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REVIEW

A fully Fermatean fuzzy multi-objective transportation model using an extended DEA technique

Muhammad Akram[1] · Sundas Shahzadi[2] · Syed Muhammad Umer Shah[1] · Tofigh Allahviranloo[3]

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Abstract

A mathematical technique called data envelope analysis is used to determine the relative efficiency of decision-making units (DMU) with numerous inputs and outputs. Compared to other DMUs, it determines how efficient the DMU is at delivering a specific level of output based on the amount of input it uses. The transportation problem is a linear programming problem for reducing the net transportation cost or maximizing the net transportation profit of moving goods from a number of sources to a number of destinations. In this manuscript, a multi-objective transportation problem is examined in which all the parameters, such as transportation cost, supply, and demand are uncertain with hesitancy and triangular Fermatean fuzzy numbers are used to represent these uncertain parameters. Using Fermatean fuzzy data envelope analysis, a new technique for determining the common set of weights is presented. The fully Fermatean fuzzy multi-objective transportation problem is then solved using a novel data envelopment analysis-based approach. To this end, two different Fermatean fuzzy efficiency scores are derived, first by considering the sources as targets and changing the destinations, and second by considering the destinations as targets and changing the sources. Next, a unique Fermatean fuzzy relative efficiency is determined for each arc by combining these two different Fermatean fuzzy efficiency scores. As a result, a single-objective Fermatean fuzzy transportation problem is constructed, which can be solved using existing techniques. A numerical illustration is provided to support the suggested methodology, and the performance of the proposed method is compared with an existing technique

Keywords Multi-objective transportation problem · Fermatean fuzzy arithmetic · Data envelopment analysis · Common set of weights

1 Introduction

Real-world problems are often tremendously complex. Complexity is partially due to a certain degree of uncertainty in the form of ambiguities, randomness, or limited understanding owing, for example, to a lack of sufficient data or poor information quality. In the majority of problems, linguistic expressions are used to define parameters that are part of the problem statement. Therefore, treating the decision-makers' knowledge as fuzzy data will provide more accurate statements and more favorable solutions. Fuzzy modeling is a mathematical method of representing uncertainty in human systems where the fuzzy set (FS) notion introduced by Zadeh (1965) plays a prominent role. Applications related to FS theory can be seen in Chen and Chen (2001), Chen and Wang (2010), Chen and Niou (2011), Chen and Phuong (2017), Akram and Bashir (2021)

✉ Muhammad Akram
m.akram@pucit.edu.pk

Sundas Shahzadi
sundas.shahzadi@ue.edu.pk

Syed Muhammad Umer Shah
m.umer25334@gmail.com

Tofigh Allahviranloo
tofigh.allahviranloo@istinye.edu.tr

[1] Department of Mathematics, University of the Punjab, New Campus, Lahore 4590, Pakistan

[2] Division of Science and Technology, Department of Mathematics, University of Education, Lahore, Pakistan

[3] Faculty of Engineering and Natural Sciences, Istinye University, Istanbul, Turkey

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and Feng et al. 2023). Although consideration of partial membership meant a striking departure from classical reasoning, there is no question that the satisfaction and dissatisfaction of human judgment cannot be assessed separately using FS theory. This drawback motivated Atanassov (1986) to propose the intuitionistic fuzzy set (IFS) theory. Since then, IFSs have been applied in a variety of optimization-related domains by numerous researchers (Malik and Gupta 2020; Mondal et al. 2021). However, IFS are not designed to handle situations where the sum of membership and non-membership grades exceeds 1 for some alternatives. To overcome this limitation, Pythagorean fuzzy sets (PFSs) were introduced by Yager (2013), who relaxed the said restriction so that it is only required that the sum of the squares of membership and non-membership should not be greater than 1 at any evaluation of an alternative. The state of the art and potential future developments for PFSs were covered by Peng and Selvachandran (2019). Several articles have recently shown the ability of PFSs to operate within well-known multi-criteria decision-making strategies. For example, a hybrid technique integrating AHP and TODIM has recently been developed by Zhou and Chen (2022) in order to choose blockchain technology providers in a Pythagorean fuzzy environment. In addition, Khan et al. (2022) have integrated VIKOR into this setting too. In continuation of this effort, Senapati and Yager (2019a, b, 2020) proposed the Fermatean fuzzy set (FFS) theory in response to the constraints of PFSs. In a FFS, it is only imposed that the cubic sum of membership and non-membership degrees be less than or equal to 1. In addition, FFS-related applications are displayed in Akram et al. (2020, 2022a, b, c, 2023a, b, c), Deveci et al. (2023), Mishra et al. (2022), Shit and Ghorai (2021), Ali and Ansari (2022), Simic et al. (2022), and Zeng et al. (2023). In addition, increasingly sophisticated models extended these ideas further. For example, Peng and Luo (2021) studied a decision-making model for China's stock market bubble warning using picture fuzzy information.

Once a decision problem has been stated for which a solution is required, we need to resort to an adequate decision-making technique that depends on the characteristics of the problem. One example from the crisp literature is the family of linear programming problems (LPPs), which are concerned with determining the optimal (either maximum or minimum) value of a given linear objective function. Bellman and Zadeh (1970) introduced the concept of decision making in a fuzzy environment, which was adapted to LPPs by Zimmerman (1978). Much later, Allahviranloo et al. (2008) used a ranking function to solve fully fuzzy LPPs. Akram et al. (2021a, b) presented various techniques to solve the LPPs in a Pythagorean fuzzy environment, and Mehmood et al. (2021a, b) studied the LPPs in a bipolar fuzzy environment. Ahmed et al. (2021) proposed a new method for evaluating bipolar single-valued neutrosophic LPPs.

An important subclass of LPPs that appears in various situations is transportation problems (TP). Hitchcock (1941) initiated this branch of operational research in 1941 to optimally distribute goods from a number of sources to a number of destinations. Thus, the main goal of classic TP is to find the minimum total transportation cost for this problem. However, in many practical applications, several features are evaluated simultaneously in one TP. These TPs are known as multi-objective TPs (MOTPs), and there are a number of methods for solving them. Ringuest and Rinks (1987) proposed two interactive algorithms for finding a satisfying solution to the MOTP while maintaining its specific structure. In 1973, Lee and Moore (1973) used the goal programming technique to calculate a suitable solution to an MOTP. This technique allows for the optimization of numerous competing objectives along with an explicit assessment of the present decision environment. Abd El-Wahed and Sang (2006) considered an interactive fuzzy goal programming technique for determining the best compromise solution for the MOTP.

Another factor to take into account when using the MOTP in practical applications is that the decision maker may not always have complete knowledge about the parameters. Fuzzy multi-objective TP (FMOTP) refers to the MOTP in which at least one of the parameters is represented by fuzzy numbers or more general cases. We can cite several solutions to FMOTPs with varying degrees of generality. Roy et al. (2018) modeled the MOTP in which transportation cost, supply, and demand variables are represented as intuitionistic fuzzy numbers. To solve this MOTP, they transformed it into a multi-objective interval optimization problem by taking a particular cut interval. To determine the most suitable solutions to their given problem, they then utilize two methodologies, namely, goal programming and intuitionistic fuzzy programming. Bera and Mondal (2022) examined a MOTP in a Gaussian fuzzy environment. Their main goal was to maximize distributor profit while minimizing retailer costs under a credit-period policy. For this purpose, they developed four models. Eslamipoor (2022) proposed a multi-objective method for selecting a product supplier in which the total cost, including supplier and buyer costs as well as environmental pollutants produced by vehicle transportation, was considered. Dutta and Borah (2023) developed multi-criteria group decision making using a trapezoidal intuitionistic fuzzy number in generalized form with the help of a new similarity measure. They applied it to several COVID-19 cases.

In 1978, Charnes et al. (1978) proposed the idea of DEA, also known as frontier analysis. It is a linear programming tool used to determine the relative efficiency of many comparable entities, or decision-making units (DMUs). Therefore, DEA is a practical approach for performance-measuring tasks whose foundations are different from traditional econometric techniques. The quantity of inputs a DMU consumes determines how well it can provide a certain level of output in comparison to other DMUs, according to DEA's estimation. The fundamental benefit of DEA is that it does not require any prior weights of inputs and outputs, allowing a single DMU to evaluate its efficiency using only the most favorable input and output weights.

In recent decades, DEA has been widely employed in a variety of problems, including supply chain management, economics, petroleum distribution system design, business research and development, military logistics, and government programs (Olson and Wu 2017). One main factor of this success is that researchers have been able to use DEA in situations where existing techniques failed to apply due to the existence of complicated (and sometimes unknown) relationships among the multiple inputs and multiple outputs (Cooper et al. 2006).

Several related techniques are based on DEA. Ahn et al. (1988) investigated how DEA might be used to study various production behavior features of higher education institutions as an alternative to more conventional methodologies like econometric-regression models. In doing so, Ahn et al. (1988) evaluated the "particularly" relative efficiencies of public and private doctoral-granting universities in the U.S., as well as the scale and technological efficiencies of such universities. Sengupta (1992) introduced a fuzzy systems approach to DEA, proposing three methods based on fuzzy set statistics. More generally, Fermatean fuzzy DEA corresponds to the DEA method for evaluating DMU efficiency using Fermatean fuzzy input and output data.

In particular, and owing to the relationship between DEA and MOTPs, DEA-based approaches are now preferred for solving practical TPs. The contribution of Zerafat et al. (2003), who developed a DEA model to assess the efficiency of each assignment with non-homogeneous costs using a common set of weights (CSW), might be noted among these methods. For assessing the efficacy of DMUs using fuzzy input and output data, several DEA approaches are used, which are known as fuzzy DEA approaches (Lotfi et al. 2020). Lotfi et al. (2009) developed a multi-activity network DEA with fuzzy inputs and outputs, then evaluated its effectiveness and efficiency. Chen and Lu (2007) proposed an approach to dealing with assignment problems with many incommensurate inputs as well as outputs for every possible assignment using the DEA. Instead of cost or profit, they applied the notion of relative efficiency in utilizing multiple resources for every possible assignment of the problem. Following that, Amirteimoori developed techniques for addressing the TP and the shortest path problems withs many characteristics related to every arc in Amirteimoori (2011) and Amirteimoori (2012), respectively. Hatami-Marbini et al. (2017) proposed a lexicographic multi-objective linear programming (MOLP) approach in which a fully fuzzy DEA model is transformed into a MOLP problem. Intuitionistic fuzzy preference relations and DEA cross-efficiency were used by Liu et al. (2019) to investigate a novel method of group decision making that may reduce information asymmetry and produce better decision-making outcomes. Azar et al. (2016) introduced a new fuzzy additive model to determine the CSW in DEA, and it is the focus of this study. More work on TPs and DEA is listed in Table 1.

In the context of fully Fermatean fuzzy multi-objective transportation problems (FFMOTPs), the main motivation in this work is to transform the Fermatean fuzzy multiple characteristics corresponding to every arc into a single Fermatean fuzzy characteristic. This concept results from treating every arc like a DMU and assessing its performance. Consequently, the DEA efficiency assessment methodologies must be applied to put the concept into practice.

The interests that motivate the proposed formulation are as follows:

1. The primary motivation behind this work is the investigation of FFMOTPs, where uncertainty occurs in all variables and parameters.
2. Fuzzy DEA approaches have been used to manage MOTPs whose variables are all fuzzy. However, these models are insufficient to handle situations where uncertainty occurs with hesitancy.
3. Here, the DEA approach to solving MOTPs based on CSW is enhanced with its extension to the Fermatean fuzzy environment.
4. The MOTP has been solved in an intuitionistic fuzzy environment by Jana and Roy (2007). These authors used membership and non-membership functions that are hyperbolic and parabolic, respectively. Being nonlinear functions, these types of evaluations are very complicated to handle. We aim to provide a more tractable approach with lower computational complexity.

