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Results in Physics

journal homepage: www.elsevier.com/locate/rinp

Fractional order age dependent Covid-19 model: An equilibria and quantitative analysis with modeling

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ARTICLE INFO

Keywords:
COVID-19
Age-dependent modeling
Epidemic model
Lyapunov function
Fixed point theorem
FFM operator

ABSTRACT

The presence of different age groups in the populations being studied requires us to develop models that account for varying susceptibilities based on age. This complexity adds a layer of difficulty to predicting outcomes accurately. Essentially, there are three main age categories: 0 – 19 years, 20 – 64 years, and > 64 years. However, in this article, we only focus on two age groups (20 – 64 years and > 64 years) because the age category 0 – 19 years is generally perceived as having a lower susceptibility to the virus due to its consistently low infection rate during the pandemic period of this research, particularly in the countries being examined. In this paper, we presented an age-dependent epidemic model for the COVID-19 Outbreak in Kuwait, France, and Cameroon in the fractal-fractional (FF) sense of derivative with the Mittag–Leffler kernel. The study includes positivity, stability, existence results, uniqueness, stability, and numerical simulations. Globally, the age-dependent COVID-19 fractal fractional model is examined using the first and second derivatives of Lyapunov. Fixed point theory is used to derive the existence and uniqueness of the fractional-order model. The numerical scheme of this paper is based on the Newton polynomial and is tested for a particular case with numerical values from Kuwait, France, and Cameroon. In our analysis, we explore the significance of these distinct parameters incorporated into the model, focusing particularly on the impact of vaccination and fractional order on the progression of the epidemic. The results are getting closer to the classical case for the orders reaching 1 while all other solutions are different with the same behavior. Consequently, the fractal fractional order model provides more substantial insights into the epidemic disease. We open a novel viewpoint on enhancing an age-dependent model and applying it to real-world data and parameters. Such a study will help determine the behavior of the virus and disease control methods for a population.

Introduction

The most dynamic and fast-increasing application of science today is mathematical biology, sometimes known as biomathematics. This is an interdisciplinary research area with applications in biology, biotechnology, and biomedical science. The data may be referred to as mathematical biology, biomathematics to pressure the mathematical side, or hypothetical biology to pressure the natural side. Mathematical areas like combinatory, logarithmic geometry, topology, dynamical frameworks, differential conditions, and coding hypothesis are now being connected in this discipline [1]. The dynamics and behavior of disease are understood through mathematical modeling, which is subsequently used to create the methods for treating it. Numerous researchers created the COVID-19 models for this purpose (see [2,3]).

The coronavirus (COVID-19) pandemic of 2019 has presented several significant challenges. As of the end of June 2022, there have been approximately 543 million cases reported worldwide and 6.3 million fatalities [4] as a result of the pandemic. Governments around the world are establishing regulations to prevent the COVID-19 pandemic from spreading further and to maintain the capability of the healthcare systems,

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Received 16 April 2023; Accepted 30 May 2023

ultimately sparing people from the disease's worst effects. The various levels of personal mobility and travel restrictions, the closing of businesses, bars, shops, and restaurants, the prohibition of gathering in parks and beaches, the closure of schools, and in the most severe situations, the cessation of industrial operations were all included in the control measures. The fact that these limits applied to the entire community, regardless of age, was one thing they all had in common. This stands in contrast to the observation that the severity of the COVID-19 disease's symptoms appears to increase with the age of the infected persons, with the unfortunate result that the elderly account for a substantial number of fatal cases [5–7]. Similar patterns in the age distribution of death have been seen across a wide range of nations [8,9]. The impacts of control measures that have a varied impact on the population's various age groups were explored in [10] about the spread of contagion. The role of several parameters, most notably vaccination, was examined in [11], where authors provided a deterministic model for the COVID-19 pandemic taking into account two age classes.

The increased severity of COVID-19 incidence across different age groups led to a surge in hospitalizations and ICU cases during the initial stages of the pandemic. Many countries swiftly responded by implementing measures such as expanding hospital capacities, increasing ventilator availability, and establishing isolation centers. These interventions aimed to slow down the virus's spread and protect particularly vulnerable populations, notably the elderly, who were disproportionately affected. Given the diversity in the ages of the populations under study (Kuwait, France, and Cameroon), it becomes essential to examine a model that factors in distinct susceptibilities for each age category. This complexity adds a layer of difficulty to predicting the growth of new infections. The COVID-19 outbreak primarily involves three age groups: 0–19 years, 20–64 years, and >64 years. However, our focus in this study is on the 20–64 years and >64 years age groups, as the 0–19 years group is generally considered to have lower susceptibility to SARS-CoV-2 [12,13], mainly due to its low infection rate observed during the study period.

Based on the power law, Riemann Liouville introduces the concept of the fractional derivative. Caputo and Fabrizio [14] proposed the new fractional derivative that applies the exponential-decay kernel. After Caputo-Fabrizio [15–18] represented the non-singular kernel fractional derivative that includes the trigonometric and exponential function, and the literature [19–22] shows some related approaches for models of the epidemic, The timeline for the COVID-19 illness conceptual model is successfully caught by the predicted breakout of this virus [23,24]. Different mathematical methodologies are described for the analysis of a mathematical model for the transmission of the COVID-19 disease [25–28].

Atangana [28] has currently developed a new strategy for the fractional calculus fractal fractional derivative. In many situations, the concept of this subject is quite suitable for handling some challenging problems. Fractional order and fractal dimension are two of the orders that the operator has. This novel method of obtaining fractal fractions is superior to the traditional method. This is possible because of fractal dimensions and fraction operators. Dealing with fractal fractional derivatives allows for simultaneous investigation. This operator has the enormous ability to allow you to build models that more accurately describe systems with memory effects. Furthermore, there are numerous issues in the real world where knowing a system's information capacity is necessary. To evaluate the dynamics of COVID-19 employing the fractals Fraction operator, Ali et al. [29] studied a SIR model. Akgul [30] highlighted some developments in fractal fractional differential equations employing power-law kernels and new applications. In [24,31] can see the fractional calculus that was utilized to model the SARS-CoV2 infection. Recent studies on COVID-19 and its implications on society have also been widely disseminated. Using a fractional order COVID-19 model with a Mittag-Leffler kernel, this pandemic was studied; the suggested approaches are described in [32,33].

Due to the flexibility of fractional differential operators, which can encompass both local and nonlocal behaviors, these operators offer effective tools for elucidating real-world complexities, making them essential for constructing mathematical models. Furthermore, the latest additions to these operators are the FF operators, which have not been explored in existing literature concerning the COVID-19 model involving the impact of vaccination across different age groups. Given these considerations, and to distinguish our work from prior research, we took a unique approach. We enhanced the COVID-19 model introduced in [11] by incorporating the FF derivative in the AB context. This choice was motivated by the fact that the FF derivative yields outcomes that are more realistic and possess a higher degree of flexibility compared to traditional integer-order derivatives. The expanded model aims to provide clearer and more detailed biological explanations. We hold the perspective that modeling holds heightened significance when addressing dynamic issues. Each model is designed with its specific goals and intentions. Due to the complexity of encompassing all dynamic facets of a disease within a single model, we assert that the "FF-COVID-19 model with an impact of vaccination on different age patients" specifically accentuates the repercussions of vaccination across various age groups of patients.

The article is structured as follows. In the Section "Introduction", an introduction and historical context are covered. In Section "Basic Concepts", a few fundamental definitions are given regarding the suggested strategies. Section "Fractal Fractional Order SIGR model" encompasses the positivity, disease-free equilibrium, and endemic equilibrium of the fractional order model. In Section "Stability", we investigate the global asymptotic stability of the fractional order COVID-19 model through the application of the Lyapunov function's first and second derivatives. Section "Existence and Uniqueness Results" employs fixed point theory to establish the existence and uniqueness of a system of solutions. Utilizing an advanced numerical technique, Section "Numerical Scheme" provides the solution to the fractal fractional-order system, enabling an examination of the effects of fractional parameters. Numerical simulations and discussions concerning select COVID-19 parameters from Kuwait, France, and Cameroon are presented in Section "Simulation and Discussion". Finally, Section "Conclusion" presents the concluding remarks.

Basic concepts

The fractal-fractional calculus's fundamental ideas are discussed in this section. The FF operators are subsequently described as:

Definition 1 (*[34]*)**.** Let $0 \le \alpha, \alpha_1 \le 1$, the fractal-fractional derivative of $X(t)$ is defined using a fractional order denoted as α and a fractal dimension indicated by α_1, employs the Riemann–Liouville approach and is characterized by a kernel exhibiting a power law nature:

$$ {}^{FFP}D_{0,t}^{\alpha,\alpha_1}(X(t)) = \frac{1}{\Gamma(m-\alpha)} \frac{d}{dt^{\alpha_1}} \int_0^t (t-\xi)^{m-\alpha-1} X(\xi) d\xi, \tag{1} $$

where $m-1 < \alpha,\ \alpha_1 \le m \in N$ and

$$ \frac{d}{d\xi^{\alpha_1}} X(\xi) = \lim_{t\to\xi} \frac{X(t)-X(\xi)}{t^{\alpha_1}-\xi^{\alpha_1}}. \tag{2} $$

Definition 2 (*[34]*)**.** Then the fractal-fractional integral of $X(t)$ with order (α, α_1) having power law type kernel is defined as:

$$ {}^{FFP}I_{0,t}^{\alpha,\alpha_1}(X(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \xi^{1-\alpha_1} X(\xi) d\xi. \tag{3} $$

Definition 3 ([34]). Let $0 \leq \alpha, \alpha_1 \leq 1$, the fractal-fractional derivative of $X(t)$ is defined using a fractional order denoted as α and a fractal dimension indicated by α_1, employs the Riemann–Liouville approach and is characterized by a kernel exhibiting an exponentially decaying nature:

$$^{FFE}D_{0,t}^{\alpha,\alpha_1}(X(t)) = \frac{\mathscr{M}(\alpha)}{\Gamma(m-\alpha)} \frac{d}{dt^{\alpha_1}} \int_0^t exp\left[-\frac{\alpha}{1-\alpha}(t-\xi)^{\mu-\alpha-1}\right] X(\xi)d\xi, \tag{4}$$

where $\alpha > 0, \alpha_1 \leq \mu \in \mathrm{N}$, and $\mathscr{M}(0) = \mathscr{M}(1) = 1$.

