2 = 0.2, \kappa_3 = 0.1, \kappa_4 = 0.3 \text{ and } \kappa_5 = 0.2.$$

As all the attributes are of same type, so the normalized decision matrix is the same as the original decision matrix.

Step 3. The performance values $\mathcal{B}_l$ of all alternatives are found by utilizing the IVFFHIWA operator for $\gamma = 2$.

Table 2 IVFFDMx

	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$
X_1	([0.3,0.4],[0.3,0.4])	([0.5,0.7],[0.6,0.7])	([0.5,0.7],[0.3,0.7])	([0.4,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])
X_2	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])
X_3	([0.7,0.8],[0.1,0.2])	([0.8,0.9],[0.1,0.2])	([0.0,0.0],[0.2,0.3])	([0.5,0.6],[0.1,0.2])	([0.7,0.8],[0.1,0.2])
X_4	([0.3,0.4],[0.5,0.7])	([0.5,0.6],[0.6,0.7])	([0.3,0.5],[0.6,0.7])	([0.4,0.6],[0.1,0.2])	([0.5,0.6],[0.1,0.2])
X_5	([0.5,0.6],[0.6,0.7])	([0.4,0.7],[0.1,0.2])	([0.3,0.4],[0.6,0.7])	([0.5,0.6],[0.3,0.4])	([0.4,0.6],[0.1,0.2])

Table 3 Normalized IVFFDMx

	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$
X_1	([0.3,0.4],[0.3,0.4])	([0.5,0.7],[0.6,0.7])	([0.5,0.7],[0.3,0.7])	([0.4,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])
X_2	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])	([0.5,0.6],[0.6,0.7])
X_3	([0.7,0.8],[0.1,0.2])	([0.8,0.9],[0.1,0.2])	([0.0,0.0],[0.2,0.3])	([0.5,0.6],[0.1,0.2])	([0.7,0.8],[0.1,0.2])
X_4	([0.3,0.4],[0.5,0.7])	([0.5,0.6],[0.6,0.7])	([0.3,0.5],[0.6,0.7])	([0.4,0.6],[0.1,0.2])	([0.5,0.6],[0.1,0.2])
X_5	([0.5,0.6],[0.6,0.7])	([0.4,0.7],[0.1,0.2])	([0.3,0.4],[0.6,0.7])	([0.5,0.6],[0.3,0.4])	([0.4,0.6],[0.1,0.2])

$$\begin{aligned}
\mathcal{B}_1 &= ([0.45, 0.61], [0.54, 0.67]), \\
\mathcal{B}_2 &= ([0.61, 0.50], [0.58, 0.70]), \\
\mathcal{B}_3 &= ([0.66, 0.76], [0.16, 0.22]), \\
\mathcal{B}_4 &= ([0.43, 0.56], [0.45, 0.58]), \\
\mathcal{B}_5 &= ([0.45, 0.61], [0.43, 0.51]).
\end{aligned}$$

Step 4. Using the score function, the score values for each possibility are:

$$\begin{aligned}
S(\mathcal{B}_1) &= -0.07, \\
S(\mathcal{B}_2) &= -0.05, \\
S(\mathcal{B}_3) &= 0.36, \\
S(\mathcal{B}_4) &= 0.02, \\
S(\mathcal{B}_5) &= 0.05.
\end{aligned}$$

Step 5. Since $X_3 > X_5 > X_4 > X_2 > X_1$, X_3 is the best alternative.

9 Comparison analysis

In the preceding section, we compare the developed technique with existing theories such as IVFFWAO (Rani et al. 2022), IVPFWAO (Garg 2016), IVPFWG (Garg 2016), IVPFWA (Liang et al. 2015), and IVPFWG (Rahman et al. 2017).