Based on these motivations, in this article, we will address a fully FFMOTP where all variables and parameters, especially all arc characteristics, are non-crisp. A novel Fermatean fuzzy DEA-based technique is presented that achieves this goal. It is a generalized version of Chen and Lu's (2007) and Amirteimoori's (2011) works that operates in the Fermatean fuzzy environment. Every arc in a fully FFMOTP is regarded as a DMU with many Fermatean

Table 1 Related work in various environments

Reference	Year	Main takeaways
Charnes et al.	(1985)	Put forward the foundations of DEA for Pareto-Koopmans
Kahraman and Tolga	(1998)	Proposed fuzzy mathematical programming by assuming that the values of inputs and outputs in DEA are not known with certainty
Kao and Liu	(2000)	Proposed a concept to transform a fuzzy DEA model into a family of conventional crisp DEA models using the α-cut method
Wang et al.	(2009)	Developed a fuzzy DEA-based application to the performance assessment of manufacturing
Lotfi et al.	(2010)	Considered relationship between MOLP and DEA on CCR dual model
Azadi et al.	(2015)	Designed an integrated DEA-improved Russell measure model in a fuzzy environment to choose the most sustainable suppliers
Ebrahimnejad	(2016a, b)	Solved fuzzy TP using LR-flat fuzzy numbers and triangular fuzzy numbers (TFNs)
Ebrahimnejad	(2017)	Presented a lexicographic ordering-based approach for solving fuzzy TP with TFNs
Tavana et al.	(2018)	Developed a hybrid DEA-MOLP model
Arya and Yadav	(2019)	Introduced DEA in intuitionistic fuzzy environment
Mahmoodirad et al.	(2019)	Proposed a novel effective method to solve the fully fuzzy TP
Song	(2021)	Determined performance of some companies using DEA and machine computing
Bagheri et al.	(2021)	Solved fuzzy multi-objective shortest path problem based on DEA method
Revuelta	(2021)	Initiated hybrid DEA-artificial neural network prediction model for COVID-19
Ebrahimnejad and Amani	(2021)	Discussed fuzzy DEA in the presence of undesirable outputs
Izadikhah et al.	(2022)	Assessed the stability of suppliers utilizing a multi-objective fuzzy voting DEA model
Das	(2022)	Presented a novel approach for optimizing the transportation cost using triangular fuzzy programming problem
Bagheri et al.	(2022)	Solved fully fuzzy MOTP using CSW in DEA
Akram et al.	(2023a, b, c)	Extended DEA method for solving MOTP with FFSs
Hosseinzadeh	(2023)	Optimized portfolio with asset preselection based on DEA

fuzzy inputs and outputs. The arc attributes corresponding to Fermatean fuzzy objectives that should be minimized are defined as Fermatean fuzzy input, whereas the arc attributes corresponding to Fermatean fuzzy objectives that should be maximized are defined as Fermatean fuzzy output. Then, two Fermatean fuzzy efficiency scores (FFESs) are calculated for each arc, according to Amirteimoori (2011). These FFESs are computed using the Fermatean fuzzy DEA approach, which is then integrated to provide a unique Fermatean fuzzy relative efficiency.

Here is a summary of this work's main contributions:

1. A multi-objective transportation problem is constructed in the Fermatean fuzzy environment in which all of the variables such as transportation cost, supply, and demands are represented as triangular Fermatean fuzzy numbers (TFFNs).
2. The CSW are determined using Fermatean fuzzy DEA. Maximization and minimization functions are simultaneously used in Fermatean fuzzy DEA for the modeling the fully FFMOTP.
3. A novel approach to solve fully FFMOTP using CSW in Fermatean fuzzy DEA is proposed. The idea of using the CSW in Fermatean fuzzy DEA is to find the two different Fermatean fuzzy efficiency scores for every arc, first by considering the sources as targets and changing the destinations, and second by considering the destinations as targets and changing the sources. Next, a unique Fermatean fuzzy relative efficiency scores will be determined for each arc by combining these two different Fermatean fuzzy efficiency scores.
4. Using these unique Fermatean fuzzy relative efficiency scores, a single-objective Fermatean fuzzy transportation problem (FFTP) will be constructed.
5. We shall give arguments proving that the Fermatean fuzzy DEA provides an effective plan for the fully FFMOTP without explicitly knowing the Fermatean fuzzy utility function of the decision maker.

6. The proposed methodology maintains the structure of a classical TP. This characteristic facilitates its solution and application as compared to other existing methodologies.
7. An example illustrates the validity and superiority of the proposed methodology. The FFESs computed by the proposed method are compared with those determined using more conventional techniques.

The rest of the paper is structured as follows: in Sect. 2, fundamental terms are defined that are relevant to the current study and to the formulation of the MOTP in a Fermatean fuzzy environment. The CSW in the Fermatean fuzzy DEA is defined in Sect. 3 with the help of TFFNs. Section 4 provides an explanation of the process for transforming the fully FFMOTP into a single-objective FFTP. A numerical example and performance analysis of the proposed methodology, plus a comparison with an existing approach, are described in Sect. 5. Conclusions and future directions for work are included in the final Sect. 6.

2 Preliminaries

Definition 1 (Ringuest and Rinks 1987) A feasible vector $\bar{y} = \{\bar{y}_{gk}\}$ gives a *non-dominated solution or Pareto optimal solution* to the MOTP if, and only if, there exist no feasible vector $y = \{y_{gk}\}$ such that

$$\sum_{g=1}^{p}\sum_{k=1}^{q} c_{gk}^{n} y_{gk} \leq \sum_{g=1}^{p}\sum_{k=1}^{q} c_{gk}^{n} \bar{y}_{gk}, \forall n$$

$$\text{and} \quad \sum_{g=1}^{p}\sum_{k=1}^{q} c_{gk}^{n} y_{gk} \neq \sum_{g=1}^{p}\sum_{k=1}^{q} c_{gk}^{n} \bar{y}_{gk}, \quad \text{for some} \quad n.$$

When this condition holds, $\bar{y}$ is regarded to be efficient solution.

Definition 2 (Senapati and Yager 2020) Suppose $\mathcal{Y}$ is a universe of discourse. A FFS $\tilde{\mathcal{A}}^F$ on $\mathcal{X}$ is defined to be a set of form

$$\tilde{\mathcal{A}}^F = \{\langle x, \mu_{\tilde{\mathcal{A}}^F}(x), v_{\tilde{\mathcal{A}}^F}(x)\rangle : x \in \mathcal{Y}\},$$

where $\mu_{\tilde{\mathcal{A}}^F} : \mathcal{Y} \to [0,1]$, $v_{\tilde{\mathcal{A}}^F} : \mathcal{Y} \to [0,1]$, and

$$0 \leq (\mu_{\tilde{\mathcal{A}}^F}(x))^3 + (v_{\tilde{\mathcal{A}}^F}(x))^3 \leq 1,$$

for all $x \in \mathcal{Y}$. The values $\mu_{\tilde{\mathcal{A}}^F}(x)$ and $v_{\tilde{\mathcal{A}}^F}(x)$ denote the membership and non-membership degree of the element x in the set $\tilde{\mathcal{A}}^F$, respectively. Further, for all $x \in \mathcal{Y}$,

$$\pi_{\tilde{\mathcal{A}}^F}(x) = \sqrt[3]{1 - (\mu_{\tilde{\mathcal{A}}^F}(x))^3 - (v_{\tilde{\mathcal{A}}^F}(x))^3}$$

denotes the degree of hesitation for the element x in $\tilde{\mathcal{A}}^F$.

Definition 3 (Senapati and Yager 2019a, b) A Fermatean fuzzy number (FFN) $\tilde{\mathcal{A}}^F$ is a FFS defined on $\mathcal{R}$ with following conditions:

1. normal, i.e., $\exists\, x \in \mathcal{R}$ such that

$$\mu_{\tilde{\mathcal{A}}^F}(x) = 1, \quad (\text{so} \quad v_{\tilde{\mathcal{A}}^F}(x) = 0),$$

2. convex for the membership function $(\mu_{\tilde{\mathcal{A}}^F})$, i.e.,

$$\mu_{\tilde{\mathcal{A}}^F}(\delta x + (1-\delta)y) \geq \min\{\mu_{\tilde{\mathcal{A}}^F}(x), \mu_{\tilde{\mathcal{A}}^F}(y)\}; \quad \forall x, y \in \mathcal{R}, \delta \in [0,1],$$

3. concave for the non-membership function $(v_{\tilde{\mathcal{A}}^F})$, i.e.,

$$v_{\tilde{\mathcal{A}}^F}(\delta x + (1-\delta)y) \leq \max\{v_{\tilde{\mathcal{A}}^F}(x), v_{\tilde{\mathcal{A}}^F}(y)\}; \quad \forall x, y \in \mathcal{R}, \delta \in [0,1].$$

Definition 4 (Akram et al. 2022a, b, c) A TFFN $\tilde{\mathcal{A}}^F = \{(a^l, a^m, a^r); p, q\}$ is a FFS with the membership function $(\mu_{\tilde{\mathcal{A}}^F})$ and non-membership function $(v_{\tilde{\mathcal{A}}^F})$ given as follows:

$$\mu_{\tilde{\mathcal{A}}^F}(y) = \begin{cases} \dfrac{(y-a^l)p}{a^m - a^l}, & a^l \leq y < a^m, \\ p, & y = a^m, \\ \dfrac{(a^r - y)p}{a^r - a^m}, & a^m < y \leq a^r, \\ 0, & y < a^l \text{ or } y > a^r, \end{cases}$$

$$v_{\tilde{\mathcal{A}}^F}(y) = \begin{cases} \dfrac{[a^m - y + q(y - a^l)]}{a^m - a^l}, & a^l \leq y < a^m, \\ q, & y = a^m, \\ \dfrac{[y - a^m + q(a^r - y)]}{a^r - a^m}, & a^m < y \leq a^r, \\ 1, & y < a^l \text{or } y > a^r. \end{cases}$$

The values p and q represent the maximum value of membership function $(\mu_{\tilde{\mathcal{A}}^F})$ and minimum value of non-membership function $(v_{\tilde{\mathcal{A}}^F})$, respectively, such that $p \in [0,1]$, $q \in [0,1,]$ and

$$0 \leq p^3 + q^3 \leq 1.$$

By taking $p = 1$ and $q = 0$ in Definition 4, the TFFN $\tilde{\mathcal{A}}^F$ assumes the form $\tilde{\mathcal{A}}^F = \{(a^l, a^m, a^r); (a^{l'}, a^m, a^{r'})\}$ whose membership $(\mu_{\tilde{\mathcal{A}}^F})$ and non-membership $(v_{\tilde{\mathcal{A}}^F})$ functions can be represented by

$$\mu_{\tilde{A}^F}(y) = \begin{cases} \dfrac{y-a^l}{a^m-a^l}, & a^l \le y < a^m, \\ 1, & y = a^m, \\ \dfrac{a^r-y}{a^r-a^m}, & a^m < y \le a^r, \\ 0, & y < a^l \text{ or } y > a^r, \end{cases}$$

$$\nu_{\tilde{A}^F}(y) = \begin{cases} \dfrac{a^m-y}{a^m-a^{l'}}, & a^{l'} \le y < a^m, \\ 0, & y = a^m, \\ \dfrac{y-a^m}{a^{r'}-a^m}, & a^m < y \le a^{r'}, \\ 1, & y < a^{l'} \text{ or } y > a^{r'}, \end{cases}$$

where $a^{l'} \le a^l \le a^m \le a^r \le a^{r'}$. The graphical representation of membership and non-membership functions of TFFN is given in Fig. 1.

If u > 0, then a TFFN A is considered non-negative.

Definition 5 (Akram et al. 2022a, b, c) If $a^{l'} \ge 0$, then a TFFN $\tilde{\mathcal{A}}^F = \{(a^l, a^m, a^r); (a^{l'}, a^m, a^{r'})\}$ is known as the non-negative (*i.e.*, $\tilde{\mathcal{A}}^F \succeq 0$).

Definition 6 (Akram et al. 2022a, b, c) If $a^{r'} \le 0$, then a TFFN $\tilde{\mathcal{A}}^F = \{(a^l, a^m, a^r); (a^{l'}, a^m, a^{r'})\}$ is known as non-positive (*i.e.*, $\tilde{\mathcal{A}}^F \preceq 0$).

Definition 7 (Akram et al. 2022a, b, c) If $a^l = 0, a^m = 0, a^r = 0, a^{l'} = 0, a^{r'} = 0$, then a TFFN $\tilde{\mathcal{A}}^F \approx \tilde{0}$.

Definition 8 (Akram et al. 2022a, b, c) The TFFNs $\tilde{\mathcal{A}}^F = \{(a^l, a^m, a^r); (a^{l'}, a^m, a^{r'})\}$, and $\tilde{\mathcal{B}}^F = \{(b^l, b^m, b^r); (b^{l'}, b^m, b^{r'})\}$ are known to be equal if, and only if, $a^l = b^l$, $a^b = \nu^m$, $a^r = b^r$, $a^{l'} = b^{l'}$, $a^{r'} = b^{r'}$.

Definition 9 (Akram et al. 2022a, b, c) Let $\tilde{\mathcal{A}}^F = \{(a^l, a^m, a^r); (a^{l'}, a^m, a^{r'})\}$, and $\tilde{\mathcal{B}}^F = \{(b^l, b^m, b^r); (b^{l'}, b^m, b^{r'})\}$ be positive TFFNs, then

1. $\tilde{\mathcal{A}}^F \oplus \tilde{\mathcal{B}}^F = \{(a^l + b^l, a^m + b^m, a^r + b^r); (a^{l'} + b^{l'}, a^m + b^m, a^{r'} + b^{r'})\}$,
2. $\tilde{\mathcal{A}}^F \ominus \tilde{\mathcal{B}}^F = \{(a^l - b^r, a^m - b^m, a^r - b^l); (a^{l'} - b^{r'}, a^m - b^m, a^{r'} - b^{l'})\}$,
3. $\tilde{\mathcal{A}}^F \otimes \tilde{\mathcal{B}}^F = \{(a^l b^l, a^m b^m, a^r b^r); (a^{l'} b^{l'}, a^m b^m, a^{r'} b^{r'})\}$,
4. $\tilde{\mathcal{A}}^F \oslash \tilde{\mathcal{B}}^F = \{(a^l/b^l, a^m/b^m, a^r/b^r); (a^{l'}/b^{l'}, a^m/b^m, a^{r'}/b^{r'})\}$.