Definition 4 ([34]). Then the fractal-fractional integral of $X(t)$ with order (α, α_1) having exponentially decaying type kernel is defined as:

$$^{FFE}I_{0,t}^{\alpha,\alpha_1}(X(t)) = \frac{\alpha_1(1-\alpha)t^{\alpha_1-1}X(t)}{\mathscr{M}(\alpha)} + \frac{\alpha\alpha_1}{\mathscr{M}(\alpha)} \int_0^t \xi^{\alpha-1} X(\xi)d\xi. \tag{5}$$

Definition 5 ([34]). Let $0 \leq \alpha, \alpha_1 \leq 1$, the fractal-fractional derivative of $X(t)$ is defined using a fractional order denoted as α and a fractal dimension indicated by α_1, employs the Riemann–Liouville approach and is characterized by a kernel exhibiting a generalized Mittag-Leffler nature:

$$^{FFM}D_{0,t}^{\alpha,\alpha_1}(X(t)) = \frac{\mathscr{AB}(\alpha)}{1-\alpha} \frac{d}{dt^{\alpha_1}} \int_0^t E_\alpha\left[-\frac{\alpha}{1-\alpha}(t-\xi)^{\alpha}\right] X(\xi)d\xi, \tag{6}$$

where $0 < \alpha, \alpha_1 \leq 1$ and $\mathscr{AB}(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Definition 6 ([34]). Then the fractal-fractional integral of $X(t)$ with order (α, α_1) having generalized Mittag-Leffler type kernel is defined as:

$$^{FFM}I_{0,t}^{\alpha,\alpha_1}(X(t)) = \frac{\alpha_1(1-\alpha)t^{\alpha_1-1}X(t)}{\mathscr{AB}(\alpha)} + \frac{\alpha\alpha_1}{\mathscr{AB}(\alpha)\Gamma(\alpha)} \int_0^t \xi^{\alpha_1-1}(t-\xi)^{\alpha-1} X(\xi)d\xi. \tag{7}$$

Fractal fractional order SIGR model

As an enhancement, to the Ross and Kermack–McKendrick models, which include age class and vaccination status for COVID-19 in a particular population, we offer a fractal fractional SIGR model. We simply keep two age groups adult and old, a and b while ignoring variations between children and young adults. A total population of N is divided into eight compartments in the suggested model [11]. We assume that every infected individual are symptomatic, and we consider the birth and natural death rates β and μ, as well as the specific fatality rate ε due to the disease. Individuals susceptible, infectious I_m, newly susceptible G_m, and fully recovered and resistant R_m are divided into age groups ($m = a, b$). For each age group ($m = a, b$), we denote transmission rates ϕ_m fertility, and loss of resistance rates β_{mm} (supposed to be equal within an age class for simplicity), natural death rates μ_b^S, μ_b^I, μ_b^G, and μ_b^R, vaccination rates group θ_m^G and θ_m^R, survival rates from age a to age b, c_a^S, c_a^I, c_a^G, and c_a^R, particular mortality rate attributed to the disease ε_m, relapsed rate η_m and recovery rate γ_m. The subsequent collection of nonlinear fractional differential equations (8) illustrates the above description.

$$\begin{aligned}
^{FFM}D_{0,t}^{\nu,\nu_1} S_a &= \beta_{aa}\left(S_a + G_a + R_a\right) + \beta_{ba}\left(S_b + G_b + R_b\right) - \left(k_1 + \phi_a\left(I_a + I_b\right)\right) S_a, \\
^{FFM}D_{0,t}^{\nu,\nu_1} S_b &= c_a^S S_a - \mu_b^S S_b - \left(k_2 + \phi_b\left(I_a + I_b\right)\right) S_b + \beta_{bb}\left(G_b + R_b\right), \\
^{FFM}D_{0,t}^{\nu,\nu_1} I_a &= \phi_a\left(I_a + I_b\right) S_a - k_3 I_a, \\
^{FFM}D_{0,t}^{\nu,\nu_1} I_b &= \phi_b\left(I_a + I_b\right) S_b + c_a^I I_a - \eta_b I_b - \mu_b^I I_b, \\
^{FFM}D_{0,t}^{\nu,\nu_1} G_a &= \eta_a I_a + \theta_a^G S_a - k_4 G_a, \\
^{FFM}D_{0,t}^{\nu,\nu_1} G_b &= \eta_b I_b + c_a^G G_a + \theta_b^G S_b - k_5 G_b, \\
^{FFM}D_{0,t}^{\nu,\nu_1} R_a &= \gamma_a G_a + \theta_a^R S_a - c_a^R R_a, \\
^{FFM}D_{0,t}^{\nu,\nu_1} R_b &= \gamma_b G_b + \theta_b^R S_b + c_a^R R_a - k_6 R_b,
\end{aligned} \quad . \tag{8}$$

with initial conditions

$$S_a(t) \geq 0,\ S_b(t) \geq 0,\ I_a(t) \geq 0,\ I_b(t) \geq 0,\ G_a(t) \geq 0,\ G_b(t) \geq 0,\ R_a(t) \geq 0,\ R_b(t) \geq 0,$$

where

$$\begin{aligned}
k_1 &= c_a^S + \theta_a^G + \theta_a^R \\
k_2 &= \theta_b^G + \theta_b^R \\
k_3 &= c_a^I + \eta_a \\
k_4 &= c_a^G + \gamma_a + \varepsilon_a \\
k_5 &= \gamma_b + \mu_b^G + \varepsilon_b + \beta_{bb} \\
k_6 &= \mu_b^R + \beta_{bb}.
\end{aligned}$$

Positivity analysis and equilibria

Equilibrium points of the proposed fractal fractional model (8) are obtained by solving the system

$$\begin{aligned}
^{FFM}D_{0,t}^{\nu,\nu_1} S_a = {}^{FFM}D_{0,t}^{\nu,\nu_1} S_b = {}^{FFM}D_{0,t}^{\nu,\nu_1} I_a = {}^{FFM}D_{0,t}^{\nu,\nu_1} I_b = {}^{FFM}D_{0,t}^{\nu,\nu_1} G_a \\
= {}^{FFM}D_{0,t}^{\nu,\nu_1} G_b = {}^{FFM}D_{0,t}^{\nu,\nu_1} R_a = {}^{FFM}D_{0,t}^{\nu,\nu_1} R_a = 0,
\end{aligned}$$

and supposing that all infectious class sizes are equal to zero which means that there is no disease eradication in the studied population, thus the disease-free equilibrium state is:

$$E^0 = \left(S_a^0, S_b^0, I_a^0, I_b^0, G_a^0, G_b^0, R_a^0, R_b^0\right) = \left(\frac{\beta_{ba} S_b}{k_1 - \beta_{aa}}, \frac{c_a^S S_a}{\mu_b^S + k_2}, 0, 0, 0, 0, 0, 0\right). \tag{9}$$

Let us fix a set of values for the model parameters, where n is a scale parameter [11]:

$\phi_a = \phi_b = \frac{4n^2}{100}$, $\mu_b^S = \epsilon_a = \beta_{bb} = 0$, $\mu_b^I = \frac{499n}{100}$, $\mu_b^G = \frac{49n}{100}$, $\mu_b^R = \frac{2n}{5}$, $\beta_{aa} = \beta_{ba} = \frac{n}{5}$, $\theta_a^G = \theta_a^R = \gamma_a = \gamma_b = 0.1n$, $\theta_b^G = \theta_b^R = n$, $c_a^S = c_a^I = c_a^G = c_a^R = \frac{98n}{96}$, $\epsilon_b = \frac{4n}{5}$, $\eta_a = \eta_b = 0.2n$. By utilizing the specified parameter values and setting $n = 1$ in the model (8), we are solving the model using Matlab and obtaining the following endemic equilibrium points:

$$E^* = \left(S_a^*, S_b^*, I_a^*, I_b^*, G_a^*, G_b^*, R_a^*, R_b^*\right) = (10, 10, 10, 10, 20, 40, 15, 15)\,.$$

Theorem 1 ([35]). *The solutions of the proposed system (8) with initial conditions are positive* $\forall\ t > 0$.

Proof. From (8), we get

$$\begin{aligned}
{}^{FFM}D_{0,t}^{\nu,\nu_1} S_a\Big|_{S_a=0} &= \beta_{aa}\left(G_a + R_a\right) + \beta_{ba}\left(S_b + G_b + R_b\right) \geq 0,\\
{}^{FFM}D_{0,t}^{\nu,\nu_1} S_b\Big|_{S_b=0} &= c_a^S S_a + \beta_{bb}\left(G_b + R_b\right) \geq 0,\\
{}^{FFM}D_{0,t}^{\nu,\nu_1} I_a\Big|_{I_a=0} &= \phi_a I_b S_a \geq 0,\\
{}^{FFM}D_{0,t}^{\nu,\nu_1} I_b\Big|_{I_b=0} &= \phi_b I_a S_b + c_a^I I_a \geq 0,\\
{}^{FFM}D_{0,t}^{\nu,\nu_1} G_a\Big|_{G_a=0} &= \eta_a I_a + \theta_a^G S_a \geq 0,\\
{}^{FFM}D_{0,t}^{\nu,\nu_1} G_b\Big|_{G_b=0} &= \eta_b I_b + c_a^G G_a + \theta_b^G S_b \geq 0,\\
{}^{FFM}D_{0,t}^{\nu,\nu_1} R_a\Big|_{R_a=0} &= \gamma_a G_a + \theta_a^R S_a \geq 0,\\
{}^{FFM}D_{0,t}^{\nu,\nu_1} R_b\Big|_{R_b=0} &= \gamma_b G_b + \theta_b^R S_b + c_a^R R_a \geq 0.
\end{aligned} \tag{10}$$

If $(S_a(0), S_b(0), I_a(0), I_b(0), G_a(0), G_b(0), R_a(0), R_b(0)) \in R_+^8$, then the result cannot escape the hyperplanes, according to (10). Consequently, R_+^8 is a set of positive invariants.

Stability

In this section, we present the global stability of (8) with the Lyapunov function.

Theorem 2 ([36]). *When the reproductive number* $R_0 > 1$*, the SIGR model's endemic equilibrium points* E^* *are globally asymptotically stable.*

Proof. To prove the theorem, we consider the Lyapunov function as follows:

$$\begin{aligned}
(S_a^*, S_b^*, I_a^*, I_b^*, G_a^*, G_b^*, R_a^*, R_b^*) = &\left(S_a - S_a^* - S_a^* \ln\frac{S_a}{S_a^*}\right) + \left(S_b - S_b^* - S_b^* \ln\frac{S_b}{S_b^*}\right)\\
&+ \left(I_a - I_a^* - I_a^* \ln\frac{I_a}{I_a^*}\right) + \left(I_b - I_b^* - I_b^* \ln\frac{I_b}{I_b^*}\right)\\
&+ \left(G_a - G_a^* - G_a^* \ln\frac{G_a}{G_a^*}\right) + \left(G_b - G_b^* - G_b^* \ln\frac{G_b}{G_b^*}\right)\\
&+ \left(R_a - R_a^* - R_a^* \ln\frac{R_a}{R_a^*}\right) + \left(R_b - R_b^* - R_b^* \ln\frac{R_b}{R_b^*}\right).
\end{aligned} \tag{11}$$

Applying the derivative with respect to t on both sides, we get

$$\begin{aligned}
\frac{dL}{dt} = \dot{L} = &\left(\frac{S_a - S_a^*}{S_a}\right)\dot{S}_a + \left(\frac{S_b - S_b^*}{S_b}\right)\dot{S}_b + \left(\frac{I_a - I_a^*}{I_a}\right)\dot{I}_a\\
&+ \left(\frac{I_b - I_b^*}{I_b}\right)\dot{I}_b + \left(\frac{G_a - G_a^*}{G_a}\right)\dot{G}_a + \left(\frac{G_b - G_b^*}{G_b}\right)\dot{G}_b\\
&+ \left(\frac{R_a - R_a^*}{R_a}\right)\dot{R}_a + \left(\frac{R_b - R_b^*}{R_b}\right)\dot{R}_b.
\end{aligned} \tag{12}$$

We can now write their derivative values as follows:

$$\begin{aligned}
\frac{dL}{dt} = \dot{L} = &\left(\frac{S_a - S_a^*}{S_a}\right)\beta_{aa}\left(S_a + G_a + R_a\right) + \beta_{ba}\left(S_b + G_b + R_b\right)\\
&- \left(k_1 + \phi_a\left(I_a + I_b\right)\right)S_a\\
&+ \left(\frac{S_b - S_b^*}{S_b}\right)\left(c_a^S S_a - \mu_b^S S_b - \left(k_2 + \phi_2\left(I_a + I_b\right)\right)S_b + \beta_{bb}\left(G_b + R_b\right)\right)\\
&+ \left(\frac{I_a - I_a^*}{I_a}\right)\left(\phi_a\left(I_a + I_b\right)S_a - k_3 I_a\right)\\
&+ \left(\frac{I_b - I_b^*}{I_b}\right)\left(\phi_b\left(I_a + I_b\right)S_b + c_a^I I_a - \eta_b I_b - \mu_b^I I_b\right)\\
&+ \left(\frac{G_a - G_a^*}{G_a}\right)\left(\eta_a I_a + \theta_a^G S_a - k_4 G_a\right)
\end{aligned}$$

$$\begin{aligned}
&+\left(\frac{G_b-G_b^*}{G_b}\right)\left(\eta_b I_b+c_a^G G_a+\theta_b^G S_b-k_5 G_b\right)\\
&+\left(\frac{R_a-R_a^*}{R_a}\right)\left(\gamma_a G_a+\theta_a^R S_a-c_a^R R_a\right)\\
&+\left(\frac{R_b-R_b^*}{R_b}\right)\left(\gamma_b G_b+\theta_b^R S_b+c_a^R R_a-k_6 R_b\right).
\end{aligned}\tag{13}$$