9.1 Comparison with existing FF operators

Assigning the MF and NMF to a given set to possess an interval value is considered as the prominent feature of IVF-

FAOs. This type of state of affairs is less or more like that encountered in FFSs, where the concept of FFSs has been extended to that of IVFFSs to describe the case of interval values that the MF and NMF of an object are assigned to a fixed set. It ought to be referred that once the upper and lower degrees of the interval values are identical, IVFFS will become FFS, indicating that the latter is a special case of the previous. Consequently, our proposed IVFF operators are more suitable in solving actual issues in comparison with FF aggregation operators (Mishra et al. 2021; Shahzadi and Akram 2021; Akram et al. 2020; Shahzadi et al. 2022; Rani and Mishra 2022, 2021).

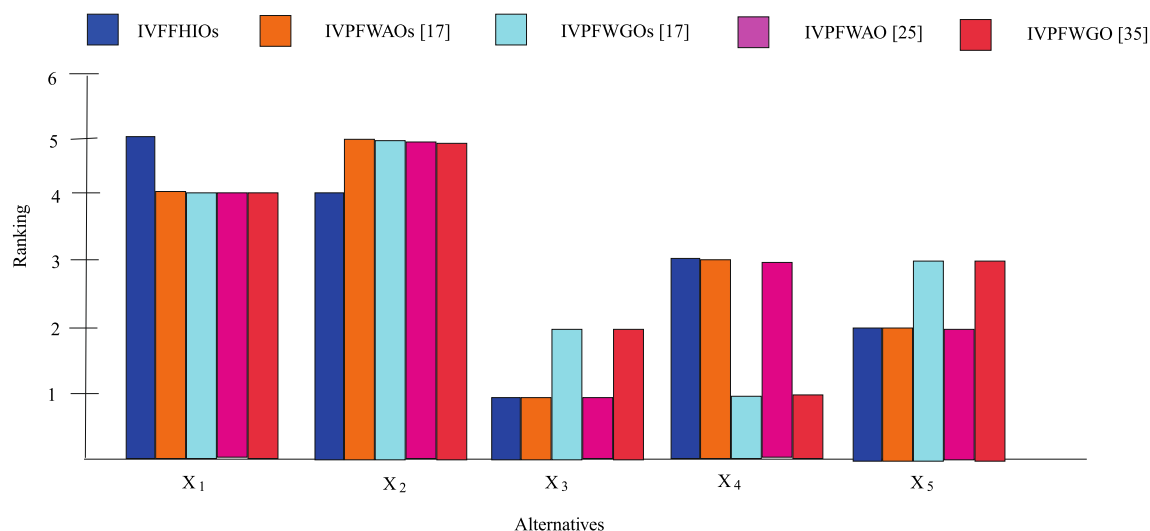
9.2 Comparison with existing IVPF operators

In this subsection, the technique developed with IVFFHIOs has been compared with IVPFWAO (Garg 2016), IVPFWG (Garg 2016), IVPFWA (Liang et al. 2015), and IVPFWG (Rahman et al. 2017). All the possible consequences are given in Table 4.

It is clear from Table 4, as a result of the comparison, numerous intriguing patterns are revealed in these results. All these techniques are compared with one another and found that the alternative X_3 is the best choice from Rani et al. (2022), Garg (2016), Liang et al. (2015) and proposed work but results from Garg (2016), Rahman et al. (2017) are different due to some of their drawbacks. For example, for some t , if $J_t = ([0, 0], [1, 1])$, then by using IVPFWG operator (Garg 2016) and IVPFWG operator (Rahman et al. 2017), we have $IVPFWG(J_1, J_2, \dots, J_q) = ([0, 0], [1, 1])$. This outcome may cause counterintuitive phenomena in MADM. That is, it is only identified by J_t to make decisions and the decision information of others can be neglected. If $J_t =$

Table 4 Comparison with existing IVPF operators

Techniques	Final ranking	Optimal alternative
Proposed method	$X_3 > X_5 > X_4 > X_2 > X_1$	X_3
IVPFWA Operator (Garg 2016)	$X_3 > X_5 > X_4 > X_1 > X_2$	X_3
IVPFWG Operator (Garg 2016)	$X_4 > X_3 > X_5 > X_1 > X_2$	X_4
IVPFWA Operator (Liang et al. 2015)	$X_3 > X_5 > X_4 > X_1 > X_2$	X_3
IVPFWG Operator (Rahman et al. 2017)	$X_4 > X_3 > X_5 > X_1 > X_2$	X_4