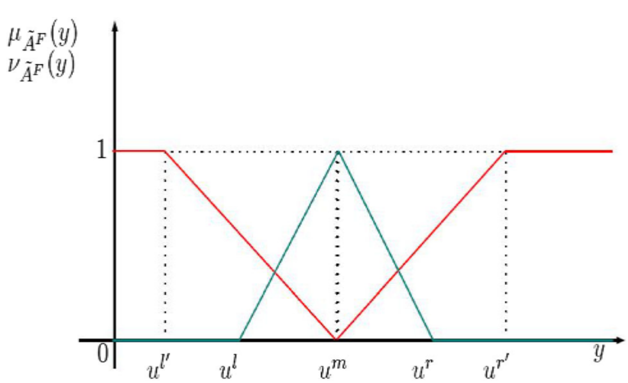

Fig. 1 Membership and non-membership functions of TFFN

Theorem 1 *Let $\tilde{\mathcal{A}}^F = \{(a^l, a^m, a^r); (a^{l'}, a^m, a^{r'})\}$, and $\tilde{\mathcal{B}}^F = \{(b^l, b^m, b^r); (b^{l'}, b^m, b^{r'})\}$ be TFFNs. Then, $\tilde{\mathcal{A}}^F \preceq \tilde{\mathcal{B}}^F$ if, and only if, $a^{l'} \le b^{l'}, a^l \le b^l, a^m \le b^m, a^r \le b^r, a^{r'} \le b^{r'}$.*

2.1 MOTP in Fermatean fuzzy environment

A MOTP is called FFMOTP if at least one of its components, such as transportation costs, supply, or demand, is taken as FFN. These FFNs can occur simultaneously in many parameters and take on a variety of shapes, including normal, triangular, or trapezoidal FFNs, as well as classical numbers. All FFMOTP parameters are represented as TFFNs in this study, and we refer to this as fully FFMOTP. Suppose p sources contain different quantities of an item which needs to be transported to q destinations. There are h Fermatean fuzzy characteristics $\tilde{c}^a_{gk} (a = 1, \ldots, h)$ for every arc (g, k) from source g to destination k. The purpose is to optimize the objective functions by designing a feasible transportation plan from sources to destinations. Suppose s_g denotes a product's supply at g^{th} source and d_k denotes a product's demand at k^{th} destination. Let x_{gk} denote the quantity of item sent from g^{th} source to k^{th} destination. Then, the FFMOTP with h Fermatean fuzzy objectives can be modeled as follows:

$$\begin{aligned} &\text{Optimize } Z_a = \sum_{g=1}^{p} \sum_{k=1}^{q} \tilde{c}^a_{gk} \times \tilde{x}_{gk}, \quad a = 1, 2, 3, \ldots, h \\ &\qquad \text{subject to } \sum_{k=1}^{q} \tilde{x}_{gk} \approx \tilde{s}_g, \quad g = 1, 2, 3, \ldots, p, \\ &\qquad\qquad \sum_{g=1}^{p} \tilde{x}_{gk} \approx \tilde{d}_k, \quad k = 1, 2, 3, \ldots, q, \\ &\qquad\qquad \tilde{x}_{gk} \succeq 0, \quad g = 1, 2, 3, \ldots, p, \\ &\qquad\qquad\qquad k = 1, 2, 3, \ldots, q. \end{aligned} \tag{1}$$

However, the best feasible solution of FFMOTP (1) is the ideal solution defined as follows:

$$\begin{aligned} \tilde{f_a} = \text{Optimize} \quad & \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{c}_{gk}^{a} \times \tilde{x}_{gk}, \quad a = 1,2,3,\ldots,h \\ \text{subject to} \quad & \sum_{k=1}^{q} \tilde{x}_{gk} \approx \tilde{s}_g, \quad g = 1,2,3,\ldots,p, \\ & \sum_{g=1}^{p} \tilde{x}_{gk} \approx \tilde{d}_k, \quad k = 1,2,3,\ldots,q, \\ & \tilde{x}_{gk} \succeq 0, \quad g = 1,2,3,\ldots,p, \\ & \qquad\qquad k = 1,2,3,\ldots,q. \end{aligned} \tag{2}$$

In addition, it is important to note that if all the parameters of a model are represented as FFNs, the optimal solution is also expected to be FFN.

Remark 1 Necessary and sufficient requirement for the feasibility of fully FFMOTP is that the problem should be balanced, that is, $\sum_{g=1}^{p} \tilde{s_g} = \sum_{k=1}^{q} \tilde{d}_k$. If the problem is unbalanced, then the approach proposed by Kumar and Kaur (2011) might be used to transform it into a balanced FFMOTP.

Remark 2 We are aware that FS only has membership grades, so it is impossible for it to assess decision making based on satisfaction and dissatisfaction. The non-membership degree in the FS was added along with the introduction of the IFS in order to address these shortcomings. Nonetheless, PFS and FFS were later introduced as extensions of FS. The rationale for selecting the FFS in this work is that it is more generic than the PFS and IFS and can manage a wider range of uncertainty.

Definition 10 (Ramezani-Tarkhorani et al. 2014) The common weight analysis (CWA) efficiency of DMU_k is defined as

$$\phi_q = \frac{\sum_{d=1}^{t} u_d^* y_{dk}}{\sum_{g=1}^{p} v_g^* x_{gk}}, \quad k = 1,2,3,\ldots,q,$$

where u_g^* and v_k^* are assigned CSW. DMU_k is CWA-efficient if $\phi_k = 1$. Otherwise DMU_k is CWA-inefficient.

The performance of DMU_k is better than DMU_k if $\phi_k > \phi_g$. Similarly, in Fermatean fuzzy environment DMU_k is CWA-efficient if $\phi_k = \tilde{1} = \{(1,1,1);(1,1,1)\}$.

3 Model for CSW in Fermatean fuzzy DEA

DEA, also known as frontier analysis, is a linear programming tool used to determine the relative efficiency of many comparable entities, or DMUs. One of the most significant advantages of DEA is that no prior knowledge is required to determine the input and output weights in conventional DEA models such as BCC (Banker et al. 1984) and CCR (Charnes et al. 1978). In practice, these models assess each DMU's efficiency under the ideal conditions. Assume there are q DMUs to be assessed, each with p inputs and t outputs. Suppose x_{gk} $(g = 1,\ldots,p)$ and y_{dk} $(d = 1,\ldots,t)$ are the input and output data of DMU_k $(k = 1,\ldots,q)$. Without loss of generality, all input and output data x_{gk} and y_{dk}, respectively, are supposed to be unclear and considered as positive TFFNs $\tilde{x}_{gk} = \{(x_{gk}^{l}, x_{gk}^{m}, x_{gk}^{r}); (x_{gk}^{l'}, x_{gk}^{m}, x_{gk}^{r'})\}$ and $\tilde{y}_{dk} = \{(y_{dk}^{l}, y_{dk}^{m}, y_{dk}^{r}); (y_{dk}^{l'}, y_{dk}^{m}, y_{dk}^{r'})\}$ for $g = 1,\ldots,p$, $d = 1,\ldots,t$ and $k = 1,\ldots,q$. Classical input and output data might be considered a type of triangular Fermatean fuzzy input and output data x_{gk} and y_{dk} with $x_{gk}^{l'} = x_{gk}^{l} = x_{gk}^{m} = x_{gk}^{r} = x_{gk}^{r'}$ and $y_{dk}^{l'} = y_{dk}^{l} = y_{dk}^{m} = y_{dk}^{r} = y_{dk}^{r'}$. The efficiency of DMU_k is defined as

$$\tilde{\phi}_k = \frac{\sum_{d=1}^{t} u_d \tilde{y}_{dk}}{\sum_{g=1}^{p} v_g \tilde{x}_{gk}}, \tag{3}$$

which is a FFN regarded to be a Fermatean fuzzy efficiency, where v_g $(g = 1,\ldots,p)$ and u_d $(d = 1,\ldots,t)$ are the weights given to the inputs and outputs, respectively. Then, the proposed additive model based on approach introduced by Azar et al. (2016) to find a CSW is

$$\begin{aligned} \max \min_{k=1,\ldots,q} \quad & \left\{ \sum_{d=1}^{t} u_d \tilde{y}_{dk} - \sum_{g=1}^{p} v_k \tilde{x}_{gk} \right\} \\ \text{subject to} \quad & \frac{\sum_{d=1}^{t} u_d \tilde{y}_{dk}}{\sum_{g=1}^{p} v_g \tilde{x}_{gk}} \preceq \tilde{1}, \quad k = 1,2,3,\ldots,q, \\ & \sum_{d=1}^{t} u_d \geq 1, \quad \sum_{g=1}^{p} v_g \geq 1, \\ & u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t, \quad g = 1,2,3,\ldots,p, \end{aligned} \tag{4}$$

where ε is a non-Archimedean positive number used to ensure that the variables are strictly positive. In the last two conditions $\sum_{d=1}^{t} u_d \geq 1$ and $\sum_{g=1}^{p} v_g \geq 1$, it can be seen that the weights assigned to inputs and outputs are unbounded, therefore, the optimum solution can be unbounded. In this way, to acquire boundedness in the optimum solution sets of our developed model, the last two conditions $\sum_{d=1}^{t} u_d \geq 1$ and $\sum_{g=1}^{p} v_g \geq 1$ are replaced by the normalized condition $\sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1$. The model (4) is simplified as follows:

$$\begin{aligned}
&\max \min_{k=1,\ldots,q}\left\{\sum_{d=1}^{t} u_d \tilde{y}_{dk} - \sum_{g=1}^{p} v_g \tilde{x}_{gk}\right\}\\
&\text{subject to } \frac{\sum_{d=1}^{t} u_d \tilde{y}_{dk}}{\sum_{g=1}^{p} v_g \tilde{x}_{gk}} \preceq \tilde{1}, \quad k = 1,2,3,\ldots,q,\\
&\sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1,\\
&u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t,\\
&\qquad g = 1,2,3,\ldots,p,
\end{aligned} \tag{5}$$

The model (5) can be written as

$$\begin{aligned}
&\max \min_{k=1,\ldots,q}\Bigg\{\sum_{d=1}^{t} u_d \Big\{(y^{l}_{dk}, y^{m}_{dk}, y^{r}_{dk}); (y^{l'}_{dk}, y^{m}_{dk}, y^{r'}_{dk})\Big\}\\
&\qquad - \sum_{g=1}^{p} v_g \Big\{(x^{l}_{gk}, x^{m}_{gk}, x^{r}_{gk}); (x^{l'}_{gk}, x^{m}_{gk}, x^{r'}_{gk})\Big\}\Bigg\}\\
&\text{subject to } \frac{\sum_{d=1}^{t} u_d \{(y^{l}_{dk}, y^{m}_{dk}, y^{r}_{dk}); (y^{l'}_{dk}, y^{m}_{dk}, y^{r'}_{dk})\}}{\sum_{g=1}^{p} v_g \{(x^{l}_{gk}, x^{m}_{gk}, x^{r}_{gk}); (x^{l'}_{gk}, x^{m}_{gk}, x^{r'}_{gk})\}} \preceq \tilde{1},\\
&\quad k = 1,2,3,\ldots,q,\\
&\sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1,\\
&u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t, \quad g = 1,2,3,\ldots,p.
\end{aligned} \tag{6}$$

Using Fermatean fuzzy arithmetic, we can write the above model as

$$\begin{aligned}
&\max \min_{k=1,\ldots,q}\Bigg\{\left(\sum_{d=1}^{t} u_d y^{l}_{dk} - \sum_{g=1}^{p} v_g x^{r}_{gk}, \sum_{d=1}^{t} u_d y^{m}_{dk} - \sum_{g=1}^{p} v_g x^{m}_{gk}, \sum_{d=1}^{t} u_d y^{r}_{dk} - \sum_{g=1}^{p} v_g x^{l}_{gk}\right);\\
&\qquad \left(\sum_{d=1}^{t} u_d y^{l'}_{dk} - \sum_{g=1}^{p} v_g x^{r'}_{gk}, \sum_{d=1}^{t} u_d y^{m}_{dk} - \sum_{g=1}^{p} v_g x^{m}_{gk}, \sum_{d=1}^{t} u_d y^{r'}_{dk} - \sum_{g=1}^{p} v_g x^{l'}_{gk}\right)\Bigg\}\\
&\text{subject to } \Bigg\{\left(\frac{\sum_{d=1}^{t} u_d y^{l}_{dk}}{\sum_{g=1}^{p} v_g x^{r}_{gk}}, \frac{\sum_{d=1}^{t} u_d y^{m}_{dk}}{\sum_{g=1}^{p} v_g x^{m}_{gk}}, \frac{\sum_{d=1}^{t} u_d y^{r}_{dk}}{\sum_{g=1}^{p} v_g x^{l}_{gk}}\right);\\
&\qquad \left(\frac{\sum_{d=1}^{t} u_d y^{l'}_{dk}}{\sum_{g=1}^{p} v_g x^{r'}_{gk}}, \frac{\sum_{d=1}^{t} u_d y^{m}_{dk}}{\sum_{g=1}^{p} v_g x^{m}_{gk}}, \frac{\sum_{d=1}^{t} u_d y^{r'}_{dk}}{\sum_{g=1}^{p} v_g x^{l'}_{gk}}\right)\Bigg\} \preceq \tilde{1}, \quad k = 1,2,3,\ldots,q,\\
&\sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1, \quad u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t, \quad g = 1,2,3,\ldots,p.
\end{aligned} \tag{7}$$