Putting $S_a = S_a - S_a^*$, $S_b = S_b - S_b^*$, $I_a = I_a - I_a^*$, $I_b = I_b - I_b^*$, $G_a = G_a - G_a^*$, $G_b = G_b - G_b^*$, $R_a = R_a - R_a^*$ and $R_b = S_b - S_b^*$ leads to

$$\begin{aligned}
\frac{dL}{dt} = \dot{L} &= \left(\frac{S_a-S_a^*}{S_a}\right)\{\beta_{aa}\left((S_a-S_a^*)+(G_a-G_a^*)+(R_a-R_a^*)\right)\\
&+\beta_{ba}\left((S_b-S_b^*)+(G_b-G_b^*)+(R_b-R_b^*)\right)-\left(k_1+\phi_a\left((I_a-I_a^*)+(I_b-I_b^*)\right)\right)(S_a-S_a^*)\}\\
&+\left(\frac{S_b-S_b^*}{S_b}\right)\{c_a^S(S_a-S_a^*)-\mu_b^S(S_b-S_b^*)-\left(k_2+\phi_b\left((I_a-I_a^*)+(I_b-I_b^*)\right)\right)(S_b-S_b^*)\\
&+\beta_{bb}\left((G_b-G_b^*)+(R_b-R_b^*)\right)\}\\
&+\left(\frac{I_a-I_a^*}{I_a}\right)\left(\phi_a\left((I_a-I_a^*)+(I_b-I_b^*)\right)(S_a-S_a^*)-k_3(I_a-I_a^*)\right)\\
&+\left(\frac{I_b-I_b^*}{I_b}\right)\left(\phi_b\left((I_a-I_a^*)+(I_b-I_b^*)\right)(S_b-S_b^*)+c_a^I(I_a-I_a^*)-\eta_b(I_b-I_b^*)-\mu_b^I(I_b-I_b^*)\right)\\
&+\left(\frac{G_a-G_a^*}{G_a}\right)\left(\eta_a(I_a-I_a^*)+\theta_a^G(S_a-S_a^*)-k_4(G_a-G_a^*)\right)\\
&+\left(\frac{G_b-G_b^*}{G_b}\right)\left(\eta_b(I_b-I_b^*)+c_a^G(G_a-G_a^*)+\theta_b^G S_b-k_5(G_b-G_b^*)\right)\\
&+\left(\frac{R_a-R_a^*}{R_a}\right)\left(\gamma_a(G_a-G_a^*)+\theta_a^R(S_a-S_a^*)-c_a^R(R_a-R_a^*)\right)\\
&+\left(\frac{R_b-R_b^*}{R_b}\right)\left(\gamma_b(G_b-G_b^*)+\theta_b^R(S_b-S_b^*)+c_a^R(R_a-R_a^*)-k_6(R_b-R_b^*)\right).
\end{aligned}\tag{14}$$

We can change the above as follows:

$$\frac{dL}{dt} = \dot{L} = f_1 - f_2 + f_3 - f_4 + f_5 - f_6 + f_7 - f_8 + f_9 - f_{10} + f_{11} - f_{12} + f_{13} - f_{14} + f_{15} - f_{16},\tag{15}$$

where

$$\begin{aligned}
f_1 &= \tfrac{\beta_{aa}(S_a-S_a^*)^2}{S_a} + \beta_{aa}G_a + \tfrac{S_a^*}{S_a}\beta_{aa}G_a^* + \beta_{aa}R_a + \tfrac{S_a^*}{S_a}\beta_{aa}R_a^* + \beta_{ba}S_b + \tfrac{S_a^*}{S_a}\beta_{ba}S_b^*\\
&+\beta_{ba}G_b + \tfrac{S_a^*}{S_a}\beta_{ba}G_b^* + \beta_{ba}R_b + \tfrac{S_a^*}{S_a}\beta_{ba}R_b^* + \tfrac{\phi_a I_a^*(S_a-S_a^*)^2}{S_a} + \tfrac{\phi_a I_b^*(S_a-S_a^*)^2}{S_a}\\
f_2 &= \beta_{aa}G_a^* + \tfrac{S_a^*}{S_a}\beta_{aa}G_a + \beta_{aa}R_a^* + \tfrac{S_a^*}{S_a}\beta_{aa}R_a + \beta_{ba}S_b^* + \tfrac{S_a^*}{S_a}\beta_{ba}S_b + \beta_{ba}G_b^* + \tfrac{S_a^*}{S_a}\beta_{ba}G_b\\
&+\beta_{ba}R_a^* + \tfrac{S_a^*}{S_a}\beta_{ba}R_b + \tfrac{(S_a-S_a^*)^2}{S_a}k_1 + \tfrac{(S_a-S_a^*)^2}{S_a}\phi_a I_a + \tfrac{(S_a-S_a^*)^2}{S_a}\phi_a I_b\\
f_3 &= c_a^S S_a + \frac{S_b^*}{S_b}c_a^S S_a^* + \frac{(S_b-S_b^*)^2}{S_b}I_a^* + \frac{(S_b-S_b^*)^2}{S_b}I_a^* + \beta_{bb}G_b + \frac{S_b^*}{S_b}\beta_{bb}G_b^* + \beta_{bb}R_b + \frac{S_b^*}{S_b}\beta_{bb}R_b^*\\
f_4 &= c_a^S S_a^* + \tfrac{S_b^*}{S_b}c_a^S S_a + \tfrac{(S_b-S_b^*)^2}{S_b}\mu_b^S + \tfrac{(S_b-S_b^*)^2}{S_b}k_2 + \tfrac{(S_b-S_b^*)^2}{S_b}\phi_b I_a + \tfrac{(S_b-S_b^*)^2}{S_b}\phi_b I_a\\
&+\beta_{bb}G_b^* + \tfrac{S_b^*}{S_b}\beta_{bb}G_b + \beta_{bb}R_b^* + \tfrac{S_b^*}{S_b}\beta_{bb}R_b\\
f_5 &= \frac{(I_a-I_a^*)^2}{I_a}\phi_a S_a + \phi_a I_b S_a + \phi_a I_b^* S_a^* + \frac{I_a^*}{I_a}\phi_a I_b S_a^* + \frac{I_a^*}{I_a}\phi_a I_b^* S_a\\
f_6 &= \frac{(I_a-I_a^*)^2}{I_a}\phi_a S_a^* + \phi_a I_b S_a^* + \phi_a I_b^* S_a + \frac{I_a^*}{I_a}\phi_a I_b S_a + \frac{I_a^*}{I_a}\phi_a I_b^* S_a^* + \frac{(I_a-I_a^*)^2}{I_a}k_3\\
f_7 &= \phi_b I_a S_b + \phi_b I_a^* S_b^* + \phi_b I_a S_b^* + \phi_b I_a^* S_b + \frac{(I_b-I_b^*)^2}{I_b}\phi_b S_b + c_a^I I_a + \frac{I_b^*}{I_b}c_a^I I_a^*\\
f_8 &= \phi_b I_a S_b^* + \phi_b I_a^* S_b + \phi_b I_a S_b + \phi_b I_a^* S_b^* + \tfrac{(I_b-I_b^*)^2}{I_b}\phi_b S_b + c_a^I I_a^* + \tfrac{I_b^*}{I_b}a_1^I I_a\\
&+\tfrac{(I_b-I_b^*)^2}{I_b}\eta_b + \tfrac{(I_b-I_b^*)^2}{I_b}\mu_b^I\\
f_9 &= \eta_a I_a + \frac{G_a^*}{G_a}\eta_a I_a^* + \theta_a^G S_a + \frac{G_a^*}{G_a}\theta_a^G S_a^*\\
f_{10} &= \eta_a I_a^* + \frac{G_a^*}{G_a}\eta_a I_a + \theta_a^G S_a^* + \frac{G_a^*}{G_a}\theta_a^G S_a + \frac{(G_a-G_a^*)^2}{G_a}k_4\\
f_{11} &= \eta_b I_b + \frac{G_b^*}{G_b}\eta_b I_b^* + c_a^G G_a + \frac{G_b^*}{G_b}c_a^G G_a^* + \theta_b^G S_b + \frac{G_b^*}{G_b}\theta_b^G S_b^*
\end{aligned}$$

$$f_{12} = \eta_b I_b^* + \frac{G_b^*}{G_b}\eta_b I_b + c_a^G G_a^* + \frac{G_b^*}{G_b} c_a^G G_a + \theta_b^G S_b^* + \frac{G_b^*}{G_b}\theta_b^G S_b + \frac{(G_a - G_a^*)^2}{G_a} k_5$$

$$f_{13} = \gamma_a G_a + \frac{R_a^*}{R_a}\gamma_a G_a^* + \theta_a^R S_a + \frac{R_a^*}{R_a}\theta_a^R S_a^*$$

$$f_{14} = \gamma_a G_a^* + \frac{R_a^*}{R_a}\gamma_a G_a + \theta_a^R S_a^* + \frac{R_a^*}{R_a}\theta_a^R S_a + \frac{(R_a - R_a^*)^2}{R_a} c_a^R$$

$$f_{15} = \gamma_b G_b + \frac{R_b^*}{R_b}\gamma_b G_b^* + \theta_b^R S_b + \frac{R_b^*}{R_b}\theta_b^R S_b^* + c_a^R R_a + \frac{R_b^*}{R_b} c_a^R R_a^*$$

$$f_{16} = \gamma_b G_b^* + \frac{R_b^*}{R_b}\gamma_b G_b + \theta_b^R S_b^* + \frac{R_b^*}{R_b}\theta_b^R S_b + c_a^R R_a^* + \frac{R_b^*}{R_b} c_a^R R_a + \frac{(R_b - R_b^*)^2}{R_b} k_6$$

The above can be expressed as follows to avoid complexity:

$$\frac{dL}{dt} = \Delta_1 - \Delta_2, \tag{16}$$

where

$$\Delta_1 = f_1 + f_3 + f_5 + f_7 + f_9 + f_{11} + f_{13} + f_{15}$$

and

$$\Delta_2 = f_2 + f_4 + f_6 + f_8 + f_{10} + f_{12} + f_{14} + f_{16}$$

It has been concluded that if $\Delta_1 < \Delta_2$, this yields $\frac{dL}{dt} < 0$, when, however $S_a = S_a^*$, $S_b = S_b^*$, $I_a = I_a^*$, $I_b = I_b^*$, $G_a = G_a^*$, $G_b = G_b^*$, $R_a = R_a^*$, $R_b = R_b^*$ is

$$0 = \Delta_1 - \Delta_2 \Rightarrow \frac{dL}{dt} = 0 \tag{17}$$

As observed, the largest compact invariant set for the proposed model is

$$\left\{ \left(S_a^*, S_b^*, I_a^*, I_b^*, G_a^*, G_b^*, R_a^*, R_b^*\right) \in \Phi : \frac{dL}{dt} = 0 \right\} \tag{18}$$

$\left\{E_*\right\}$ represents the endemic equilibrium of the nonlinear model. Lasalle's invariance principle demonstrates that E_* is globally asymptotic and stable in Φ if $\Delta_1 < \Delta_2$.