**Fig. 3** Comparison of rankings between IVPFAOs and proposed operators

$([0, 0], [\wp^{lb}, \wp^{ub}])$, the coupled value is $IVPFWG(J_1, J_2, \dots, J_q) = ([0, 0], [\wp^{lb}, \wp^{ub}])$. That is, the membership part of coupled value is to be zero. This outcome may create a counterintuitive situation in some circumstances. Therefore, it is unacceptable and improper to apply operators given in all (Garg 2016; Rahman et al. 2017) to aggregate the data in MADM when meet the special issues discussed above. In this application, the quantity of alternatives is restricted to five; the result of presented work might not be observed as conclusive but if the number of choices is increased, the outcome will become clearer. As a result, the established theory is sound and applicable to problems with decision-making. The proposed theory offers a solution to some of the problems with the current ideas. The graphical representation of comparison of rankings between IVPFAOs and proposed operators is shown in Fig. 3.

9.3 Comparison with IVFFWAOs and IVFF-TOPSIS method

TOPSIS is the commonly used distance-based MADM technique. It ranges the alternatives by calculating their closeness to the positive ideal solution and distance to the negative-ideal solution. Here, we study the comparison of the presented theory with the IVFF-TOPSIS method (Ilieva and Yankova

2022). The procedure to find out the appropriate alternative consists of the following steps:

1. Table 2 illustrates the IVFF decision matrix wherein all entries represent IVFFN.
2. The IVFFPIS $\mathfrak{S}^+$ and IVFFNIS $\mathfrak{S}^-$ are:

$$\mathfrak{S}^+ = \{([0.7, 0.8], [0.1, 0.2]), ([0.8, 0.9], [0.1, 0.2]), ([0.5, 0.7], [0.3, 0.7]), ([0.5, 0.6], [0.3, 0.4]), ([0.5, 0.6], [0.1, 0.2])\}$$

$$\mathfrak{S}^- = \{([0.3, 0.4], [0.5, 0.7]), ([0.5, 0.6], [0.6, 0.7]), ([0.3, 0.5], [0.6, 0.7]), ([0.4, 0.6], [0.6, 0.7]), ([0.5, 0.6], [0.6, 0.7])\}.$$

3. The Euclidean distance between the alternative X_t and $\mathfrak{S}^+$ along with $\mathfrak{S}^-$ is stated in Table 5.
4. The revised closeness degree for all alternatives:

$$\Theta(X_1) = 0.335,$$

$$\Theta(X_2) = 0.239,$$

$$\Theta(X_3) = 0.609,$$

$$\Theta(X_4) = 0.338,$$

$$\Theta(X_5) = 0.339.$$

5. Order the alternatives in the ascending order of $\Theta(X_t)$ for finding the best one:

$$X_3 > X_5 > X_4 > X_1 > X_2.$$

Table 5 Distance of alternatives from $\mathfrak{S}^+$ and $\mathfrak{S}^-$

$\mathfrak{D}(X_t, \mathfrak{S}^+)$	$\mathfrak{D}(X_t, \mathfrak{S}^-)$
0.487	0.246
0.489	0.154
0.390	0.608
0.485	0.248
0.484	0.249

6. X_3 is the optimal solution. The ranking results of alternatives through the proposed technique, IVFF-TOPSIS method, and IVFFWAOs are given in Table 6.

The graphical representation of comparison of rankings between IVFFWAOs (Rani et al. 2022), IVFF-TOPSIS (Ilieva and Yankova 2022), and proposed operators is shown in Fig. 4.