In the model (7), if upper bound $\frac{u_d y^{r'}_{dk}}{v_g x^{l'}_{gk}} \leq 1$ then $\frac{u_d y^{l'}_{dk}}{v_g x^{r'}_{gk}} \leq 1$, $\frac{u_d y^{l}_{dk}}{v_g x^{r}_{gk}} \leq 1$, $\frac{u_d y^{m}_{dk}}{v_g x^{m}_{gk}} \leq 1$ and $\frac{u_d y^{r}_{dk}}{v_g x^{l}_{gk}}$ will also be satisfied. Therefore, this model can be simplified as follows:

$$\begin{aligned}
&\max \min_{k=1,\ldots,q}\Bigg\{\left(\sum_{d=1}^{t} u_d y^{l}_{dk} - \sum_{g=1}^{p} v_g x^{r}_{gk}, \sum_{d=1}^{t} u_d y^{m}_{dk} - \sum_{g=1}^{p} v_g x^{m}_{gk}, \sum_{d=1}^{t} u_d y^{r}_{dk} - \sum_{g=1}^{p} v_g x^{l}_{gk}\right);\\
&\qquad \left(\sum_{d=1}^{t} u_d y^{l'}_{dk} - \sum_{g=1}^{p} v_q x^{r'}_{gk}, \sum_{d=1}^{t} u_d y^{m}_{dk} - \sum_{g=1}^{p} v_g x^{m}_{gk}, \sum_{d=1}^{t} u_d y^{r'}_{dk} - \sum_{g=1}^{p} v_g x^{l'}_{gk}\right)\Bigg\}\\
&\text{subject to } \frac{\sum_{d=1}^{t} u_d y^{r'}_{dk}}{\sum_{g=1}^{p} v_g x^{l'}_{gk}} \leq 1, \quad k = 1,2,3,\ldots,q,\\
&\sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1, \quad u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t, \quad g = 1,2,3,\ldots,p.
\end{aligned} \tag{8}$$

The above model can be written as

$$\begin{aligned}
&\max \min_{k=1,\ldots,q}\left\{\left(\sum_{d=1}^{t}u_d y_{dk}^{l}-\sum_{g=1}^{p}v_g x_{gk}^{r},\sum_{d=1}^{t}u_d y_{dk}^{m}-\sum_{g=1}^{p}v_g x_{gk}^{m},\sum_{d=1}^{t}u_d y_{dk}^{r}-\sum_{g=1}^{p}v_g x_{gk}^{l}\right);\right.\\
&\qquad\left.\left(\sum_{d=1}^{t}u_d y_{dk}^{l'}-\sum_{g=1}^{p}v_g x_{gk}^{r'},\sum_{d=1}^{t}u_d y_{dk}^{m}-\sum_{g=1}^{p}v_g x_{gk}^{m},\sum_{d=1}^{t}u_d y_{dk}^{r'}-\sum_{g=1}^{p}v_g x_{gk}^{l'}\right)\right\}\\
&\text{subject to}\quad \sum_{d=1}^{t}u_d y_{dk}^{r'}-\sum_{g=1}^{p}v_g x_{gk}^{l'}\le 0,\quad k=1,2,3,\ldots,q,\\
&\qquad \sum_{d=1}^{t}u_d+\sum_{g=1}^{p}v_g=1,\quad u_d,v_g\ge\varepsilon,\quad d=1,2,3,\ldots,t,\quad g=1,2,3,\ldots,p.
\end{aligned} \tag{9}$$

To transform the model (9) into a LP model, we assume that ϕ represents the term of

$$\begin{aligned}
&\min_{k=1,\ldots,q}\left\{\left(\sum_{d=1}^{t}u_d y_{dk}^{l}-\sum_{g=1}^{p}v_g x_{gk}^{r},\sum_{d=1}^{t}u_d y_{dk}^{m}-\sum_{g=1}^{p}v_g x_{gk}^{m},\sum_{d=1}^{t}u_d y_{dk}^{r}-\sum_{g=1}^{p}v_g x_{gk}^{l}\right);\right.\\
&\qquad\left.\left(\sum_{d=1}^{t}u_d y_{dk}^{l'}-\sum_{g=1}^{p}v_g x_{gk}^{r'},\sum_{d=1}^{t}u_d y_{dk}^{m}-\sum_{g=1}^{p}v_g x_{gk}^{m},\sum_{d=1}^{t}u_d y_{dk}^{r'}-\sum_{g=1}^{p}v_g x_{gk}^{l'}\right)\right\},
\end{aligned}$$

then, the model (9) can be written as

$$\begin{aligned}
&\max\quad \phi\\
&\text{subject to}\quad \phi=\min_{k=1,\ldots,q}\left\{\left(\sum_{d=1}^{t}u_d y_{dk}^{l}-\sum_{g=1}^{p}v_g x_{gk}^{r},\sum_{d=1}^{t}u_d y_{dk}^{m}-\sum_{g=1}^{p}v_g x_{gk}^{m},\sum_{d=1}^{t}u_d y_{dk}^{r}-\sum_{g=1}^{p}v_g x_{gk}^{l}\right);\right.\\
&\qquad\left.\left(\sum_{d=1}^{t}u_d y_{dk}^{l'}-\sum_{g=1}^{p}v_g x_{gk}^{r'},\sum_{d=1}^{t}u_d y_{dk}^{m}-\sum_{g=1}^{p}v_g x_{gk}^{m},\sum_{d=1}^{t}u_d y_{dk}^{r'}-\sum_{g=1}^{p}v_g x_{gk}^{l'}\right)\right\},\\
&\qquad \sum_{d=1}^{t}u_d y_{dk}^{r'}-\sum_{g=1}^{p}v_g x_{gk}^{l'}\le 0,\quad k=1,2,3,\ldots,q,\\
&\qquad \sum_{d=1}^{t}u_d+\sum_{g=1}^{p}v_g=1,\quad u_d,v_g\ge\varepsilon,\quad d=1,2,3,\ldots,t,\quad g=1,2,3,\ldots,p.
\end{aligned} \tag{10}$$

Above model in LP form can be written as

$$
\begin{aligned}
&\max \quad \phi \\
&\text{subject to} \quad \sum_{d=1}^{t} u_d y_{dk}^{l} - \sum_{g=1}^{p} v_g x_{gk}^{r} \geq \phi, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d y_{dk}^{m} - \sum_{g=1}^{p} v_g x_{gk}^{m} \geq \phi, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d y_{dk}^{r} - \sum_{g=1}^{p} v_g x_{gk}^{l} \geq \phi, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d y_{dk}^{l'} - \sum_{g=1}^{p} v_g x_{gk}^{r'} \geq \phi, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d y_{dk}^{r'} - \sum_{g=1}^{p} v_i x_{gk}^{l'} \geq \phi, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d y_{dk}^{r'} - \sum_{g=1}^{p} v_g x_{gk}^{l'} \leq 0, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1, \\
&\qquad u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t, \quad g = 1,2,3,\ldots,p.
\end{aligned}
\tag{11}
$$

Since $\sum_{d=1}^{t} u_d y_{dk}^{l'} - \sum_{g=1}^{p} v_g x_{gk}^{r'} \geq \phi$, therefore, $\sum_{d=1}^{t} u_d y_{dk}^{l} - \sum_{g=1}^{p} v_g x_{gk}^{r} \geq \phi$, $\sum_{d=1}^{t} u_d y_{dk}^{m} - \sum_{g=1}^{p} v_g x_{gk}^{m} \geq \phi$, $\sum_{d=1}^{t} u_d y_{dk}^{r} - \sum_{g=1}^{p} v_g x_{gk}^{l} \geq \phi$ and $\sum_{d=1}^{t} u_d y_{dk}^{r'} - \sum_{g=1}^{p} v_g x_{gk}^{l'} \geq \phi$ will be constantly followed, so these constraints are redundant and deleted. Then, the final model for calculating CSW in Fermatean fuzzy environment is described as follows:

$$
\begin{aligned}
&\max \quad \phi \\
&\text{subject to} \quad \sum_{d=1}^{t} u_d y_{dk}^{l'} - \sum_{g=1}^{p} v_g x_{gk}^{r'} \geq \phi; \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d y_{dk}^{r'} - \sum_{g=1}^{p} v_g x_{gk}^{l'} \leq 0, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1, \\
&\qquad u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t, \quad g = 1,2,3,\ldots,p.
\end{aligned}
\tag{12}
$$

Using an optimum solution of the aforementioned model, such as $(u^*, v^*) = (u_1^*, \ldots, u_t^*, v_1^*, \ldots, v_p^*)$, as CSW, the Fermatean fuzzy efficiency of $DMU_k (k = 1,2,3,\ldots,q)$ can be calculated as follows:

$$
\frac{\sum_{d=1}^{t} u_b^* \tilde{y}_{dk}}{\sum_{g=1}^{p} v_p^* \tilde{x}_{gk}} = \left\{ \left(\frac{\sum_{d=1}^{t} u_b^* y_{dk}^{l}}{\sum_{g=1}^{p} v_p^* x_{gk}^{r}}, \frac{\sum_{d=1}^{t} u_b^* y_{dk}^{m}}{\sum_{g=1}^{p} v_p^* x_{gk}^{m}}, \frac{\sum_{d=1}^{t} u_b^* y_{dk}^{r}}{\sum_{g=1}^{p} v_p^* x_{gk}^{l}} \right) \left(\frac{\sum_{d=1}^{t} u_b^* y_{dk}^{l'}}{\sum_{g=1}^{p} v_p^* x_{gk}^{r'}}, \frac{\sum_{d=1}^{t} u_b^* y_{dk}^{m}}{\sum_{g=1}^{p} v_p^* x_{gk}^{m}}, \frac{\sum_{d=1}^{t} u_b^* y_{dk}^{r'}}{\sum_{g=1}^{p} v_p^* x_{gk}^{l'}} \right) \right\}
\tag{13}
$$

It should be noted that there is no assurance that the model (12) will provide a unique optimal solution. Therefore, different FFESs for specific DMUs may be achieved. To overcome this dilemma, related to Ramezani-Tarkhorani et al. (2014), lexicographic technique is used to solve the model (14). Then, a unique optimum solution with minimal output weights and maximum input weights is determined. This is because, similar to Obata and Ishii (2003), a maximum number of possible output indexes (a minimum percentage of input indexes) is preferred by the decision maker. As a result, for output (input) weights, it is desirable to utilize a smaller (greater) range of weights:

$$
\begin{aligned}
&\min(Lex\{u_1, u_2, \ldots, u_t, -v_1, -v_2, \ldots, -v_p\}) \\
&\text{subject to} \quad \sum_{d=1}^{t} u_d y_{dk}^{l'} - \sum_{g=1}^{p} v_g x_{gk}^{r'} \geq \phi^*; \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d y_{dk}^{r'} - \sum_{g=1}^{p} v_g x_{gk}^{l'} \leq 0, \quad k = 1,2,3,\ldots,q, \\
&\qquad \sum_{d=1}^{t} u_d + \sum_{g=1}^{p} v_g = 1, \\
&\qquad u_d, v_g \geq \varepsilon, \quad d = 1,2,3,\ldots,t, \quad g = 1,2,3,\ldots,p,
\end{aligned}
\tag{14}
$$

where ϕ^* is the optimum value of the model (12).