Second derivative of Lyapunov function

Although employing the first derivative of a Lyapunov function aids in stability assessment, it is important to acknowledge that this analysis does not capture the complete intricacies of the function's changes. To achieve a comprehensive comprehension of all variation aspects, further exploration is required. Therefore, we proceed to scrutinize the second derivative of the associated Lyapunov function of our model.

$$\begin{aligned} \frac{d\dot{L}}{dt} &= \frac{d}{dt}\{(\frac{S_a - S_a^*}{S_a})\dot{S}_a + (\frac{S_b - S_b^*}{S_b})\dot{S}_b + (\frac{I_a - I_a^*}{I_a})\dot{I}_a + (\frac{I_b - I_b^*}{I_b})\dot{S}_b \\ &+ (\frac{G_a - G_a^*}{G_a})\dot{G}_a + (\frac{G_b - G_b^*}{G_b})\dot{G}_b + (\frac{R_a - R_a^*}{R_a})\dot{R}_a + (\frac{R_b - R_b^*}{R_b})\dot{R}_b\} \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{d\dot{L}}{dt} &= (\frac{\dot{S}_a}{S_a})^2 S_a^* + (\frac{\dot{S}_b}{S_b})^2 S_b^* + (\frac{\dot{I}_a}{I_a})^2 I_a^* + (\frac{\dot{I}_b}{I_b})^2 I_b^* \\ &+ (\frac{\dot{G}_a}{G_a})^2 G_a^* + (\frac{\dot{G}_b}{G_b})^2 G_b^* + (\frac{\dot{R}_a}{R_a})^2 R_a^* + (\frac{\dot{R}_b}{R_b})^2 R_b^* \\ &+ (1 - \frac{S_a^*}{S_a})\ddot{S}_a + (1 - \frac{S_b^*}{S_b})\ddot{S}_b + (1 - \frac{I_a^*}{I_a})\ddot{I}_a + (1 - \frac{I_b^*}{I_b})\ddot{I}_b \\ &+ (1 - \frac{G_a^*}{G_a})\ddot{G}_a + (1 - \frac{G_b^*}{G_b})\ddot{G}_b + (1 - \frac{R_a^*}{R_a})\ddot{R}_a + (1 - \frac{R_b^*}{R_b})\ddot{R}_b \end{aligned} \tag{20}$$

here

$$\begin{aligned} \dot{S}_a &= \beta_{aa}\left(\dot{S}_a + \dot{G}_a + \dot{R}_a\right) + \beta_{ba}\left(\dot{S}_b + \dot{G}_b + \dot{R}_b\right) - \phi_a\left(\dot{I}_a + \dot{I}_b\right) S_a - \left(c_a^S + \theta_a^G + \phi_a\left(I_a + I_b\right)\right)\dot{S}_a \\ \dot{S}_b &= c_a^S \dot{S}_a - \mu_b^S \dot{S}_b - \phi_b\left(I_a + I_b\right) S_b - \left(\theta_b^G + \theta_b^R + \phi_b\left(I_a + I_b\right)\right)\dot{S}_b + \beta_{bb}\left(\dot{G}_b + \dot{R}_b\right) \\ \dot{I}_a &= \phi_a\left(\dot{I}_a + \dot{I}_b\right) S_a + \phi_a\left(I_a + I_b\right)\dot{S}_a - \left(c_a^I + \eta_a\right)\dot{I}_a \\ \dot{I}_b &= \phi_b\left(\dot{I}_a + \dot{I}_b\right) S_b + \phi_b\left(I_a + I_b\right)\dot{S}_b + c_a^I \dot{I}_a - \eta_b \dot{I}_b - \mu_b^I \dot{I}_b \\ \dot{G}_a &= \eta_a \dot{I}_a + \theta_a^G \dot{S}_a - \left(c_a^G + \gamma_a + \varepsilon_a\right)\dot{G}_a \\ \dot{G}_b &= \eta_b \dot{I}_b + c_a^G \dot{G}_a + \theta_b^G \dot{S}_b - \left(\gamma_b + \mu_b^G + \varepsilon_b + \beta_{bb}\right)\dot{G}_b \\ \dot{R}_a &= \gamma_a \dot{G}_a + \theta_a^R \dot{S}_a - c_a^R \dot{R}_a \\ \dot{R}_b &= \gamma_b \dot{G}_b + \theta_b^R \dot{S}_b + c_a^R \dot{R}_a - \left(\mu_b^R + \beta_{bb}\right)\dot{R}_b. \end{aligned} \tag{21}$$

After simplification

$$\begin{aligned}\frac{d\dot{L}}{dt} &= \dot{\Lambda}(S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) + h_1 - h_2 + h_3 - h_4 + h_5 - h_6 + h_7 \\ &- h_8 + h_9 - h_{10} + h_{11} - h_{12} + h_{13} - h_{14} + h_{15} - h_{16}\end{aligned} \tag{22}$$

where

$$\begin{aligned}h_1 &= \beta_{aa}^2(S_a + G_a + R_a) + \beta_{aa}\beta_{ba}(S_b + G_b + R_b) + \beta_{aa}\eta_a I_a + \beta_{aa}\theta_a^G S_a \\ &+ \beta_{aa}\gamma_a G_a + \beta_{aa}\theta_a^R S_a + \beta_{ba}c_a^S S_a + \beta_{ba}\beta_{bb}(G_b + R_b) + \beta_{ba}\eta_b I_b + c_a^G G_a \\ &+ \beta_{ba}\theta_b^G S_b + \beta_{ba}\gamma_b G_b + \beta_{ba}\theta_b^R S_b + \beta_{ba}c_a^R R_a + k_3\phi_a I_a S_a + \phi_a\eta_b I_b S_a + \phi_a\mu_b^I I_b S_a \\ &+ (k_1 + \phi_a(I_a + I_b))^2 S_a + S_a^*\beta_{aa}(k_1 + \phi_a(I_a + I_b)) + \frac{S_a^*}{S_a}\beta_{aa}k_4 G_a + \frac{S_a^*}{S_a}\beta_{aa}c_a^R R_a \\ &+ \frac{S_a^*}{S_a}\beta_{ba}\mu_b^S S_b + \frac{S_a^*}{S_a}\beta_{ba}(k_1 + \phi_b(I_a + I_b))S_b + \frac{S_a^*}{S_a}\beta_{ba}k_5 G_b + \frac{S_a^*}{S_a}\beta_{ba}k_6 R_b \\ &+ \phi_a^2(I_a + I_b)S_a S_a^* + \phi_a\phi_b(I_a + I_b)S_b S_a^* + \phi_a c_a^I I_a S_a^* + \beta_{aa}(k_1 + \phi_a(I_a + I_b))S_a^* \\ &+ \frac{S_a^*}{S_a}\beta_{aa}(k_1 + \phi_a(I_a + I_b))G_a + \frac{S_a^*}{S_a}\beta_{aa}(k_1 + \phi_a(I_a + I_b))R_a + \frac{S_a^*}{S_a}\beta_{ba}(k_1 + \phi_a(I_a + I_b))S_b \\ &+ \frac{S_a^*}{S_a}\beta_{ba}(k_1 + \phi_a(I_a + I_b))G_b + \frac{S_a^*}{S_a}\beta_{ba}(k_1 + \phi_a(I_a + I_b))R_b,\end{aligned} \tag{23}$$

$$\begin{aligned}h_2 &= \beta_{aa}^2 S_a^* + \frac{S_a^*}{S_a}\beta_{aa}^2 G_a + \frac{S_a^*}{S_a}\beta_{aa}^2 R_a + \frac{S_a^*}{S_a}\beta_{aa}\beta_{ba}(S_b + G_b + R_b) + \frac{S_a^*}{S_a}\beta_{aa}\eta_a I_a \\ &+ \beta_{aa}\theta_a^G S_a^* + \frac{S_a^*}{S_a}\beta_{aa}\gamma_a G_a + \beta_{aa}\theta_a^R S_a^* + \beta_{ba}c_a^S S_a^* + \frac{S_a^*}{S_a}\beta_{ba}\beta_{bb}(G_b + R_b) + \frac{S_a^*}{S_a}\beta_{ba}\eta_b I_b \\ &+ \frac{S_a^*}{S_a}c_a^G G_a + \frac{S_a^*}{S_a}\beta_{ba}\theta_b^G S_b + \frac{S_a^*}{S_a}\beta_{ba}\gamma_b G_b + \frac{S_a^*}{S_a}\beta_{ba}\theta_b^R S_b + \frac{S_a^*}{S_a}\beta_{ba}c_a^R R_a + k_3\phi_a I_a S_a^* \\ &+ \phi_a\eta_b I_b S_a^* + \phi_a\mu_b^I I_b S_a^* + (k_1 + \phi_a(I_a + I_b))^2 S_a^* + S_a\beta_{aa}(k_1 + \phi_a(I_a + I_b)) + \beta_{aa}k_4 G_a \\ &+ \beta_{aa}c_a^R R_a + \beta_{ba}\mu_b^S S_b + \beta_{ba}(k_1 + \phi_b(I_a + I_b))S_b + \beta_{ba}k_5 G_b + \beta_{ba}k_6 R_b + \phi_a^2(I_a + I_b)S_a^2 \\ &+ \phi_a\phi_b(I_a + I_b)S_a S_b + \phi_1 a_1^I I_a S_a + \beta_{aa}(k_1 + \phi_a(I_a + I_b))S_a + \beta_{aa}(k_1 + \phi_a(I_a + I_b))G_a \\ &+ \beta_{aa}(k_1 + \phi_a(I_a + I_b))R_a + \beta_{ba}(k_1 + \phi_a(I_a + I_b))S_b + \beta_{ba}(k_1 + \phi_a(I_a + I_b))G_b \\ &+ \beta_{ba}(k_1 + \phi_a(I_a + I_b))R_b\end{aligned} \tag{24}$$

$$\begin{aligned}h_3 &= c_a^S\beta_{aa}(S_a + G_a + R_a) + c_a^S\beta_{ba}(S_b + G_b + R_b) + (\mu_b^S)^2 S_b + \mu_b^S(k_2 + \phi_b(I_a + I_b))S_b \\ &+ \phi_b k_3 I_a S_b + \phi_b\eta_b I_b S_b + \phi_b\mu_b^I I_b S_b + (k_2 + \phi_b(I_a + I_b))\mu_b^S S_b + (k_2 + \phi_b(I_a + I_b)^2)S_b \\ &+ \beta_{bb}\eta_b I_b + \beta_{bb}c_a^G G_a + \beta_{bb}\theta_b^G S_b + \beta_{bb}\gamma_b G_b + \beta_{bb}\theta_b^R S_b + \beta_{bb}c_a^R R_a + \frac{S_b^*}{S_b}(k_1 + \phi_a(I_a + I_b))c_a^S S_a \\ &+ \frac{S_b^*}{S_b}\mu_b^S c_a^S S_a + \frac{S_b^*}{S_b}\mu_b^S\beta_{bb}(G_b + R_b) + \phi_a\phi_b(I_a + I_b)S_a S_b^* + \phi_b^2(I_a + I_b)S_b S_b^* + \phi_b c_a^I I_a S_b^* \\ &+ \frac{S_b^*}{S_b}(k_2 + \phi_b(I_a + I_b))a_1^S S_a + \frac{S_b^*}{S_b}(k_2 + \phi_b(I_a + I_b))\beta_{bb}(G_b + R_b) + \frac{S_b^*}{S_b}\beta_{bb}k_4 G_a + \frac{S_b^*}{S_b}\beta_{bb}k_6 R_b\end{aligned} \tag{25}$$