9.4 Results and discussions

In the proposed study, we have considered the decision-making for emergency problems of mine accidents. The proposed algorithm for decision-making analysis that is based on IVFFHI operators has been employed to find out the optimal solution. In case of mine accidents, the mine explosion is considered as the most dangerous hazard. The mine explosion can threaten the safety of the workplace and lives of the workers as well as the safe production of mines. Since the occurrence of these explosions is often sudden and unexpected, it is difficult to predict these explosion accidents and adopt auxiliary actions and have adequate solutions. Consequently, the emergency reaction plans and the simulations of the injuries are a requisite technique in catastrophe preparedness and appropriate responses. The excessive nice and feasibility of the emergency plans will at once impact the later emergency actions, and affect the evolution of failures. Therefore, the assessment and decision of the given emergency plans with simulations are taken into consideration essential for the disaster control of mine injuries. In the proposed work, a novel MADM technique based on IVFFHIWA and IVFFHIWG operators is developed to handle mine emergency problems. Building the database of expert assessments initially, taking into account the specifications for each possibility and obtain a IVFFDMx. Then, the normalization of the obtained matrix is done for benefit criteria as well as for benefit criteria. The IVFFHIWA operator is applied to the IVFFDMx to determine the overall preference values of each alternative. The developed score functions and accuracy values are then applied to find out the final ranking of alternatives. In the given case study, we have considered the five alternative plans to overcome the explosion accidents in mines. Applying the steps of Algorithm 1, we obtain the

ranking of alternatives as $X_3 > X_5 > X_4 > X_1 > X_2$. That is, the obtained conclusion by proposed technique reveals that X_3 is the most appropriate plan for this mine emergency problem. Hence, the proposed model has consequential information that can be employed by DMs in taking strategic or operational selections in mine emergencies assessment. Further, we have proved the consistency of our obtained results through the comparison of proposed method with existing techniques and theories. Note that the limits of the results obtained in this study are the same as the limits of the score functions and accuracy value, that is, $[-1, 1]$ or $[0, 1]$. If the ranking of alternatives has been obtained through score function, then the results will lie between -1 and 1 . On the other hand, if the score function fails to provide the ranking evaluation, then accuracy values are used and the obtained results will lie between 0 and 1 in this case.

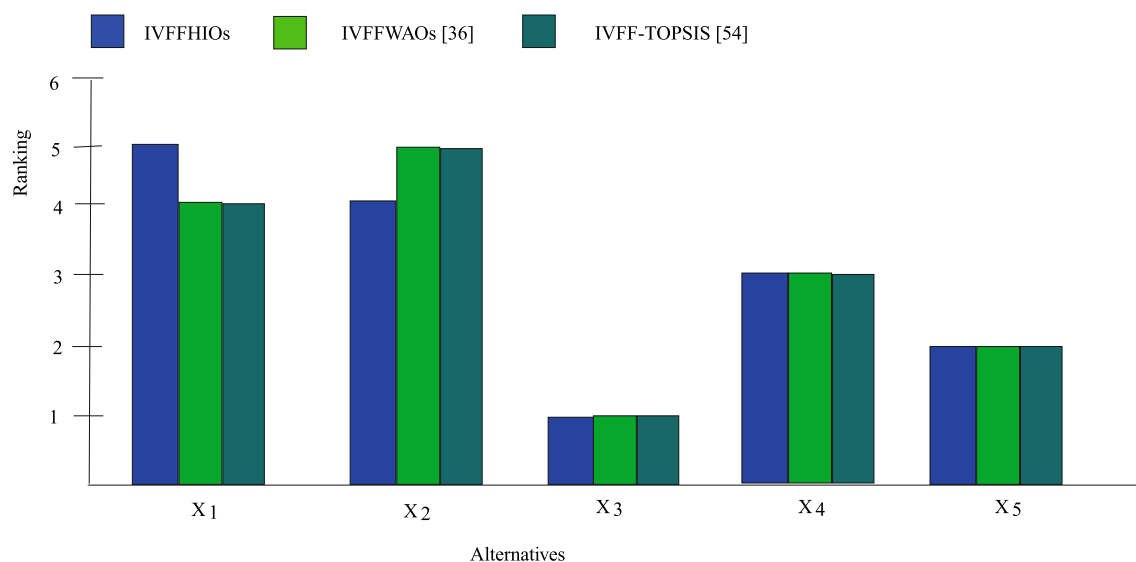
9.5 Benefits of the suggested method

We come to the conclusion that the work offers the following benefits:

- As they build the appropriate field of unsure facts and figures, the IVFF MF generalizes the IVIF and IVPF MFs.
- IVFFWA (Rani et al. 2022), IVPFWA (Garg 2016), and IVPWA (Liang et al. 2015) operators are best ways for classifying possibilities and selecting the top option. As Table 6 provides the results derived from these three operators, we find out the best choice X_3 . The consequences acquired for the finest option through the initiated operators are also identical, but the IVPFWA (Garg 2016) and IVPFWA (Liang et al. 2015) operators are limited to the domain where the square sum of the upper bound of MF and NMF should be less than 1 , so when this boundary is exceeded by 1 , the operators given in Garg (2016), Liang et al. (2015) are unable to deal with such problems and if $J_t = ([\mathfrak{I}^{lb}, \mathfrak{I}^{ub}], [0, 0])$, the coupled value is $([\mathfrak{I}^{lb}, \mathfrak{I}^{ub}], [0, 0])$. That is, the non-membership part of coupled value is to be zero. This outcome may create counterintuitive situation in some circumstances. Therefore, it is unacceptable and improper to apply operators given in all (Rani et al. 2022; Garg 2016; Liang et al. 2015) to aggregate the data in MADM when meeting the special issues discussed above. In each of these cases, we rank the choices using the suggested operators.
- The relative proximity degree of the alternatives is used in the IVFF-TOPSIS approach to evaluate the alternatives. Justifying how excellent or bad an option is, it is not appropriate. In the suggested theory, the cost and benefit parameters are both heavily weighted with proposed AOs on IVFFSs, which produces a more accurate result than only addressing the cost or benefit criteria.

Table 6 Ranking through the proposed technique, IVFF-TOPSIS method, and IVFFWAOs

Techniques	Final ranking	Optimal alternative
Proposed method	$X_3 > X_5 > X_4 > X_2 > X_1$	X_3
IVFFWA Operator (Rani et al. 2022)	$X_3 > X_5 > X_4 > X_1 > X_2$	X_3
IVFF-TOPSIS (Ilieva and Yankova 2022)	$X_3 > X_5 > X_4 > X_1 > X_2$	X_3

**Fig. 4** Comparison of rankings between IVFFWAOs (Rani et al. 2022), IVFF-TOPSIS (Ilieva and Yankova 2022), and proposed operators

- The conversion of “IVIFNs and IVPFNs to IVFFNs” improves the usability of the presented ambiguous data and the applicability of operators to the machine where the membership capabilities are challenging or impractical to specify exactly.
- The HIAOs give more precise and exact choice values in decision results when applied to MCDM framework.
- The proposed HIAOs deal with the relationship between the MF and NMF.
- The proposed operators possess the flexibility of parameter λ . We are not restricted to any domain for parameter λ . For $\lambda = 1$, it reduces to algebraic t -norm and algebraic t -conorm. For $\lambda = 2$, it becomes Einstein t -norm and Einstein t -conorm.

10 Sensitivity analysis

We review a sensitivity study in respect of distinct values of parameter λ . In the following, we constantly view the impacts of the parameter λ on the mine emergency decision making in more detail. Multiple values of parameter λ are taken for exploration. This analysis is considered to declare the functioning of the presented framework. The different values of λ can support to assess the sensitivity of the introduced model. All the sensitivity behavior is shown in Fig 5, which shows

that the optimal alternative is X_3 for all values of parameter, whereas the overall ordering of the alternatives varies over different values of the parameter. From Table 7 and Fig 5, the order of preference is $X_3 > X_5 > X_4 > X_1 > X_2$ for $\lambda = 1, 3, 5, 6, 7$ and $X_3 > X_5 > X_4 > X_2 > X_1$ for $\lambda = 2$ and $X_3 > X_5 > X_2 > X_4 > X_1$ for $\lambda = 4$. Therefore, it is deduced that decision-making of mine emergency study is dependent on and sensitive to the parameter. That’s why, the presented model has an agreeable constancy over different values of parameter. At the end, we can conclude that the use of different values of parameter will improve the stability of the presented work. Note that the values of parameter λ are taken from the positive domain of real numbers, that is, the values of λ can be considered from the interval $(0, \infty)$. On the other hand, if negative values are assigned to λ , then the resulting values of IVFFNs can be negative. That provides the contradiction against the constraint of FFSs. Hence, the parameter λ lies only on the positive real line.