Theorem 2 *The lexicographic technique to solve the model (14) provides a unique optimal solution.*

Proof Assume that $\{u_1^*, \ldots, u_t^*, v_1^*, \ldots, v_p^*\}$ are the optimal values for steps 1 to $t+p$, respectively. The feasible region in step $t+p$ is thus

$$
\begin{aligned}
R_{t+p} = \{[U,V] | [U,V] \in R^* \\
u_1 = u_1^*, \ldots, u_t = u_t^*, v_1 = v_1^*, \ldots, v_p = v_p^*\},
\end{aligned}
$$

where R^* is the optimal solutions set for the model (12). According to the lexicographic technique, if one optimal solution is obtained in step $i (i \leq t+p)$, then we are done. On contrary suppose that, there are at least two optimal solutions in step $t+p$, say $(\dot{U}, \dot{V})$ and $(\ddot{U}, \ddot{V})$. It is understood that $\{\dot{u}_1 = \ddot{u}_1 = u_1^*, \ldots, \dot{u}_t = \ddot{u}_t = u_t^*, \quad \dot{v}_1 = \ddot{v}_1 = v_1^*, \ldots, \dot{v}_p = \ddot{v}_p = v_p^*\}$, therefore, $(\dot{U}, \dot{V}) = (\ddot{U}, \ddot{V})$. Therefore, $(\dot{U}, \dot{V}) = (\ddot{U}, \ddot{V})$, which is a contradiction. □

Theorem 3 *The Fermatean fuzzy efficiency of* DMU_q *provided by Eq. (13) preserves the shape of non-negative TFFN.*

Proof Assume that $(u^*, v^*) = (u_1^*, \ldots, u_t^*, v_1^*, \ldots, v_p^*)$ is the optimum solution of the model (14). The model's last constraints imply that u^* and v^* are both greater than or equal to zero. On the other hand, because of the non-negative Fermatean fuzzy input $\tilde{x}_{gk} = \{(x_{gk}^l, x_{gk}^m, x_{gk}^r); (x_{gk}^{l'}, x_{gk}^m, x_{gk}^{r'})\}$ and output $\tilde{y}_{dk} = \{(y_{dk}^l, y_{dk}^m, y_{dk}^r); (y_{dk}^{l'}, y_{dk}^m, y_{dk}^{r'})\}$, we have

$$\begin{aligned}
&0 \le x_{gk}^{l'} \le x_{gk}^l \le x_{gk}^m \le x_{gk}^r \le x_{gk}^{r'},\\
&\quad g = 1,2,3,\ldots,p, \quad k = 1,2,3,\ldots,q,\\
&0 \le y_{dk}^{l'} \le y_{dk}^l \le y_{dk}^m \le y_{dk}^r \le y_{dk}^{r'},\\
&\quad d = 1,2,3,\ldots,t, \quad k = 1,2,3,\ldots,q.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&0 \le \sum_{g=1}^{p} x_{gk}^{l'} \le \sum_{g=1}^{p} x_{gk}^{l} \le \sum_{g=1}^{p} x_{gk}^{m} \le \sum_{g=1}^{p} x_{gk}^{r} \le \sum_{g=1}^{p} x_{gk}^{r'},\\
&\quad k = 1,2,3,\ldots,q,\\
&0 \le \sum_{d=1}^{t} y_{dk}^{l'} \le \sum_{d=1}^{t} y_{dk}^{l} \le \sum_{d=1}^{t} y_{dk}^{m} \le \sum_{d=1}^{t} y^{r}_{dk} \le \sum_{d=1}^{t} y_{dk}^{r'},\\
&\quad k = 1,2,3,\ldots,q.
\end{aligned}$$

Hence,

$$\begin{aligned}
0 &\le \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{l'}}{\sum_{g=1}^{p} v_g^* x_{gk}^{r'}} \le \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{l}}{\sum_{g=1}^{p} v_g^* x_{gk}^{r}} \le \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{r}}{\sum_{g=1}^{p} v_g^* x_{gk}^{l}}\\
&\le \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{r}}{\sum_{g=1}^{p} v_g^* x_{gk}^{l}} \le \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{r'}}{\sum_{g=1}^{p} v_g^* x_{qp}^{l'}}, \quad k = 1,2,3,\ldots,q.
\end{aligned}$$

This shows that

$$\frac{\sum_{d=1}^{t} u_d^* \tilde{y}_{dk}}{\sum_{g=1}^{p} v_g^* \tilde{x}_{gk}} \approx \left\{ \left(\frac{\sum_{d=1}^{t} u_d^* y_{dk}^{l}}{\sum_{g=1}^{p} v_g^* x_{gk}^{r}}, \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{m}}{\sum_{g=1}^{p} v_g^* x_{gk}^{m}}, \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{r}}{\sum_{g=1}^{p} v_g^* x_{gk}^{l}} \right); \left(\frac{\sum_{d=1}^{t} u_d^* y_{dk}^{l'}}{\sum_{g=1}^{p} v_g^* x_{gk}^{r'}}, \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{m}}{\sum_{g=1}^{p} v_g^* x_{gk}^{m}}, \frac{\sum_{d=1}^{t} u_d^* y_{dk}^{r'}}{\sum_{g=1}^{p} v_g^* x_{gk}^{l'}} \right) \right\}$$

preserve the form of non-negative TFFN. □

4 Solving the fully FFMOTP based on CSW

Considering the fully FFMOTP as described in the model (1), having h Fermatean fuzzy attributes which should be maximized or minimized. In this part, we propose an approach to solve the fully FFMOTP using CSW in Fermatean fuzzy DEA. Every arc is considered as a DMU to achieve this goal. The arc attributes corresponding to Fermatean fuzzy objectives that should be minimized are defined as Fermatean fuzzy input, whereas the arc attributes corresponding to Fermatean fuzzy objectives that should be maximized are defined as Fermatean fuzzy output. In this way, using CSW, two FFESs are calculated for every arc as a criterion for the performance evaluation of the single-objective transportation from g $(g = 1,\ldots,p)$ to k $(k = 1,\ldots,q)$. Next, the mean of these FFESs is used to calculate arc's new FFES. Then, the h Fermatean fuzzy attributes associated with each arc are converted into a single Fermatean fuzzy attribute, and the fully FFMOTP is converted into a single-objective FFTP. To proceed with this procedure, the steps are as follows:

Step 1 For each $k = 1,\ldots,q$, the performance evaluation of single-objective transportation from g^{th} source to k^{th} destination, i.e., $\tilde{E}_{gk}^{1^*}$, is determined by taking the g^{th} origin as the target and varying the k^{th} location. This is achieved using the CSW obtained as follows:

$$\begin{aligned}
&\max \min_{k=1,\ldots,q} \left\{ \sum_{d=1}^{t} u_d \tilde{y}_{gk}^{b} - \sum_{e=1}^{r} v_e \tilde{x}_{gk}^{d} \right\}\\
&\text{subject to } \frac{\sum_{d=1}^{t} u_d \tilde{y}_{gk}^{b}}{\sum_{e=1}^{r} v_e \tilde{x}_{gk}^{d}} \preceq \tilde{1}, \quad k = 1,2,3,\ldots,q,\\
&\qquad \sum_{d=1}^{t} u_d + \sum_{e=1}^{r} v_e = 1,\\
&\qquad u_d, v_e \ge \varepsilon, \quad d = 1,2,3,\ldots,t,\\
&\qquad e = 1,2,3,\ldots,r.
\end{aligned} \tag{15}$$

Using Fermatean fuzzy arithmetic, the LP model (15) can be written as

$$\begin{aligned}
&\max \quad \phi_1\\
&\text{subject to } \sum_{d=1}^{t} u_d y_{gk}^{d,l'} - \sum_{e=1}^{r} v_e x_{gk}^{e,r'} \ge \phi_1;\\
&\qquad k = 1,2,3,\ldots,q,\\
&\qquad \sum_{d=1}^{t} u_d y_{gk}^{d,r'} - \sum_{e=1}^{r} v_e x_{gk}^{e,l'} \le 0,\\
&\qquad k = 1,2,3,\ldots,q,\\
&\qquad \sum_{d=1}^{t} u_d + \sum_{e=1}^{r} v_e = 1,\\
&\qquad u_d, v_e \ge \varepsilon, \quad d = 1,2,3,\ldots,t,\\
&\qquad e = 1,2,3,\ldots,r.
\end{aligned} \tag{16}$$

The model (16) has alternative optimum solutions. Therefore, to find a unique optimum solution, this model is solved using a lexicographic strategy, such as

$$\begin{aligned}
&\min(Lex\{u_1, u_2, \ldots, u_s, -v_1, -v_2, \ldots, -v_r\})\\
&\text{subject to}\quad \sum_{d=1}^{t} u_d y_{gk}^{d,l'} - \sum_{e=1}^{r} v_e x_{gk}^{e,r'} \geq \phi_1^*;\\
&\qquad k = 1, 2, 3, \ldots, q,\\
&\qquad \sum_{d=1}^{t} u_d y_{gk}^{d,r'} - \sum_{e=1}^{r} v_e x_{gk}^{e,l'} \leq 0,\\
&\qquad k = 1, 2, 3, \ldots, q,\\
&\qquad \sum_{d=1}^{t} u_d + \sum_{e=1}^{r} v_e = 1,\\
&\qquad u_d, v_e \geq \varepsilon, \quad d = 1, 2, 3, \ldots, t,\\
&\qquad e = 1, 2, 3, \ldots, r,
\end{aligned} \tag{17}$$

where ϕ_1^* is the model (16)'s optimal value. Finally, using the unique optimal solution of the model (17) represented by $(u^*, v^*) = (u_1^*, \ldots, u_t^*, v_1^*, \ldots, v_r^*)$, as CSW, the FFES $\tilde{E}_{gk}^{1^*}$ of arc (g, k) $(k = 1, 2, 3, \ldots, q)$ can be determined using the following relation:

$$\begin{aligned}
\tilde{E}_{gk}^{1^*} &= \frac{\sum_{d=1}^{t} u_d^* \tilde{y}_{gk}^{d}}{\sum_{e=1}^{r} v_e^* \tilde{x}_{gk}^{e}}\\
&= \left\{ \left(\frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,l}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,r}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,m}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,m}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,r}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,l}} \right); \right.\\
&\quad \left. \left(\frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,l'}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,r'}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,m}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,m}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,r'}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,l'}} \right) \right\}.
\end{aligned} \tag{18}$$

Step 2 For each $g = 1, \ldots, p,$, the performance evaluation $\tilde{E}_{gk}^{2^*}$ of single-objective transportation from g^{th} source to k^{th} destination is calculated by taking the k^{th} location as the target and varying the g^{th} origin. This is done by utilizing the CSW determined as follows:

$$\begin{aligned}
&\max \min_{g=1,\ldots,p} \left\{ \sum_{d=1}^{t} u_d \tilde{y}_{gk}^{d} - \sum_{e=1}^{r} v_e \tilde{x}_{gk}^{e} \right\}\\
&\text{subject to}\quad \frac{\sum_{d=1}^{t} u_d \tilde{y}_{gk}^{d}}{\sum_{e=1}^{r} v_e \tilde{x}_{gk}^{e}} \preceq \tilde{1},\\
&\qquad g = 1, 2, 3, \ldots, p,\\
&\qquad \sum_{d=1}^{t} u_d + \sum_{e=1}^{r} v_e = 1,\\
&\qquad u_d, v_e \geq \varepsilon, \quad d = 1, 2, 3, \ldots, t,\\
&\qquad e = 1, 2, 3, \ldots, r.
\end{aligned} \tag{19}$$

Using Fermatean fuzzy arithmetic, the LP model (19) can be written as

$$\begin{aligned}
&\max \quad \phi_2\\
&\text{subject to}\quad \sum_{d=1}^{t} u_d y_{gk}^{d,l'} - \sum_{e=1}^{r} v_e x_{gk}^{e,r'} \geq \phi_2;\\
&\qquad g = 1, 2, 3, \ldots, p,\\
&\qquad \sum_{d=1}^{t} u_d y_{gk}^{d,r'} - \sum_{e=1}^{r} v_e x_{gk}^{e,l'} \leq 0,\\
&\qquad g = 1, 2, 3, \ldots, p,\\
&\qquad \sum_{d=1}^{t} u_d + \sum_{e=1}^{r} v_e = 1,\\
&\qquad u_d, v_e \geq \varepsilon, \quad d = 1, 2, 3, \ldots, t,\\
&\qquad e = 1, 2, 3, \ldots, r.
\end{aligned} \tag{20}$$

The model (20) has alternative optimum solutions. Therefore, to find a unique optimum solution, this model is solved using a lexicographic technique, such as

$$\begin{aligned}
&\min(Lex\{u_1, u_2, \ldots, u_t, -v_1, -v_2, \ldots, -v_r\})\\
&\text{subject to}\quad \sum_{d=1}^{t} u_d y_{gk}^{d,l'} - \sum_{e=1}^{r} v_e x_{gk}^{e,r'} \geq \phi_2^*;\\
&\qquad g = 1, 2, 3, \ldots, p,\\
&\qquad \sum_{d=1}^{t} u_d y_{gk}^{d,r'} - \sum_{e=1}^{r} v_e x_{gk}^{e,l'} \leq 0,\\
&\qquad g = 1, 2, 3, \ldots, p,\\
&\qquad \sum_{d=1}^{t} u_d + \sum_{e=1}^{r} v_e = 1,\\
&\qquad u_d, v_e \geq \varepsilon, \quad d = 1, 2, 3, \ldots, t,\\
&\qquad e = 1, 2, 3, \ldots, r,
\end{aligned} \tag{21}$$

where ϕ_2^* is the model (20)'s optimal value. Then, using the unique optimal solution of the model (20) denoted by $(u^*, v^*) = (u_1^*, \ldots, u_t^*, v_1^*, \ldots, v_r^*)$, as CSW, the FFES $\tilde{E}_{gk}^{2^*}$ of arc (g, k) $(g = 1, 2, 3, \ldots, p)$ can be computed using the following relation:

$$\begin{aligned}
\tilde{E}_{gk}^{2^*} &= \frac{\sum_{d=1}^{t} u_d^* \tilde{y}_{gk}^{d}}{\sum_{e=1}^{r} v_e^* \tilde{x}_{gk}^{e}}\\
&= \left\{ \left(\frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,l}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,r}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,m}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,m}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,r}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,l}} \right); \right.\\
&\quad \left. \left(\frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,l'}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,r'}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,m}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,m}}, \frac{\sum_{d=1}^{t} u_d^* y_{gk}^{d,r'}}{\sum_{e=1}^{r} v_e^* x_{gk}^{e,l'}} \right) \right\}.
\end{aligned} \tag{22}$$

For each arc (g, k), $g = 1, \ldots, p$ and $k = 1, \ldots, q$, two FFESs $\tilde{E}_{gk}^{1^*}$ and $\tilde{E}_{gk}^{2^*}$ are obtained.