$$\begin{aligned}h_4 &= \frac{S_b^*}{S_b}a_1^S\beta_{aa}(S_a + G_a + R_a) + \frac{S_b^*}{S_b}c_a^S\beta_{ba}(S_b + G_b + R_b) + (\mu_b^S)^2 S_b^* + \mu_b^S(k_2 + \phi_b(I_a + I_b))S_b^* \\ &+ \phi_b k_3 I_a S_b^* + \phi_b\eta_b I_b S_b^* + \phi_b\mu_b^I I_b S_b^* + (k_2 + \phi_b(I_a + I_b))\mu_b^S S_b^* + (k_2 + \phi_b(I_a + I_b)^2)S_b^* \\ &+ \frac{S_b^*}{S_b}\beta_{bb}\eta_b I_b + \frac{S_b^*}{S_b}\beta_{bb}c_a^G G_a + \beta_{bb}\theta_b^G S_b^* + \frac{S_b^*}{S_b}\beta_{bb}\gamma_b G_b + \beta_{bb}\theta_b^R S_b^* + \frac{S_b^*}{S_b}\beta_{bb}c_a^R R_a + (k_1 + \phi_a(I_a + I_b))c_a^S S_a \\ &+ \mu_b^S c_a^S S_a + \mu_b^S\beta_{bb}(G_b + R_b) + \phi_a\phi_b(I_a + I_b)S_a + \phi_b^2(I_a + I_b)S_b^2 + \phi_b c_a^I I_a \\ &+ (k_2 + \phi_b(I_a + I_b))c_a^S S_a + (k_2 + \phi_b(I_a + I_b))\beta_{bb}(G_b + R_b) + \beta_{bb}k_4 G_a + \beta_{bb}k_6 R_b\end{aligned} \tag{26}$$

$$\begin{aligned}h_5 &= \phi_a^2(I_a + I_b)S_a^2 + \phi_a\phi_b(I_a + I_b)S_a S_b + c_a^I\phi_a I_a S_a + \phi_a(I_a + I_b)\beta_{aa}(S_a + G_a + R_a) \\ &+ \phi_a(I_a + I_b)\beta_{ba}(S_b + G_b + R_b) + k_3^2 I_a + \phi_a k_3 S_a I_a^* + \frac{I_a^*}{I_a}\phi_a\eta_b I_b S_a + \frac{I_a^*}{I_a}\phi_a\mu_b^I I_b S_a \\ &+ \frac{I_a^*}{I_a}\phi_a(I_a + I_b)(k_1 + \phi_a(I_a + I_b))S_a + \frac{I_a^*}{I_a}k_3\phi_a(I_a + I_b)S_a\end{aligned} \tag{27}$$

$$\begin{aligned} h_6 = {} & \frac{I_a^*}{I_a}\phi_a^2(I_a+I_b)S_a^2 + \frac{I_a^*}{I_a}\phi_a\phi_b(I_a+I_b)S_aS_b + a_1^I\phi_a I_a^* S_a + \frac{I_a^*}{I_a}\phi_a(I_a+I_b)\beta_{aa}(S_a+G_a+R_a) \\ & + \frac{I_a^*}{I_a}\phi_a(I_a+I_b)\beta_{ba}(S_b+G_b+R_b) + k_3^2 I_a^* + \phi_a k_3 S_a I_a + \phi_a\eta_b I_b S_a + \phi_a\mu_b^I I_b S_a \\ & + \phi_a(I_a+I_b)(k_1+\phi_a(I_a+I_b))S_a + k_3\phi_a(I_a+I_b)S_a \end{aligned} \tag{28}$$

$$\begin{aligned} h_7 = {} & \phi_a\phi_b(I_a+I_b)S_aS_b + \phi_b^2(I_a+I_b)S_b^2 + c_a^I\phi_b I_a S_b + \phi_b(I_a+I_b)c_a^S S_a \\ & + \phi_b(I_a+I_b)\beta_{bb}(G_b+R_b) + c_a^I\phi_a(I_a+I_b)S_a + (\eta_b+\mu_b^I)^2 I_b + \frac{I_b^*}{I_b}\phi_b k_3 I_a S_b \\ & + \phi_b\eta_b S_b I_b^* + \phi_b\mu_b^I S_b I_b^* + \frac{I_b^*}{I_b}\phi_b(I_a+I_b)\mu_b^S S_b + \frac{I_b^*}{I_b}\phi_b(I_a+I_b)(k_2+\phi_b(I_a+I_b))S_b \\ & + \frac{I_b^*}{I_b}c_a^I k_3 I_a + \frac{I_b^*}{I_b}(\eta_b+\mu_b^I)\phi_b(I_a+I_b)S_b + \frac{I_b^*}{I_b}(\eta_b+\mu_b^I)c_a^I I_a \end{aligned} \tag{29}$$

$$\begin{aligned} h_8 = {} & \frac{I_b^*}{I_b}\phi_a\phi_b(I_a+I_b)S_aS_b + \frac{I_b^*}{I_b}\phi_b^2(I_a+I_b)S_b^2 + \frac{I_b^*}{I_b}c_a^I\phi_b I_a S_b + \frac{I_b^*}{I_b}\phi_b(I_a+I_b)c_a^S S_a \\ & + \frac{I_b^*}{I_b}\phi_b(I_a+I_b)\beta_{bb}(G_b+R_b) + \frac{I_b^*}{I_b}c_a^I\phi_a(I_a+I_b)S_a + (\eta_b+\mu_b^I)^2 I_b^* + \phi_b k_3 I_a S_b \\ & + \phi_b\eta_b S_b I_b + \phi_b\mu_b^I S_b I_b + \phi_b(I_a+I_b)\mu_b^S S_b + \phi_b(I_a+I_b)(k_2+\phi_b(I_a+I_b))S_b \\ & + a_1^I k_3 I_a + (\eta_b+\mu_b^I)\phi_b(I_a+I_b)S_b + (\eta_b+\mu_b^I)c_a^I I_a \end{aligned} \tag{30}$$

$$\begin{aligned} h_9 = {} & \eta_a\phi_a(I_a+I_b)S_a + \theta_a^G\beta_{aa}(S_a+G_a+R_a) + \theta_a^G\beta_{ba}(S_b+G_b+R_b) + k_4^2 G_a \\ & + \frac{G_a^*}{G_a}\eta_a k_3 I_a + \frac{G_a^*}{G_a}\theta_a^G(k_1+\phi_a(I_a+I_b))S_a + \frac{G_a^*}{G_a}k_4\eta_a I_a + \frac{G_a^*}{G_a}k_4\theta_a^G S_a \end{aligned} \tag{31}$$

$$\begin{aligned} h_{10} = {} & \frac{G_a^*}{G_a}\eta_a\phi_a(I_a+I_b)S_a + \frac{G_a^*}{G_a}\theta_a^G\beta_{aa}(S_a+G_a+R_a) + \frac{G_a^*}{G_a}\theta_a^G\beta_{ba}(S_b+G_b+R_b) + k_4^2 G_a^* \\ & + \eta_a k_3 I_a + \theta_a^G(k_1+\phi_a(I_a+I_b))S_a + k_4\eta_a I_a + k_4\theta_a^G S_a \end{aligned} \tag{32}$$

$$\begin{aligned} h_{11} = {} & \eta_b\phi_b(I_a+I_b)S_b + c_a^I\eta_b I_a + c_a^G\eta_a I_a + c_a^G\theta_a^G S_a + \theta_b^G c_a^S S_a + \theta_b^G\beta_{bb}(G_b+R_b) \\ & + k_5^2 G_b + \frac{G_b^*}{G_b}\eta_b^2 I_b + \frac{G_b^*}{G_b}\eta_b\mu_b^I I_b + \frac{G_b^*}{G_b}c_a^G k_4 G_a + \frac{G_b^*}{G_b}\theta_b^G\mu_b^S S_b + \frac{G_b^*}{G_b}\theta_b^G(k_2+\phi_b(I_a+I_b))S_b \\ & + \frac{G_b^*}{G_b}k_5\eta_b I_b + \frac{G_b^*}{G_b}k_5 c_a^G G_a + \frac{G_b^*}{G_b}k_5\theta_b^G S_b \end{aligned} \tag{33}$$

$$\begin{aligned} h_{12} = {} & \frac{G_b^*}{G_b}\eta_b\phi_b(I_a+I_b)S_b + \frac{G_b^*}{G_b}c_a^I\eta_b I_a + \frac{G_b^*}{G_b}c_a^G\eta_a I_a + \frac{G_b^*}{G_b}c_a^G\theta_a^G S_a + \frac{G_b^*}{G_b}\theta_b^G c_a^S S_a \\ & + \frac{G_b^*}{G_b}\theta_b^G\beta_{bb}(G_b+R_b) + k_5^2 G_b^* + \eta_b^2 I_b + \eta_b\mu_b^I I_b + c_a^G k_4 G_a + \theta_b^G\mu_b^S S_b + \theta_b^G(k_2+\phi_b(I_a+I_b))S_b \\ & + k_5\eta_b I_b + k_5 c_a^G G_a + k_5\theta_b^G S_b \end{aligned} \tag{34}$$

$$\begin{aligned} h_{13} = {} & \gamma_a\eta_a I_a + \gamma_a\theta_a^G S_a + \theta_a^R\beta_{aa}(S_a+G_a+R_a) + \theta_a^R\beta_{ba}(S_b+G_b+R_b) \\ & + (c_a^R)^2 R_a + \frac{R_a^*}{R_a}\gamma_a k_4 G_a + \frac{R_a^*}{R_a}\theta_a^R(k_1+\phi_a(I_a+I_b))S_a + \frac{R_a^*}{R_a}c_a^R\gamma_a G_a \\ & + \frac{R_a^*}{R_a}c_a^R\theta_a^R S_a \end{aligned} \tag{35}$$

$$\begin{aligned} h_{14} = {} & \frac{R_a^*}{R_a}\gamma_a\eta_a I_a + \frac{R_a^*}{R_a}\gamma_a\theta_a^G S_a + \frac{R_a^*}{R_a}\theta_a^R\beta_{aa}(S_a+G_a+R_a) + \frac{R_a^*}{R_a}\theta_a^R\beta_{ba}(S_b+G_b+R_b) \\ & + (c_a^R)^2 R_a^* + \gamma_a k_4 G_a + \theta_a^R(k_1+\phi_a(I_a+I_b))S_a + a_1^R\gamma_a G_a + c_a^R\theta_a^R S_a \end{aligned} \tag{36}$$

$$\begin{aligned} h_{15} = {} & \gamma_b\eta_b I_b + \gamma_b c_a^G S_a + \theta_b^R c_a^S S_a + \theta_b^R\beta_{bb}(G_b+R_b) + c_a^R\gamma_a G_a + c_a^R\theta_a^R S_a \\ & + k_6^2 R_b + \frac{R_b^*}{R_b}\gamma_b k_4 G_a + \frac{R_b^*}{R_b}\theta_b^R\mu_b^S S_b + \frac{R_b^*}{R_b}\theta_b^R(k_2+\phi_b(I_a+I_b))S_b \\ & + \frac{R_b^*}{R_b}(c_a^R)^2 R_a + \frac{R_b^*}{R_b}k_6\gamma_b G_b + \frac{R_b^*}{R_b}k_6\theta_b^R S_b + \frac{R_b^*}{R_b}k_6 a_1^R R_a \end{aligned} \tag{37}$$

$$\begin{aligned} h_{16} = {} & \frac{R_b^*}{R_b}\gamma_b\eta_b I_b + \frac{R_b^*}{R_b}\gamma_b c_a^G S_a + \frac{R_b^*}{R_b}\theta_b^R c_a^S S_a + \frac{R_b^*}{R_b}\theta_b^R\beta_{bb}(G_b+R_b) \\ & + \frac{R_b^*}{R_b}c_a^R\gamma_a G_a + \frac{R_b^*}{R_b}c_a^R\theta_a^R S_a + k_6^2 R_b^* + \gamma_b k_4 G_a + \theta_b^R\mu_b^S S_b \\ & + \theta_b^R(k_2+\phi_b(I_a+I_b))S_b + (c_a^R)^2 R_a + k_6\gamma_b G_b + k_6\theta_b^R S_b + k_6 c_a^R R_a \end{aligned} \tag{38}$$

For simplicity, we can write

$$\frac{d\dot{L}}{dt} = \Psi_1 - \Psi_2 \tag{39}$$

where

$$\Psi_1 = \dot{\Lambda}(S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) + h_1 + h_3 + h_5 + h_7 + h_9 + h_{11} + h_{13} + h_{15} \tag{40}$$

and

$$\Psi_2 = h_2 + h_4 + h_6 + h_8 + h_{10} + h_{12} + h_{14} + h_{16} \tag{41}$$

It can be seen that if $\Psi_1 > \Psi_2$ then $\frac{d\dot{L}}{dt} > 0$, if $\Psi_1 < \Psi_2$ then $\frac{d\dot{L}}{dt} < 0$ and if $\Psi_1 = \Psi_2$ then $\frac{d\dot{L}}{dt} = 0$.