11 Conclusions

The purpose of this study is to introduce AOs for IVFFSs, a theory that enables decision-making experts to convey a wider range of ambiguous information by assigning MF and NMF to a group of alternatives in the form of inter-

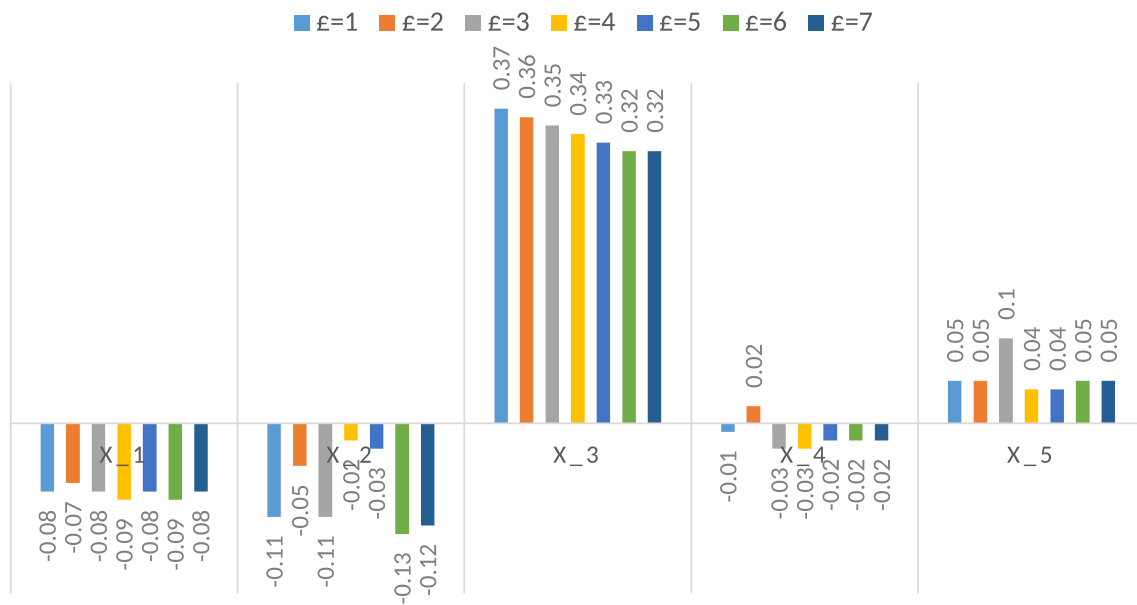


Fig. 5 Ranking of alternatives for different MADM techniques

vals. The conventional operating laws corresponding to FFSs and IVFFSs have been assessed. We have created multiple aggregation operators to aggregate the IVFF data based on the operations of IVFFSs and have explained their key aspects. These operators have also been applied to MCGDM. Additionally, a few of these geometric and aggregation operators' features have been briefly examined. Finally, in order to demonstrate the effectiveness and coherence of the suggested models, we took into consideration a case study of a mine emergency decision-making situation. In the proposed case study, five alternative plans have been taken into consideration to overcome the explosion accidents in mines. Applying the steps of Algorithm 1, the ranking of alternatives as $X_3 > X_5 > X_4 > X_1 > X_2$ has been obtained. That is, the obtained conclusion by proposed technique revealed that X_3 is the most appropriate plan for this mine emergency problem. Hence, the proposed model has consequential information that can be employed by DMs in taking strategic or operational selections in mine emergencies assessment as well as in case of any other MADM issues. Additionally, sen-

sitivity and comparison analysis was used to show the results' dependability and robustness. We have also looked into how parameters affect judgment. Finally, we have compared the new model to a few of the existing models, demonstrating its benefits and applicability.