Step 3 At this stage, we have two FFESs, but our motive is to transform the fully FFMOTP into a single-objective FFTP. To do this, it is necessary to convert these two FFESs into one. Therefore, in order to do that, the average of these FFESs $\tilde{E}^{1^*}_{gk}$ and $\tilde{E}^{2^*}_{gk}$ is utilized to calculate a novel FFES for every arc (g, k) as

$$\tilde{E}_{gk} = \frac{\tilde{E}^{1^*}_{gk} + \tilde{E}^{2^*}_{gk}}{2}. \tag{23}$$

Finally, the h Fermatean fuzzy attributes associated with each arc are converted into a single Fermatean fuzzy attribute $\tilde{E}_{gk}$, and the corresponding fully FFMOTP is converted into a single-objective FFTP:

$$\begin{aligned} E^* &= \max \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{E}_{gk} \otimes \tilde{x}_{gk} \\ \text{subject to } & \sum_{k=1}^{q} \tilde{x}_{gk} \approx \tilde{s}_g, \quad g = 1,\ldots,p, \\ & \sum_{g=1}^{p} \tilde{x}_{gk} \approx \tilde{d}_k, \quad k = 1,\ldots,q, \\ & \tilde{x}_{gk} \succeq 0, \quad g = 1,\ldots,p, \quad k = 1,\ldots,q. \end{aligned} \tag{24}$$

Remark 3 As negative quantities of products are physically senseless, all of the variables $\tilde{s}_g$s, $\tilde{d}_k$s are taken as non-negative TFFNs. Furthermore, it is obvious that the FFESs specified in the Eq. (23) are non-negative TFFNs.

To solve the model (24), we develop a method based on approach [39]. Using the Remark 3, $\tilde{s}_g$, $\tilde{d}_k$, $\tilde{E}_{gk}$ and $\tilde{x}_{gk}$ all represented by non-negative TFFNs as $\{(s^l_g, s^m_g, s^r_g); (s^{l'}_g, s^m_g, s^{r'}_g)\}$, $\{(d^l_k, d^m_k, d^r_k); (d^{l'}_k, d^m_k, d^{r'}_k)\}$, $\{(E^l_{gk}, E^m_{gk}, E^r_{gk}); (E^{l'}_{gk}, E^m_{gk}, E^{r'}_{gk})\}$ and $\{(x^l_{gk}, x^m_{gk}, x^r_{gk}); (x^{l'}_{gk}, x^m_{gk}, x^{r'}_{gk})\}$, respectively. Then, FFTP (24) may be rewritten as

$$\begin{aligned} E^* &= \max \sum_{g=1}^{p}\sum_{k=1}^{q} \{(E^l_{gk}, E^m_{gk}, E^r_{gk}); (E^{l'}_{gk}, E^m_{gk}, E^{r'}_{gk})\} \\ &\qquad \otimes \{(x^l_{gk}, x^m_{gk}, x^r_{gk}); (x^{l'}_{gk}, x^m_{gk}, x^{r'}_{gk})\} \\ \text{subject to } & \sum_{k=1}^{q} \{(x^l_{gk}, x^m_{gk}, x^r_{gk}); (x^{l'}_{gk}, x^m_{gk}, x^{r'}_{gk})\} \\ &\qquad \approx \{(s^l_g, s^m_g, s^r_g); (s^{l'}_g, s^m_g, s^{r'}_g)\}, \quad g = 1,\ldots,p, \\ & \sum_{g=1}^{p} \{(x^l_{gk}, x^m_{gk}, x^r_{gk}); (x^{l'}_{gk}, x^m_{gk}, x^{r'}_{gk})\} \\ &\qquad \approx \{(d^l_k, d^m_k, d^r_k); (d^{l'}_k, d^m_k, d^{r'}_k)\}, \quad k = 1,\ldots,q, \\ & \{(x^l_{gk}, x^m_{gk}, x^r_{gk}); (x^{l'}_{gk}, x^m_{gk}, x^{r'}_{gk})\} \succeq 0, \quad g = 1,\ldots,p, \\ & k = 1,\ldots,q. \end{aligned} \tag{25}$$

From Definitions 8 and 9, the above FFTP can be written as

$$\begin{aligned} E^* &= \max \Bigg\{ \left(\sum_{g=1}^{p}\sum_{k=1}^{q} E^l_{gk} x^l_{gk}, \sum_{g=1}^{p}\sum_{k=1}^{q} E^m_{gk} x^m_{gk}, \sum_{g=1}^{p}\sum_{k=1}^{q} E^r_{gk} x^r_{gk} \right); \\ & \left(\sum_{g=1}^{p}\sum_{k=1}^{q} E^{l'}_{gk} x^{l'}_{gk}, \sum_{g=1}^{p}\sum_{k=1}^{q} E^m_{gk} x^m_{gk}, \sum_{g=1}^{p}\sum_{k=1}^{q} E^{r'}_{gk} x^{r'}_{gk} \right) \Bigg\} \\ \text{subject to } & \sum_{k=1}^{q} x^l_{gk} = s^l_g, \quad g = 1,\ldots,p, \\ & \sum_{k=1}^{q} x^m_{gk} = s^m_g, \quad \sum_{k=1}^{q} x^r_{gk} = s^r_g, \quad g = 1,\ldots,p, \\ & \sum_{k=1}^{q} x^{l'}_{gk} = s^{l'}_g, \quad \sum_{k=1}^{q} x^{r'}_{gk} = s^{r'}_g, \quad g = 1,\ldots,p, \\ & \sum_{g=1}^{p} x^l_{gk} = d^l_k, \quad \sum_{g=1}^{p} x^m_{gk} = d^m_k, \quad k = 1,\ldots,q, \\ & \sum_{g=1}^{p} x^r_{gk} = d^r_k, \quad \sum_{g=1}^{p} x^{l'}_{gk} = d^{l'}_k, \quad k = 1,\ldots,q, \\ & \sum_{g=1}^{p} x^{r'}_{gk} = d^{r'}_k, \quad k = 1,\ldots,q, \\ & x^l_{gk} - x^{l'}_{gk} \geq 0, \quad x^m_{gk} - x^l_{gk} \geq 0, \quad g = 1,\ldots,p, \\ & k = 1,\ldots,q, \\ & x^r_{gk} - x^m_{gk} \geq 0, \quad x^{r'}_{gk} - x^r_{gk} \geq 0, \quad g = 1,\ldots,p, \\ & k = 1,\ldots,q, \\ & x^{l'}_{gk} \geq 0, \quad g = 1,\ldots,p, \quad k = 1,\ldots,q. \end{aligned} \tag{26}$$

It should be noted that the first ten constraints of the model (26) are due to the equality of two TFFNs, and the last five are to impose non-negative parameters $\tilde{x}_{gk}$ while maintaining their shape as non-negative TFFNs.

Theorem 4 *The optimum solution of the FFTP model (24) is an efficient solution of the fully FFMOTP model (2).*

Proof It should be noted that the objective function of model (24) is the weighted sum objective function of the fully FFMOTP (2). Since we know that, the optimum solution of a weighted sum problem with positive weights is always the most efficient solution of the multi-objective LPP in discussion. Since the values of $\tilde{E}_{gk}$ in Eq. (23) are taken as the weights of the weighted sum problem (24), therefore, it is enough to prove that $\tilde{E}_{gk} \succeq 0$. By equation $\tilde{E}_{gk} = \dfrac{\tilde{E}^{1^*}_{gk} + \tilde{E}^{2^*}_{gk}}{2}$, we have to show that $\tilde{E}^{1^*}_{gk} \succeq 0$ and $\tilde{E}^{2^*}_{gk} \succeq 0$. According to the given condition $\sum_{d=1}^{t} u_d + \sum_{e=1}^{r} v_e = 1$, $v_e, u_d \geq \varepsilon$ and the non-negativity of the input and output values, it is obvious that $\sum_{d=1}^{t} u^*_d y^{l'}_{gk} > 0$ and $\sum_{e=1}^{r} v^*_e x^{r'}_{gk} > 0$. Therefore, the first variable of $\tilde{E}^{1^*}_{gk}$, i.e., $\dfrac{\sum_{d=1}^{t} u^*_d y^{l'}_{gk}}{\sum_{e=1}^{r} v^*_e x^{r'}_{gk}}$, is positive. Hence, $\tilde{E}^{1^*}_{gk} \succeq 0$. Similarly, $\tilde{E}^{2^*}_{gk} \succeq 0$ can be proved. □

Theorem 5 *The optimum solution of model (26) is a Fermatean fuzzy optimum solution of FFTP (24).*

Proof Suppose that $\{(\vec{x}^{l}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r}_{gk}); (\vec{x}^{l'}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r'}_{gk})\}$ is the optimum solution of the model (26). The first five constraints of model (26) states that

$$\left\{ \left(\sum_{k=1}^{q} \vec{x}^{l}_{gk}, \sum_{k=1}^{q} \vec{x}^{m}_{gk}, \sum_{k=1}^{q} \vec{x}^{r}_{gk} \right); \left(\sum_{k=1}^{q} \vec{x}^{l'}_{gk}, \sum_{k=1}^{q} \vec{x}^{m}_{gk}, \sum_{k=1}^{q} \vec{x}^{r'}_{gk} \right) \right\} \approx \{(s^{l}_{g}, s^{m}_{g}, s^{r}_{g}); (s^{l'}_{g}, s^{m}_{g}, s^{r'}_{g})\}$$

and second five constraints of model (26) states that

$$\left\{ \left(\sum_{g=1}^{p} \vec{x}^{l}_{gk}, \sum_{g=1}^{p} \vec{x}^{m}_{gk}, \sum_{g=1}^{p} \vec{x}^{r}_{gk} \right); \left(\sum_{g=1}^{p} \vec{x}^{l'}_{gk}, \sum_{g=1}^{p} \vec{x}^{m}_{gk}, \sum_{g=1}^{p} \vec{x}^{r'}_{gk} \right) \right\} \approx \{(d^{l}_{k}, d^{m}_{k}, d^{r}_{k}); (d^{l'}_{k}, d^{m}_{k}, d^{r'}_{k})\}.$$

Moreover, the remaining constraints states that

$$0 \leq \vec{x}^{l'}_{gk} \leq \vec{x}^{l}_{gk} \leq \vec{x}^{m}_{gk} \leq \vec{x}^{r}_{gk} \leq \vec{x}^{r'}_{gk}.$$

Therefore, $\tilde{\vec{x}}_{gk} = \{(\vec{x}^{l}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r}_{gk}); (\vec{x}^{l'}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r'}_{gk})\}$ fulfills the Fermatean fuzzy limitations of FFTP (24). Therefore, it is a Fermatean fuzzy feasible solution of FFTP (24). Now, suppose that $\tilde{\hat{x}}_{gk} = \{(\hat{x}^{l}_{gk}, \hat{x}^{m}_{gk}, \hat{x}^{r}_{gk}); (\hat{x}^{l'}_{gk}, \hat{x}^{m}_{gk}, \hat{x}^{r'}_{gk})\}$ is an arbitrary feasible solution of FFTP model (24). Because of the non-negative form of TFFN $\tilde{\hat{x}}_{gk} = \{(\hat{x}^{l}_{gk}, \hat{x}^{m}_{gk}, \hat{x}^{r}_{gk}); (\hat{x}^{l'}_{gk}, \hat{x}^{m}_{gk}, \hat{x}^{r'}_{gk})\}$, we have

$$0 \leq \hat{x}^{l'}_{gk} \leq \hat{x}^{l}_{gk} \leq \hat{x}^{m}_{gk} \leq \hat{x}^{r}_{gk} \leq \hat{x}^{r'}_{gk}.$$