Existence and uniqueness results

Using fixed point results, we show that the model (8) under investigation has at least one and unique solution. Because the integral is differentiable, the proposed model may be written as

$$\begin{cases} {}^{FFM}D_{0,t}^{\nu,\nu_1} S_a(t) = \nu_1 t^{\nu_1 - 1} J(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ {}^{FFM}D_{0,t}^{\nu,\nu_1} S_b(t) = \nu_1 t^{\nu_1 - 1} K(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ {}^{FFM}D_{0,t}^{\nu,\nu_1} I_a(t) = \nu_1 t^{\nu_1 - 1} L(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ {}^{FFM}D_{0,t}^{\nu,\nu_1} I_b(t) = \nu_1 t^{\nu_1 - 1} M(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ {}^{FFM}D_{0,t}^{\nu,\nu_1} G_a(t) = \nu_1 t^{\nu_1 - 1} N(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ {}^{FFM}D_{0,t}^{\nu,\nu_1} G_b(t) = \nu_1 t^{\nu_1 - 1} O(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ {}^{FFM}D_{0,t}^{\nu,\nu_1} R_a(t) = \nu_1 t^{\nu_1 - 1} P(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ {}^{FFM}D_{0,t}^{\nu,\nu_1} R_b(t) = \nu_1 t^{\nu_1 - 1} Q(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \end{cases} \tag{42}$$

where

$$\begin{aligned}
&J(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = \beta_{aa}\left(S_a + G_a + R_a\right) + \beta_{ba}\left(S_b + G_b + R_b\right) - \left(c_a^S + \theta_a^G + \phi_a\left(I_a + I_b\right)\right) S_a, \\
&K(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = c_a^S S_a - \mu_b^S S_b - \left(\theta_b^G + \theta_b^R + \phi_b\left(I_a + I_b\right)\right) S_b + \beta_{bb}\left(G_b + R_b\right), \\
&L(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = \phi_a\left(I_a + I_b\right) S_a - \left(c_a^I + \eta_a\right) I_a, \\
&M(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = \phi_b\left(I_a + I_b\right) S_b + c_a^I I_a - \eta_b I_b - \mu_b^I I_b, \\
&N(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = \eta_a I_a + \theta_a^G S_a - \left(c_a^G + \gamma_a + \varepsilon_a\right) G_a, \\
&O(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = \eta_b I_b + c_a^G G_a + \theta_b^G S_b - \left(\gamma_b + \mu_b^G + \varepsilon_b + \beta_{bb}\right) G_b, \\
&P(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = \gamma_a G_a + \theta_a^R S_a - c_a^R R_a, \\
&Q(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) = \gamma_b G_b + \theta_b^R S_b + c_a^R R_a - \left(\mu_b^R + \beta_{bb}\right) R_b.
\end{aligned}$$

We can write system (42) as:

$$\begin{cases} {}^{FFM}D_{0,t}^{\nu,\nu_1} h(t) = \nu_1 t^{\nu_1 - 1} h(t, \mathscr{C}(t)), \\ \mathscr{C}(0) = \mathscr{C}_0. \end{cases} \tag{43}$$

The utilization of fractional integrals yields the subsequent outcomes:

$$\mathscr{C}(t) = \mathscr{C}(0) + \frac{\nu_1 t^{\nu_1 - 1}(1 - \nu)}{AB(\nu)} h(t, C(t)) + \frac{\nu \nu_1}{AB(\nu)\Gamma(\nu)} \int_0^t \sigma^{\nu_1 - 1} h(\sigma, C(\sigma))(1 - \sigma)^{\nu - 1} d\sigma, \tag{44}$$

where

$$\mathscr{C}(t) = \begin{cases} S_a(t) \\ S_b(t) \\ I_a(t) \\ I_b(t) \\ G_a(t) \\ G_b(t) \\ R_a(t) \\ R_b(t) \end{cases}, \mathscr{C}(0) = \begin{cases} S_a(0) \\ S_b(0) \\ I_a(0) \\ I_b(0) \\ G_a(0) \\ G_b(0) \\ R_a(0) \\ R_b(0) \end{cases}, h(t, \mathscr{C}(t)) = \begin{cases} J(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ K(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ L(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ M(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ N(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ O(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ P(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b), \\ Q(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b). \end{cases}$$

We define a Banach space $E = \mathscr{B} \times \mathscr{B} \times \mathscr{B} \times \mathscr{B} \times \mathscr{B} \times \mathscr{B} \times \mathscr{B} \times \mathscr{B}$ for existence theory, where $\mathscr{B} = [0, T]$ under the norm:

$$\|\mathscr{C}(t)\| = \max_{t \in [0,T]} \left| S_a(t) + S_b(t) + I_a(t) + I_b(t) + G_a(t) + G_b(t) + R_a(t) + R_b(t) \right|. \tag{45}$$

The operator $\Theta : \mathrm{E} \rightarrow \mathrm{E}$ can be written as follows:

$$\Theta(\mathscr{C})(t) = \mathscr{C}(0) + \frac{\nu_1 t^{\nu_1 - 1}(1-\nu)}{AB(\nu)} h(t, \mathscr{C}(t)) + \frac{\nu \nu_1}{AB(\nu)\Gamma(\nu)} \int_0^t \sigma^{\nu_1 - 1} h(\sigma, \mathscr{C}(\sigma))(1-\sigma)^{\nu - 1} d\sigma. \tag{46}$$

Assume that the non-linear function $h(t, \mathscr{C}(t))$ satisfy the growth and Lipschitz conditions. Some positive constants exist for $\mathscr{C} \in \mathrm{E}$ such that H_h and M_h:

$$|h(t, \mathscr{C}(t))| \leq \mathrm{H}_h\, |\mathscr{C}(t)| + \mathrm{M}_h, \tag{47}$$

and for $\mathscr{C}, \bar{\mathscr{C}} \in \mathrm{E}$ there exists a constant $\mathrm{N}_h > 0$ such that

$$\left| h(t, \mathscr{C}(t)) - h(t, \bar{\mathscr{C}}(t)) \right| \leq \mathrm{N}_h \left| \mathscr{C}(t) - \bar{\mathscr{C}}(t) \right|. \tag{48}$$

Theorem 3 ([37]). *Suppose that the condition (47) is satisfied. Let $h : [0,T] \times \mathrm{E} \rightarrow \mathfrak{R}$ be a continuous function. Then the offered model has at least one solution.*

Proof. The operator Θ defined by (46) should first be proven to be completely continuous. Given the continuity of the function h, it follows that the operator Θ must also exhibit continuity. Let $\mathrm{Z} = \{\mathscr{C} \in \mathrm{E} : \|\mathscr{C}\| \leq \mathfrak{R}, \mathfrak{R} > 0\}$. Now for any $\mathscr{C} \in \mathrm{E}$, we have

$$\begin{aligned}
|\Theta(\mathscr{C})| &= \max_{t\in[0,T]} \left| \mathscr{C}(0) + \tfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} h(t, \mathscr{C}(t)) + \tfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \textstyle\int_0^t \sigma^{\nu_1-1} h(\sigma, \mathscr{C}(\sigma))(t-\sigma)^{\nu-1} d\sigma \right| \\
&\leq \mathscr{C}(0) + \tfrac{\nu_1 T^{\nu_1-1}(1-\nu)}{AB(\nu)} (H_h \|\mathscr{C}\| + \mathrm{M}_h) + \max_{t\in[0,T]} \tfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \textstyle\int_0^t \sigma^{\nu_1-1} |h(\sigma, \mathscr{C}(\sigma))| (t-\sigma)^{\nu-1} d\sigma \\
&\leq \mathscr{C}(0) + \tfrac{\nu_1 T^{\nu_1-1}(1-\nu)}{AB(\nu)} (H_h \|\mathscr{C}\| + \mathrm{M}_h) + \tfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} (H_h \|\mathscr{C}\| + \mathrm{M}_h) T^{\nu+\nu_1-1} \mathrm{Z}(\nu, \nu_1) \\
&\leq \mathfrak{R}.
\end{aligned}$$

With $\mathrm{Z}(\nu, \nu_1)$ denoting for the beta function, the operator Θ is hence uniformly bounded. For equicontinuity of Θ, let us take $t_1 < t_2 \leq T$. Then consider

$$\begin{aligned}
\left|\Theta(\mathscr{C})(t_2) - \Theta(\mathscr{C})(t_1)\right| &= \left| \tfrac{\nu_1 t_2^{\nu_1-1}(1-\nu)}{AB(\nu)} h(t_2, \mathscr{C}(t_2)) + \tfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \textstyle\int_0^t \sigma^{\nu_1-1} h(\sigma, \mathscr{C}(\sigma))(t_2-\sigma)^{\nu-1} d\sigma \right. \\
&\quad \left. - \tfrac{\nu_1 t_1^{\nu_1-1}(1-\nu)}{AB(\nu)} h(t_2, \mathscr{C}(t_2)) + \tfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \textstyle\int_0^t \sigma^{\nu_1-1} h(\sigma, \mathscr{C}(\sigma))(t_2-\sigma)^{\nu-1} d\sigma \right| \\
&\leq \tfrac{\nu_1 t_2^{\nu_1-1}(1-\nu)}{AB(\nu)} (H_h \|\mathscr{C}\| + \mathrm{M}_h) + \tfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} (H_h \|\mathscr{C}\| + \mathrm{M}_h) t_2^{\nu+\nu_1-1} \mathrm{Z}(\nu, \nu_1) \\
&\quad - \tfrac{\nu_1 t_1^{\nu_1-1}(1-\nu)}{AB(\nu)} (H_h \|\mathscr{C}\| + \mathrm{M}_h) + \tfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} (H_h \|\mathscr{C}\| + \mathrm{M}_h) t_1^{\nu+\nu_1-1} \mathrm{Z}(\nu, \nu_1),
\end{aligned}$$

when $t_1 \rightarrow t_2$ then $\left|\Theta(\mathscr{C})(t_2) - \Theta(\mathscr{C})(t_1)\right| \rightarrow 0$. Thus, we can conclude that $\left|\Theta(\mathscr{C})(t_2) - \Theta(\mathscr{C})(t_1)\right| \rightarrow 0$ as $t_1 \rightarrow t_2$. It follows that N is equicontinuous. So, by Arzela–Ascoli theorem is completely continuous. Thus, the suggested model has at least one solution according to Schauder's fixed point result.

Theorem 4 ([37]). *Assume that (48) holds. Let $\zeta < 1$, where*

$$\zeta = \left(\frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} T^{\nu+\nu_1-1} \mathrm{Z}(\nu, \nu_1) \right) \mathrm{N}_h.$$

Then the considered model has a unique solution.

Proof. For $\mathscr{C}, \bar{\mathscr{C}} \in \mathrm{E}$, we have

$$\begin{aligned}
\left|\Theta(\mathscr{C}) - \Theta(\bar{\mathscr{C}})\right| &= \max_{t\in[0,T]} \left| \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} (h(t, \mathscr{C}(t)) - h(t, \bar{\mathscr{C}}(t))) \right. \\
&\quad \left. + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \int_0^t \sigma^{\nu_1-1} (h(\sigma, \mathscr{C}(\sigma)) - h(\sigma, \bar{\mathscr{C}}(t)))(1-\sigma)^{\nu-1} d\sigma \right| \\
&\leq \left[\frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} T^{\nu+\nu_1-1} \mathrm{Z}(\nu, \nu_1) \right] \left\| \mathscr{C} - \bar{\mathscr{C}} \right\| \\
&\leq \zeta \left\| \mathscr{C} - \bar{\mathscr{C}} \right\|.
\end{aligned}$$

As a result, Θ is a contraction. Therefore, according to the Banach contraction principle, the proposed model has a unique solution.