11.1 Limitations

The following are the proposed methodology's main drawbacks:

- Since IVFFNs are a generalization of IVIFNs and IVPFNs, they do not work in circumstances where $(\rho^{ub})^3 + (\varrho^{ub})^3 \geq 1$.
- In such circumstances, where we presume the parameters for the assessment of anything, the proposed operators are not appropriate.
- This work lacks the parameterization property as well. Hence, when someone is working in parameterized environment, where the distinct parameters or attributes are

Table 7 Ranking of alternatives for different values of ϵ

ϵ	X_1	X_2	X_3	X_4	X_5	Ranking order
$\epsilon = 1$	-0.08	-0.11	0.37	-0.01	0.05	$X_3 > X_5 > X_4 > X_1 > X_2$
$\epsilon = 2$	-0.07	-0.05	0.36	0.02	0.05	$X_3 > X_5 > X_4 > X_2 > X_1$
$\epsilon = 3$	-0.08	-0.11	0.35	-0.03	0.10	$X_3 > X_5 > X_4 > X_1 > X_2$
$\epsilon = 4$	-0.09	-0.02	0.34	-0.03	0.04	$X_3 > X_5 > X_2 > X_4 > X_1$
$\epsilon = 5$	-0.08	-0.03	0.33	-0.02	0.04	$X_3 > X_5 > X_4 > X_1 > X_2$
$\epsilon = 6$	-0.09	-0.13	0.32	-0.02	0.05	$X_3 > X_5 > X_4 > X_1 > X_2$
$\epsilon = 7$	-0.08	-0.12	0.32	-0.02	0.05	$X_3 > X_5 > X_4 > X_1 > X_2$

assigned to different criteria, the proposed methodology lacks certain information or data in such cases.

11.2 Future directions

The under consideration area can be expanded in the successive schemes to address the shortcomings of the suggested and prior studies. More AOs for IVFFSs can be developed in the future. At the same time, we'll use these operators to create new MCDA models and look into a number of applications, such as MCDM issues, game theory, cluster analysis, and medical diagnosis. To obtain the desired results, the under consideration work can be extended to:

1. The IVFFAOs would be extended to IV q -ROFAOs to overcome the limiting property of the proposed operators, that is, $(\rho^{ub})^3 + (q^{ub})^3 \geq 1$.
2. In case of parameterized theories, the current work can be extended to hybrid theories involving the generalizations of soft set theories.
3. Additionally, the proposed technique can be used to handle a variety of MAGDM applications in different industries, such as agricultural, healthcare, construction firms, etc.
4. In fuzzy setting of FFS and IVFFSs, new techniques, named as DNMA (double normalization-based multiple aggregation), GLDS (gained and lost dominance score), and MARCOS (Measurement of Alternatives and Ranking according to the Compromise Solution)" can be applied in the future.

The FFS theory, a generalization of the PFS and IFS theories, is the foundation of the suggested approach. Therefore, the adoption of the suggested approach currently has no significant constraints. The choice of attributes is the only apparent genuine restriction. Only four characteristics were chosen, despite the fact that numerous other aspects were taken into account such as "groundwater depth, proximity to surface water, elevation, land slope, soil permeability, soil stability, flooding susceptibility, lithology and stratification, faults, land use type, nearby settlements and urbanization, proximity to cultural and protected sites, wind direction, roads, railroads, proximity to building materials, pipelines, and power lines, as well as proximity to airports." Future studies can take these elements into account.

Author Contributions The project was developed and designed by Gulfam Shahzadi, Anam Luqman, and Faruk Karaaslan. They also conducted the data analysis and prepared the report.

Funding This initiative receives no specific support.

Data Availability Statement This study was not supported by any data.

Declarations

Conflict of interest The authors declare no conflict of interest.

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