From $\sum_{k=1}^{q} \tilde{\hat{x}}_{gk} = \tilde{s}_g$, we have

$$\left\{ \left(\sum_{k=1}^{q} \hat{x}^{l}_{gk}, \sum_{k=1}^{q} \hat{x}^{m}_{gk}, \sum_{k=1}^{q} \hat{x}^{r}_{gk} \right); \left(\sum_{k=1}^{q} \hat{x}^{l'}_{gk}, \sum_{k=1}^{q} \hat{x}^{m}_{gk}, \sum_{k=1}^{q} \hat{x}^{r'}_{gk} \right) \right\} \approx \{(s^{l}_{g}, s^{m}_{g}, s^{r}_{g}); (s^{l'}_{g}, s^{m}_{g}, s^{r'}_{g})\}$$

and $\sum_{k=1}^{q} \hat{x}^{l'}_{gk} = s^{l'}_{g}$, $\sum_{k=1}^{q} \hat{x}^{l}_{gk} = s^{l}_{g}$, $\sum_{k=1}^{q} \hat{x}^{m}_{gk} = s^{m}_{g}$, $\sum_{k=1}^{q} \hat{x}^{r}_{gk} = s^{r}_{g}$ and $\sum_{k=1}^{q} \hat{x}^{r'}_{gk} = s^{r'}_{g}$. Similarly, from $\sum_{g=1}^{p} \tilde{\hat{x}}_{gk} = \tilde{d}_k$, we get $\sum_{g=1}^{p} \hat{x}^{l'}_{gk} = d^{l'}_{k}$ $\sum_{g=1}^{p} \hat{x}^{l}_{gk} = d^{l}_{k}$, $\sum_{g=1}^{p} \hat{x}^{m}_{gk} = d^{m}_{k}$, $\sum_{g=1}^{p} \hat{x}^{r}_{gk} = d^{r}_{k}$ and $\sum_{g=1}^{p} \hat{x}^{r'}_{gk} = d^{r'}_{k}$. Therefore, $\tilde{\hat{x}}_{gk} = \{(\hat{x}^{l}_{gk}, \hat{x}^{m}_{gk}, \hat{x}^{r}_{gk}); (\hat{x}^{l'}_{gk}, \hat{x}^{m}_{gk}, \hat{x}^{r'}_{gk})\}$ is a feasible solution of model (26). By the optimality of $\tilde{\vec{x}}_{gk} = \{(\vec{x}^{l}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r}_{gk}); (\vec{x}^{l'}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r'}_{gk})\}$ for model (26), it is find that

$$\sum_{g=1}^{p} \sum_{k=1}^{q} E^{l'}_{gk} \hat{x}^{l'}_{gk} \leq \sum_{g=1}^{p} \sum_{k=1}^{q} E^{l'}_{gk} \vec{x}^{l'}_{gk},$$

$$\sum_{g=1}^{p} \sum_{k=1}^{q} E^{l}_{gk} \hat{x}^{l}_{gk} \leq \sum_{g=1}^{p} \sum_{k=1}^{q} E^{l}_{gk} \vec{x}^{l}_{gk},$$

$$\sum_{g=1}^{p} \sum_{k=1}^{q} E^{m}_{gk} \hat{x}^{m}_{gk} \leq \sum_{g=1}^{p} \sum_{k=1}^{q} E^{m}_{gk} \vec{x}^{m}_{gk},$$

$$\sum_{g=1}^{p} \sum_{k=1}^{q} E^{r}_{gk} \hat{x}^{r}_{gk} \leq \sum_{g=1}^{p} \sum_{k=1}^{q} E^{r}_{gk} \vec{x}^{r}_{gk},$$

$$\sum_{g=1}^{p} \sum_{k=1}^{q} E^{r'}_{gk} \hat{x}^{r'}_{gk} \leq \sum_{g=1}^{p} \sum_{k=1}^{q} E^{r'}_{gk} \vec{x}^{r'}_{gk}.$$

From Theorem 1, it follows that

$$\sum_{g=1}^{p} \sum_{k=1}^{q} \tilde{E}_{gk} \tilde{\hat{x}}_{gk} \prec \sum_{g=1}^{p} \sum_{k=1}^{q} \tilde{E}_{gk} \tilde{\vec{x}}_{gk}.$$

Hence, $\tilde{\vec{x}}_{gk} = \{(\vec{x}^{l}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r}_{gk}); (\vec{x}^{l'}_{gk}, \vec{x}^{m}_{gk}, \vec{x}^{r'}_{gk})\}$ is the Fermatean fuzzy optimum solution of FFTP (24). □

By solving the model (26) based on approach (Akram et al. 2023a) steps are as follows:

Step 1 Solve the classical TP using the crisp transportation technique:

$$\begin{aligned} E^*_{l'} &= \max \sum_{g=1}^{p} \sum_{k=1}^{q} E^{l'}_{gk} x^{l'}_{gk} \\ \text{subject to } & \sum_{k=1}^{q} x^{l'}_{gk} = s^{l'}_{g}, \quad g = 1, \ldots, p, \\ & \sum_{g=1}^{p} x^{l'}_{gk} = d^{l'}_{k}, \quad k = 1, \ldots, q, \\ & x^{l'}_{gk} \geq 0, \quad g = 1, \ldots, p, \quad k = 1, \ldots, q. \end{aligned} \tag{27}$$

Step 2 Solve the bounded classical transportation model (28) using the bounded simplex algorithm:

$$E_l^* = \max \sum_{g=1}^{p}\sum_{k=1}^{q} E_{gk}^l x_{gk}^l$$
$$\text{subject to } \sum_{k=1}^{q} x_{gk}^l = s_g^l, \quad g = 1,\ldots,p,$$
$$\sum_{g=1}^{p} x_{gk}^l = d_k^l, \quad k = 1,\ldots,q,$$
$$x_{gk}^l \geq x_{gk}^{l'^*}, \quad g = 1,\ldots,p, \quad k = 1,\ldots,q, \tag{28}$$

where $x^{l'^*} = (x_{gk}^{l'^*})_{kj\times 1}$ is an optimum solution of model (27).

Step 3 Solve the bounded classical transportation model (29) using the bounded simplex algorithm:

$$E_m^* = \max \sum_{g=1}^{p}\sum_{k=1}^{q} E_{gk}^m x_{gk}^m$$
$$\text{subject to } \sum_{k=1}^{q} x_{gk}^m = s_g^m, \quad g = 1,\ldots,p,$$
$$\sum_{g=1}^{p} x_{gk}^m = d_k^m, \quad k = 1,\ldots,q,$$
$$x_{gk}^m \geq x_{gk}^{l^*}, \quad g = 1,\ldots,p, \quad k = 1,\ldots,q, \tag{29}$$

where $x^{l^*} = (x_{gk}^{l^*})_{kj\times 1}$ is an optimum solution of model (28).

Step 4 Solve the bounded classical transportation model (30) using the bounded simplex algorithm:

$$E_r^* = \max \sum_{g=1}^{p}\sum_{k=1}^{q} E_{gk}^r x_{gk}^r$$
$$\text{subject to } \sum_{k=1}^{q} x_{gk}^r = s_g^r, \quad g = 1,\ldots,p,$$
$$\sum_{g=1}^{p} x_{gk}^r = d_k^r, \quad k = 1,\ldots,q,$$
$$x_{gk}^r \geq x_{gk}^{m^*}, \quad g = 1,\ldots,p, \quad k = 1,\ldots,q, \tag{30}$$

where $x^{m^*} = (x_{gk}^{m^*})_{kj\times 1}$ is an optimum solution of the model (29).

Step 5 Solve the bounded classical transportation model (31) using the bounded simplex algorithm:

$$E_{r'}^* = \max \sum_{g=1}^{p}\sum_{k=1}^{q} E_{gk}^{r'} x_{gk}^{r'}$$
$$\text{subject to } \sum_{k=1}^{q} x_{gk}^{r'} = s_g^{r'}, \quad g = 1,\ldots,p,$$
$$\sum_{g=1}^{p} x_{gk}^{r'} = d_k^{r'}, \quad k = 1,\ldots,q,$$
$$x_{gk}^{r'} \geq x_{gk}^{r^*}, \quad g = 1,\ldots,p, \quad k = 1,\ldots,q, \tag{31}$$

where $x^{r^*} = (x_{gk}^{r^*})_{kj\times 1}$ is an optimum solution of the transportation model (30).

Step 6 Determine the Fermatean fuzzy optimum solution $\tilde{x}_{gk}^*$ by inserting the values of $\tilde{x}_{gk}^{l'^*}, \tilde{x}_{gk}^{l^*}, \tilde{x}_{gk}^{m^*}, \tilde{x}_{gk}^{r^*}$ and $\tilde{x}_{gk}^{r'^*}$ (obtained from step 1 to 5, respectively) in $\tilde{x}_{gk}^* = \{(\tilde{x}_{gk}^{l^*}, \tilde{x}_{gk}^{m^*}, \tilde{x}_{gk}^{r^*}); (\tilde{x}_{gk}^{l'^*}, \tilde{x}_{gk}^{m^*}), \tilde{x}_{gk}^{r'^*}\}$.

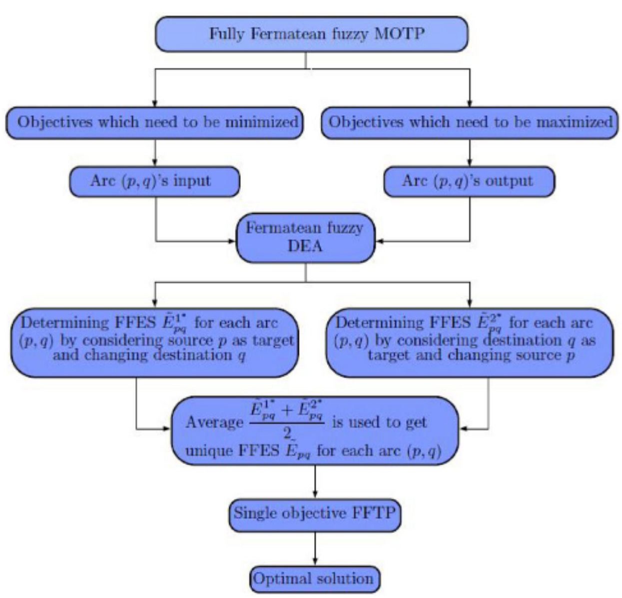

Fig. 2 Criteria for converting the Fermatean fuzzy MOTP into single-objective FFTP

The criteria for converting the Fermatean fuzzy MOTP into single-objective FFTP are given in Fig. 2.

Remark 4 It can be verified that $0 \leq E_{l'}^* \leq E_l^* \leq E_m^* \leq E_r^* \leq E_{r'}^*$. Consequently, the non-negative form of the TFFN is preserved by the total transportation FFES $\tilde{E}^* = \{(E_l^*, E_m^*, E_r^*); (E_{l'}^*, E_m^*, E_{r'}^*)\}$.

5 Numerical example

Example 1 An online dairy shop is selling fresh dairy items such as yoghurts and milk-based desserts in three blocks A_1, A_2 and A_3 of a town. They must deliver their goods in four different blocks B_1, B_2, B_3 and B_4 of another town within 12 h after purchasing online. The primary goal is to maximize profit while reducing time for delivery and transportation cost along a certain route. Table 2 shows the time for delivery, transportation cost, and net transportation profit per unit from one town to another as TFFNs, symbolized by $\{(a_l, a_m, a_r); (a_l', a_m, a_r')\}$. Furthermore, the last

Table 2 Data of example

		B_1	B_2	B_3	B_4	s_g
	Profit	{(120, 125, 130); (115, 125, 135)}	{(110, 115, 120); (105, 115, 125)}	{(107, 112, 119); (102, 112, 123)}	{(95, 106, 115); (90, 106, 123)}	
A_1	Delivery t	{(2.5, 3, 3.5); (2.3, 3, 3.8)}	{(2.9, 3.5, 4); (2.5, 3.5, 4.4)}	{(3.3, 4, 4.9); (3, 4, 5.3)}	{(3, 3.3, 4); (2.8, 3.3, 4.2)}	{(105, 110, 115); (100, 110, 120)}
	Trans. cost	{(3.5, 8, 9); (3, 8, 9.3)}	{(4.5, 7, 8.6); (3, 7, 9)}	{(5.5, 9, 10.5); (3, 9, 11)}	{(4.6, 6, 7.5); (4, 6, 8.1)}	
	Profit	{(107,116,125);(105,116,130)}	{(78,85,92);(75,85,98)}	{(69,75,80);(65,75,84)}	{(82,87,90);(79,87,95)}	
A_2	Delivery t	{(3.7,4,4.3);(3.4,4,4.6)}	{(4.5,5,6);(3.7,5,5.9)}	{(2.4,3.4,3.9);(2.1,3.4,4)}	{(3.7,4.1,4.5);(3.4,4.1,4.8)}	{(64,70,75);(60,70,80)}
	Trans. cost	{(8,11.5,12.8);(7.5,11.5,13)}	{(7,8,9);(6.7,8,9.5)}	{(8.3,9,9.7);(8,9,10)}	{(8,8.6,9.1);(7.5,8.6,9.5)}	
	Profit	{(119,125,132);(115,125,135)}	{(106,114,119);(102,114,123)}	{(109,115,121);(105,115,126)}	{(89,95,99);(85,95,104)}	
A_3	Delivery t	{(2.3,2.5,2.8);(2,2.5,3)}	{(2.5,2.9,3.2);(2.3,2.9,3.5)}	{(3.1,3.4,3.7);(3,3.4,4)}	{(3.5,4,4.5);(3.2,4,4.9)}	{(69,80,89);(65,80,97)}
	Trans. cost	{(7.1,7.5,8);(6.8,7.5,8.2)}	{(6.3,6.7,6.9);(6,6.7,7.2)}	{(7.5,8,8.5);(7,8,9)}	{(8,8.3,8.7);(7.8,8.3,9)}	
	d_k	{(74,85,94);(70,85,102)}	{(59,65,70);(55,65,75)}	{(60,62,64);(58,62,67)}	{(45,48,51);(42,48,53)}	

column and row of this table show the Fermatean fuzzy demand and supply amounts, respectively.