Numerical scheme

In this section, we go discuss a numerical method for solving model (8) that is based on a Newton polynomial. On the proposed model in this case, we apply the new integral and differential operators. The following results are obtained by using a fractal-fractional integral with a

Mittag-Leffler kernel:

$$
\begin{cases}
S_a(t) = S_a(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_1(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_1(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
S_b(t) = S_b(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_2(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_2(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
I_a(t) = I_a(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_3(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_3(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
I_b(t) = I_b(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_4(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_4(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
G_a(t) = G_a(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_5(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_5(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
G_b(t) = G_b(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_6(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_6(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
R_a(t) = R_a(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_7(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_7(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
R_b(t) = R_b(0) + \dfrac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_8(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) \\
\quad + \dfrac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \displaystyle\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_8(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma.
\end{cases}
\tag{49}
$$

The numerical method is now derived for the time instance $t = t_{p+1}$, where $p = 0, 1, 2, \ldots$.

$$
\begin{cases}
S_a^{p+1} = S_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_1(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_1(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
S_b^{p+1} = S_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_2(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_2(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
I_a^{p+1} = I_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_3(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_3(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
I_b^{p+1} = I_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_4(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_4(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
G_a^{p+1} = G_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_5(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_5(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
G_b^{p+1} = G_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_6(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_6(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
R_a^{p+1} = R_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_7(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_7(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\
R_b^{p+1} = R_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_8(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\
\int_0^t \sigma^{\nu_1-1}(t-\sigma)^{\nu-1} A_8(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma.
\end{cases}
\tag{50}
$$

By approximating the integrals on the right-hand side of the system (60), we derive the following expression:

$$\begin{cases} S_a^{p+1} = S_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_1(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{p+1}-\sigma)^{\nu-1} A_1(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\ S_b^{p+1} = S_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_2(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{p+1}-\sigma)^{\nu-1} A_2(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\ I_a^{p+1} = I_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_3(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{n+1}-\sigma)^{\nu-1} A_3(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\ I_b^{p+1} = I_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_4(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{p+1}-\sigma)^{\nu-1} A_4(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\ G_a^{p+1} = G_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_5(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{p+1}-\sigma)^{\nu-1} A_5(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\ G_b^{p+1} = G_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_6(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{p+1}-\sigma)^{\nu-1} A_6(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\ R_a^{p+1} = R_a^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_7(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{p+1}-\sigma)^{\nu-1} A_7(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma, \\ R_b^{p+1} = R_b^0 + \frac{\nu_1 t^{\nu_1-1}(1-\nu)}{AB(\nu)} A_8(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1}{AB(\nu)\Gamma(\nu)} \\ \sum_{u=0}^{p} \int_{t_u}^{t_{u+1}} \sigma^{\nu_1-1}(t_{p+1}-\sigma)^{\nu-1} A_8(t, S_a, S_b, I_a, I_b, G_a, G_b, R_a, R_b) d\sigma. \end{cases} \tag{51}$$

We employ Newton polynomial to approximate the kernel within the integrals. This approximation is applied within the finite interval $[t_u, t_{u+1}]$ and is expressed as follows:

$$\begin{cases} S_a(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_1(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_1(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}), \\ S_b(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_2(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_2(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}), \\ I_a(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_3(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_3(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}), \\ I_b(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_4(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_4(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}), \\ G_a(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_5(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_5(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}), \\ G_b(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_6(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_6(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}), \\ R_a(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_7(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_7(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}), \\ R_b(\sigma) = \frac{\sigma-t_{u-1}}{t_u-t_{u-1}} t_u^{\nu_1-1} A_8(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) - \frac{\sigma-t_u}{t_u-t_{u-1}} t_{u-1}^{\nu_1-1} \\ A_8(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}). \end{cases} \tag{52}$$

Now, using Newton polynomial, we get the following

$$
\begin{cases}
S_a^{p+1} = S_a^0 + \frac{\nu_1 t_p^{\nu_1-1}(1-\nu)}{AB(\nu)} A_1(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_u^{\nu_1-1} A_1(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_1(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix} \\
S_b^{p+1} = S_b^0 + \frac{\nu_1 t_p^{\nu_1-1}(1-\nu)}{AB(\nu)} A_2(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_u^{\nu_1-1} A_2(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_2(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix} \\
I_a^{p+1} = I_a^0 + \frac{\nu_1 t_p^{\nu_1-1}(1-\nu)}{AB(\nu)} A_3(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_u^{\nu_1-1} A_3(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_3(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix} \\
I_b^{p+1} = I_b^0 + \frac{\nu_1 t_n^{\nu_1-1}(1-\nu)}{AB(\nu)} A_4(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_s^{\nu_1-1} A_4(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_4(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix} \\
G_a^{p+1} = G_a^0 + \frac{\nu_1 t_p^{\nu_1-1}(1-\nu)}{AB(\nu)} A_5(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_u^{\nu_1-1} A_5(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_5(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix} \\
G_b^{p+1} = G_b^0 + \frac{\nu_1 t_p^{\nu_1-1}(1-\nu)}{AB(\nu)} A_6(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_u^{\nu_1-1} A_6(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_6(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix} \\
R_a^{p+1} = R_a^0 + \frac{\nu_1 t_p^{\nu_1-1}(1-\nu)}{AB(\nu)} A_7(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_u^{\nu_1-1} A_7(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_7(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix} \\
R_b^{p+1} = R_b^0 + \frac{\nu_1 t_p^{\nu_1-1}(1-\nu)}{AB(\nu)} A_8(t^p, S_a^p, S_b^p, I_a^p, I_b^p, G_a^p, G_b^p, R_a^p, R_b^p) + \frac{\nu\nu_1(\Delta t)^\nu}{AB(\nu)\Gamma(\nu+2)} \\
\sum_{u=0}^{p} \begin{bmatrix} t_u^{\nu_1-1} A_8(t^u, S_a^u, S_b^u, I_a^u, I_b^u, G_a^u, G_b^u, R_a^u, R_b^u) \times (q_1^\nu(q_1+1+\nu) - q_2^\nu(q_1+2\nu+1)) \\ -t_{u-1}^{\nu_1-1} A_8(t^{u-1}, S_a^{u-1}, S_b^{u-1}, I_a^{u-1}, I_b^{u-1}, G_a^{u-1}, G_b^{u-1}, R_a^{u-1}, R_b^{u-1}) \times (q_1^{\nu+1} - q_2^\nu(q_1+\nu)) \end{bmatrix}
\end{cases}
\tag{53}
$$

where

$p + 1 - u = q_1$ and $p - u = q_2$.

Simulation and discussion

Numerical simulations were conducted for the model using parameters obtained from Kuwait, France, and Cameroon. According to the COVID-19 outbreak at the beginning of the fourth wave in Kuwait in December 2021, in France at the beginning of the fifth wave in December 2021, and in Cameroon at the beginning of the second wave in January 2021 we have selected the following set of parameter and initial values: (see Table 1).

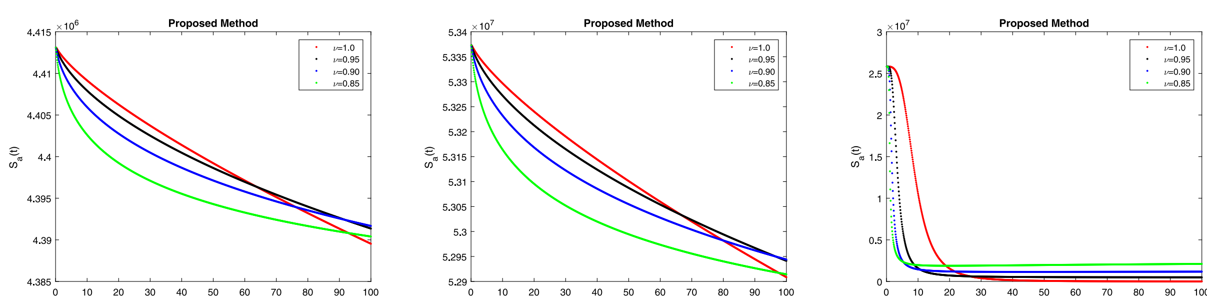
The results for the three countries under consideration show some commonalities as well as some discrepancies, which we shall go through below.

Figs. 1a–8c depicted the numerical simulation for a fractal fractional model for Kuwait, France, and Cameroon. Figs. 1a–2c show that the simulation of S_a and S_b for Kuwait, France, and Cameroon. Figs. 3a–4c for Kuwait demonstrates that infectious (I) spreads more quickly in the younger class (≤65 years) than in the older class (>65 years), which is also true for France and Cameroon. On the other hand, this phenomenon is inverted in the three countries about the expansion of populations who have received both temporary (G) (Figs. 5a–6c and permanent (R) (Figs. 7a–8c) vaccinations. The phenomenon is more pronounced in Kuwait than in France, and in France itself there is a more pronounced difference than in Cameroon. This is partially explained by the fact that Kuwait has a higher vaccination rate in the young population compared to France and Cameroon, even though the effectiveness of vaccinations in all three nations was previously assumed to be the same (although it declines in the older population). Also, we plot the findings for a variety of fractional orders to examine the impact of the order of derivation on the outcomes, and we observe that while the outcomes exhibit the same behavior, the numerical value attained varies for each order. Figs. 1a–2c shows increasing behavior of the susceptible population for decreasing values of fractional order ν. Figs. 3a–4c illustrate the impact of the fractional order parameter ν on the symptomatic population of the model. The plot demonstrates a notable reduction in the count of infected individuals as ν decreases. In contrast, the remaining state variables within the model exhibit a distinct pattern. Figs. 5a–6c and 7a–8c reveal a declining trend in the count of new susceptible and recovered individuals, respectively. The plots highlight a considerable decrease in the number of newly susceptible and recovered individuals as ν decreases. The influence of the derivation order on the numerical results produced amply demonstrates the significance of choosing the proper fractional order for the virus propagation model.

The numerical simulation results presented here agree with those obtained in previous work related to COVID-19. For instance, in [38] the authors study an SEIR age-structured multi-group epidemic model to understand the impact of social contact patterns in controlling the disease.

Table 1
Parameters values from Kuwait, France and Cameroon.

Parameter	Values from Kuwait	Values from France	Values from Cameroon	Source
ϕ_a	1.7	1.2	1.3	[11]
ϕ_b	0.9	0.9	0.9	[11]
μ_b^S	0.003	0.009	0.009	[11]
ϵ_a	0.28	0.003	0.28	[11]
β_{bb}	0	0	0	[11]
μ_b^I	0.0025	0.0025	0.0025	[11]
μ_b^G	0.002	0.002	0.002	[11]
μ_b^R	0.0021	0.0021	0.0021	[11]
β_{aa}	2.1	1.9	4.5	[11]
β_{ba}	2.3	2.2	4.1	[11]
θ_a^G	0.765	0.735	0.024	[11]
θ_a^R	0.678	0.678	0.678	[11]
γ_a	0.62	0.62	0.62	[11]
γ_b	0.74	0.74	0.74	[11]
θ_b^G	0.45	0.75	0.45	[11]
θ_b^R	0.33	0.33	0.33	[11]
c_a^S	0.7	0.7	0.7	[11]
c_a^I	0.54	0.54	0.54	[11]
c_a^G	0.65	0.65	0.65	[11]
c_a^R	0.8	0.8	0.8	[11]
ϵ_b	0.19	0.19	0.19	[11]
η_a	0.3	0.3	0.3	[11]
η_b	0.38	0.38	0.38	[11]
S_a	4 413 099	53 372 880	25 828 973	[11]
S_b	51 422	14 017 120	721 027	[11]
I_a	65	3912	151	[11]
I_b	194	1757	21	[11]
G_a	20	200	75	[11]
G_b	17	170	33	[11]
R_a	30	300	56	[11]
R_b	73	730	64	[11]



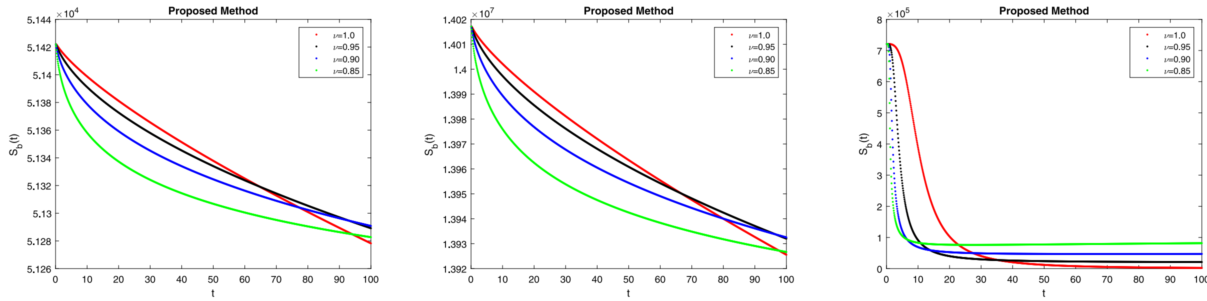
(a) Kuwait (b) France (c) Cameroon

Fig. 1. Simulation of S_a for Kuwait, France and Cameroon with different fractional order ν.

In [39], the authors considered an individual-level model for SARS-CoV2 transmission which accounts for location-dependent distributions of age and household structure. Incorporating age as a factor in the modeling of the COVID-19 pandemic enables the simulation of distinct dynamic behaviors in the growth of infectious and immune populations across different age groups, spanning both younger and older individuals. This consideration holds the potential for adjusting vaccination strategies based on age-related factors [11]. Our study substantiates this notion through both theoretical findings and numerical simulations.

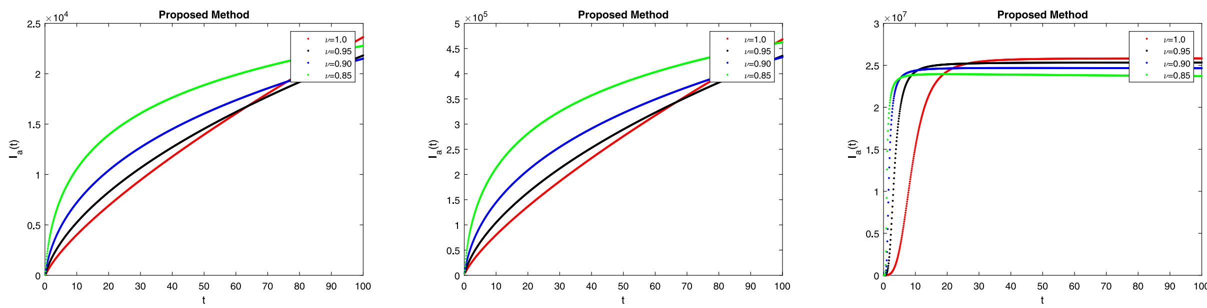
Conclusion

In this paper, we presented an age-dependent epidemic model for the COVID-19 Outbreak in Kuwait, France, and Cameroon in the fractal–fractional (FF) sense of derivative with the Mittag-Leffler kernel. By including age in the COVID-19 pandemic modeling, it can imitate the varied dynamic behavior of the rise of infectious and immunological populations, young and old, in order, for example, to modify the vaccination policy according to age. The model equilibria, their steady state, and their fundamental characteristics are examined. Utilizing the Lyapunov function, the analysis of global stability is performed. Using fixed point theory, it is thoroughly demonstrated that solutions to the fractal fractional model



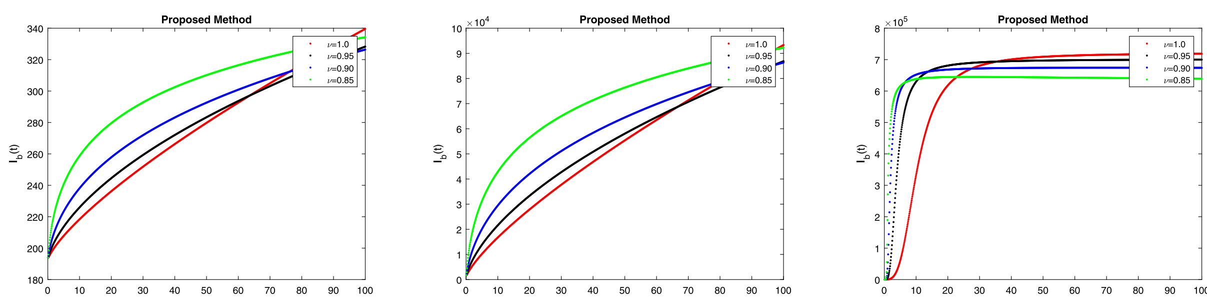
(a) Kuwait (b) France (c) Cameroon

Fig. 2. Simulation of S_b for Kuwait, France and Cameroon with different fractional order ν.



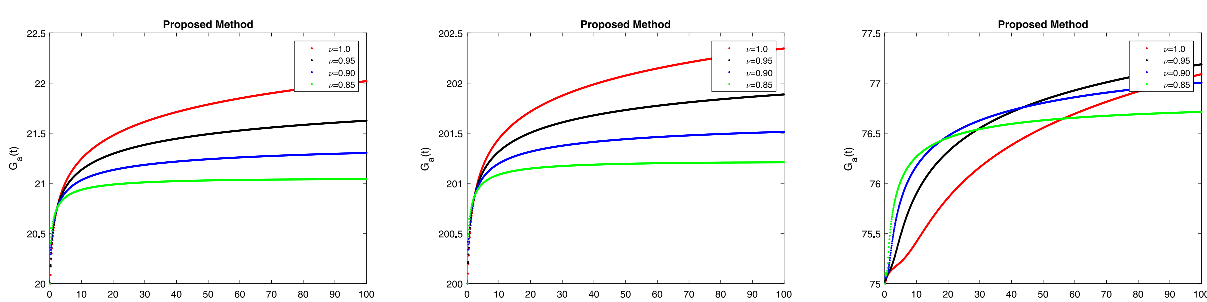
(a) Kuwait (b) France (c) Cameroon

Fig. 3. Simulation of I_a for Kuwait, France and Cameroon with different fractional order ν.



(a) Kuwait (b) France (c) Cameroon

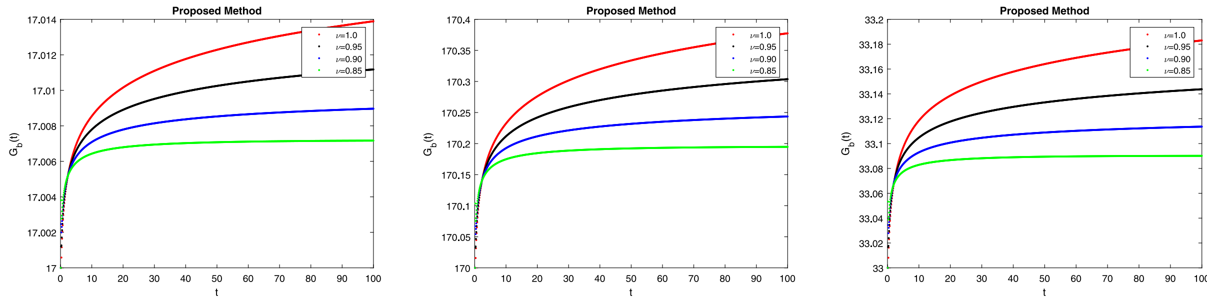
Fig. 4. Simulation of I_b for Kuwait, France and Cameroon with different fractional order ν.



(a) Kuwait (b) France (c) Cameroon

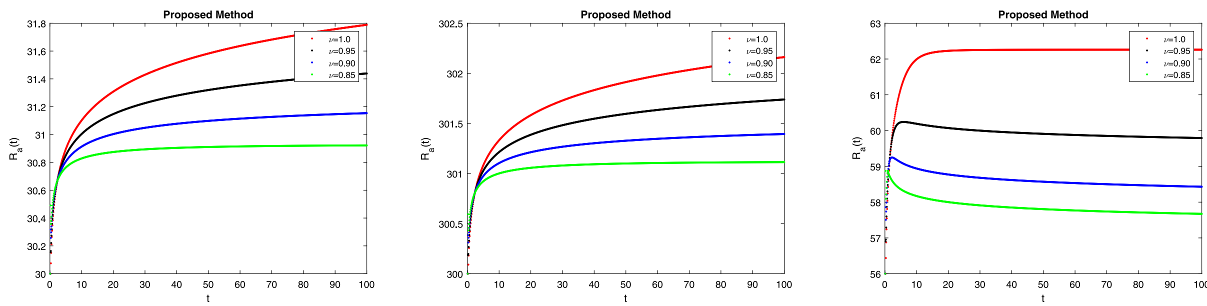
Fig. 5. Simulation of G_a for Kuwait, France and Cameroon with different fractional order ν.

exist and are distinct. The model's numerical solution is produced using Newton's polynomial scheme. The deterministic mathematical model for the impact of COVID-19 on different ages is further analyzed with various fractional parameters. The graphical outcomes demonstrate the new fractional model's continuous dependence on the fractional parameter and its ability to offer better and more adaptable data for studying the



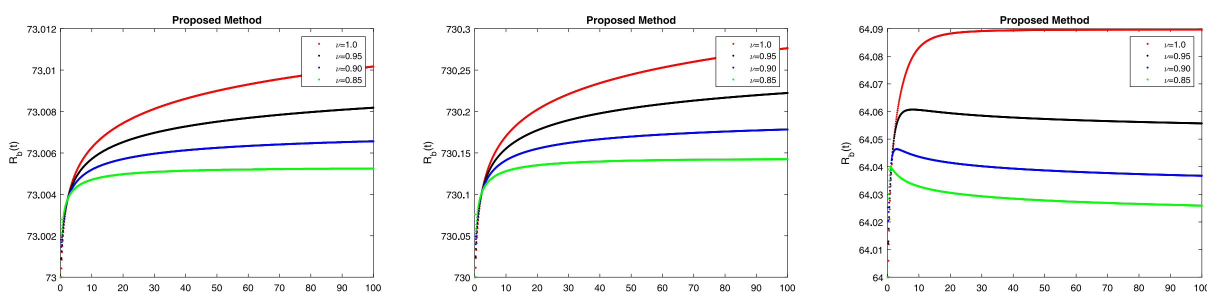
(a) Kuwait (b) France (c) Cameroon

Fig. 6. Simulation of G_b for Kuwait, France and Cameroon with different fractional order ν.



(a) Kuwait (b) France (c) Cameroon

Fig. 7. Simulation of R_a for Kuwait, France and Cameroon with different fractional order ν.



(a) Kuwait (b) France (c) Cameroon

Fig. 8. Simulation of R_b for Kuwait, France and Cameroon with different fractional order ν.

dynamics of Covid-19. The findings are very valuable for formulating plans, and decisions, and creating control methods to mitigate the impact of COVID-19 on society.

CRediT authorship contribution statement

Saba Jamil: Analysis, Preparing figures. **Muhammad Farman:** Analysis, Preparing figures. **Ali Akgül:** Supervision, Writing – review & editing, Investigation. **Muhammad Umer Saleem:** Analysis, Preparing figures. **Evren Hincal:** Analysis, Preparing figures. **Sayed M. El Din:** Analysis, Preparing figures.

Declaration of competing interest

There is no conflict of interest between the authors.

Data availability

No data was used for the research described in the article.

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