Model (16) may be used to calculate the FFESs $\tilde{E}^{1^*}_{A_gB_k}$s $(g = 1, 2, 3, k = 1, 2, 3, 4)$ by taking the time for delivery and transportation cost as input 1 and input 2, respectively, and transportation profit as output 1.

Step 1 The values of $\tilde{E}^{1^*}_{A_gB_k}$ are determined by treating the source A_g as the target and varying the destinations. For example, the model to find $\tilde{E}^{1^*}_{A_1B_k}$s $(k = 1, 2, 3, 4)$ is as follows (where, $\varepsilon = 10^{-6}$):

$$
\begin{aligned}
&\max \quad \phi_1\\
&\text{subject to} \quad 115u_1 - 3.8v_1 - 9.3v_2 \geq \phi_1,\\
&\qquad 105u_1 - 4.4v_1 - 9v_2 \geq \phi_1,\\
&\qquad 102u_1 - 5.3v_1 - 11v_2 \geq \phi_1,\\
&\qquad 90u_1 - 4.2v_1 - 8.1v_2 \geq \phi_1,\\
&\qquad 135u_1 - 2.3v_1 - 3v_2 \leq 0,\\
&\qquad 125u_1 - 2.5v_1 - 3v_2 \leq 0,\\
&\qquad 123u_1 - 3v_1 - 3v_2 \leq 0,\\
&\qquad 123u_1 - 2.8v_1 - 4v_2 \leq 0,\\
&\qquad u_1 + v_1 + v_2 = 1, \quad u_1, v_1, v_2 \geq \varepsilon.
\end{aligned}
\tag{32}
$$

Optimum value of model is $\phi_1^* = -3.50255$. The MOLP model (33) can be solved using a lexicographic technique to find a unique optimum solution of the above model:

$$
\begin{aligned}
&\min \quad (Lex\{u_1, -v_1, -v_2\})\\
&\text{subject to} \quad 115u_1 - 3.8v_1 - 9.3v_2 \geq -3.50255,\\
&\qquad 105u_1 - 4.4v_1 - 9v_2 \geq -3.50255,\\
&\qquad 102u_1 - 5.3v_1 - 11v_2 \geq -3.50255,\\
&\qquad 90u_1 - 4.2v_1 - 8.1v_2 \geq -3.50255,\\
&\qquad 135u_1 - 2.3v_1 - 3v_2 \leq 0,\\
&\qquad 125u_1 - 2.5v_1 - 3v_2 \leq 0,\\
&\qquad 123u_1 - 3v_1 - 3v_2 \leq 0,\\
&\qquad 123u_1 - 2.8v_1 - 4v_2 \leq 0,\\
&\qquad u_1 + v_1 + v_2 = 1, \quad u_1, v_1, v_2 \geq \varepsilon.
\end{aligned}
\tag{33}
$$

The above model yields a unique optimum solution $(u_1^*, v_1^*, v_2^*) = (0.016752, 0.98325, 10^{-6})$. Therefore, the FFES can be calculated using the equation as follows:

$$\tilde{E}^{1^*}_{A_1B_k} = \frac{0.016752\tilde{y}^1_{A_1B_k}}{0.98325\tilde{x}^1_{A_1B_k} + 10^{-6}\tilde{x}^2_{A_1B_k}}$$
$$= \left\{ \left(\frac{0.016752\tilde{y}^{1,l}_{A_1B_k}}{0.98325\tilde{x}^{1,r}_{A_1B_k} + 10^{-6}\tilde{x}^{2,r}_{A_1B_k}}, \frac{0.016752\tilde{y}^{1,m}_{A_1B_k}}{0.98325\tilde{x}^{1,m}_{A_1B_k} + 10^{-6}\tilde{x}^{2,m}_{A_1B_k}}, \frac{0.016752\tilde{y}^{1,r}_{A_1B_k}}{0.98325\tilde{x}^{1,l}_{A_1B_k} + 10^{-6}\tilde{x}^{2,r}_{A_1B_k}} \right); \left(\frac{0.016752\tilde{y}^{1,l'}_{A_1B_k}}{0.98325\tilde{x}^{1,r'}_{A_1B_k} + 10^{-6}\tilde{x}^{2,r'}_{A_1B_k}}, \frac{0.016752\tilde{y}^{1,m}_{A_1B_k}}{0.98325\tilde{x}^{1,m}_{A_1B_k} + 10^{-6}\tilde{x}^{2,m}_{A_1B_k}}, \frac{0.016752\tilde{y}^{1,r'}_{A_1B_k}}{0.98325\tilde{x}^{1,l'}_{A_1B_k} + 10^{-6}\tilde{x}^{2,l'}_{A_1B_k}} \right) \right\}. \tag{34}$$

Similarly, the values of the $\tilde{E}^{1^*}_{A_gB_k}$ can be determined for remaining arcs by altering the source A_g. The required results are given in Table 3.

Step 2 Similarly, the values of $\tilde{E}^{2^*}_{A_gB_k}(g = 1, 2, 3, k = 1, 2, 3)$ are obtained by treating destinations k as target and varying the sources. To obtain $\tilde{E}^{2^*}_{A_gB_1}(g = 1, 2, 3)$, we solve

$$\begin{aligned} \max \quad & \phi_2 \\ \text{subject to} \quad & 115u_1 - 3.8v_1 - 9.3v_2 \geq \phi_2, \\ & 105u_1 - 4.6v_1 - 13v_2 \geq \phi_2, \\ & 115u_1 - 3v_1 - 8.2v_2 \geq \phi_2, \\ & 135u_1 - 2.3v_1 - 3v_2 \leq 0, \\ & 130u_1 - 3.4v_1 - 7.5v_2 \leq 0, \\ & 135u_12 - v_1 - 6.8v_2 \leq 0, \\ & u_1 + v_1 + v_2 = 1, \quad u_1, v_1, v_2 \geq \varepsilon. \end{aligned} \tag{35}$$

Optimum value of model is $\phi_2^* = -3$. The MOLP model (36) is solved using a lexicographic technique to find a unique optimum solution of this model:

$$\begin{aligned} \min \quad & (Lex\{u_1, -v_1, -v_2\}) \\ \text{subject to} \quad & 115u_1 - 3.8v_1 - 9.3v_2 \geq -3, \\ & 105u_1 - 4.6v_1 - 13v_2 \geq -3, \\ & 115u_1 - 3v_1 - 8.2v_2 \geq -3, \\ & 135u_1 - 2.3v_1 - 3v_2 \leq 0, \\ & 130u_1 - 3.4v_1 - 7.5v_2 \leq 0, \\ & 135u_12 - v_1 - 6.8v_2 \leq 0, \\ & u_1 + v_1 + v_2 = 1, \quad u_1, v_1, v_2 \geq \varepsilon. \end{aligned} \tag{36}$$

The above model provides a unique optimum solution $(u_1^*, v_1^*, v_2^*) = (0.014599, 0.9854, 10^{-6})$. Then, the FFES can be calculated using the equation as follows:

$$\tilde{E}^{2^*}_{A_gB_1} = \frac{0.014599\tilde{y}^1_{A_gB_1}}{0.9854\tilde{x}^1_{A_gB_1} + 10^{-6}\tilde{x}^2_{A_gB_1}}$$
$$= \left\{ \left(\frac{0.014599\tilde{y}^{1,l}_{A_gB_1}}{0.9854\tilde{x}^{1,r}_{A_gB_1} + 10^{-6}\tilde{x}^{2,r}_{A_gB_1}}, \frac{0.014599\tilde{y}^{1,m}_{A_gB_1}}{0.9854\tilde{x}^{1,m}_{A_gB_1} + 10^{-6}\tilde{x}^{2,m}_{A_gB_1}}, \frac{0.014599\tilde{y}^{1,r}_{A_gB_1}}{0.9854\tilde{x}^{1,l}_{A_gB_1} + 10^{-6}\tilde{x}^{2,l}_{A_gB_1}} \right); \left(\frac{0.014599\tilde{y}^{1,l'}_{A_gB_1}}{0.9854\tilde{x}^{1,r'}_{A_gB_1} + 10^{-6}\tilde{x}^{2,r'}_{A_gB_1}}, \frac{0.014599\tilde{y}^{1,m}_{A_gB_1}}{0.9854\tilde{x}^{1,m}_{A_gB_1} + 10^{-6}\tilde{x}^{2,m}_{A_gB_1}}, \frac{0.014599\tilde{y}^{1,r'}_{A_gB_1}}{0.9854\tilde{x}^{1,l'}_{A_gB_1} + 10^{-6}\tilde{x}^{2,l'}_{A_gB_1}} \right) \right\}. \tag{37}$$

By changing the destinations B_k, the values of $\tilde{E}^{2^*}_{A_gB_k}$ are determined for remaining arcs listed in Table 4.

Step 3 Now, by finding the mean of FFESs $\tilde{E}^{1^*}_{A_gB_k}$s and $\tilde{E}^{2^*}_{A_gB_k}$s, the FFESs $\tilde{E}^{*}_{A_gB_k}$s can be determined, that are represented in Table 5.

Table 3 Values of $\tilde{E}^{1^*}_{A_g B_k}$

	B_1	B_2	B_3	B_4
A_1	{(0.58, 0.71, 0.89); (0.52, 0.71, 1.00)}	{(0.47,0.56,0.70);(0.41,0.56,0.85)}	{(0.37,0.48,0.61);(0.33,0.48,0.70)}	{(0.40,0.55,0.64);(0.37,0.55,0.75)}
A_2	{(0.59,0.69,0.80);(0.54,0.69,0.91)}	{(0.33,0.40,0.55);(0.30,0.40,0.63)}	{(0.42,0.52,0.79);(0.39,0.52,0.95)}	{(0.43,0.50,0.58);(0.39,0.50,0.67)}
A_3	{(0.63,0.74,0.85);(0.57,0.74,1.00)}	{(0.49,0.58,0.71);(0.43,0.58,0.79)}	{(0.44,0.50,0.58);(0.39,0.50,0.62)}	{(0.29,0.35,0.42);(0.26,0.35,0.48)}

Table 4 Values of $\tilde{E}^{2^*}_{A_g B_k}$

	B_1	B_2	B_3	B_4
A_1	{(0.51,0.62,0.77);(0.45,0.62,0.87)}	{(0.51,0.61,0.77);(0.45,0.61,0.93)}	{(0.52,0.67,0.86);(0.46,0.67,0.98)}	{(0.54,0.73,0.86);(0.49,0.73,1.00)}
A_2	{(0.37,0.43,0.50);(0.34,0.43,0.57)}	{(0.26,0.32,0.43);(0.24,0.32,0.50)}	{(0.42,0.52,0.79);(0.39,0.52,0.94)}	{(0.41,0.48,0.55);(0.37,0.48,0.64)}
A_3	{(0.63,0.74,0.85);(0.57,0.74,1.00)}	{(0.62,0.74,0.89);(0.54,0.74,1.00)}	{(0.70,0.80,0.93);(0.63,0.80,1.00)}	{(0.45,0.54,0.64);(0.39,0.54,0.74)}

Table 5 Values of $\tilde{E}^{2^*}_{A_gB_k}$

	B_1	B_2	B_3	B_4
A_1	{(0.55,0.67,0.83);(0.48,0.67,0.94)}	{(0.49,0.59,0.74);(0.43,0.59,0.89)}	{(0.45,0.58,0.74);(0.40,0.58,0.84)}	{(0.47,0.64,0.75);(0.43,0.64,0.88)}
A_2	{(0.48,0.56,0.65);(0.44,0.56,0.74)}	{(0.30,0.36,0.49);(0.27,0.36,0.57)}	{(0.42,0.52,0.79);(0.39,0.52,0.95)}	{(0.42,0.49,0.57);(0.38,0.49,0.66)}
A_3	{(0.63,0.74,0.85);(0.57,0.74,1.00)}	{(0.56,0.66,0.8);(0.49,0.66,0.90)}	{(0.57,0.65,0.76);(0.51,0.65,0.81)}	{(0.37,0.45,0.53);(0.33,0.45,0.61)}

Therefore, the three Fermatean fuzzy objectives for every arc are transformed into unique positive Fermatean fuzzy attribute $\tilde{E}^*_{A_gB_k}$. Therefore, to determine a solution of the current fully FFMOTP, it is enough to solve the model:

$$\begin{aligned}
\tilde{E}^* &= \max \quad \sum_{g=1}^{3}\sum_{k=1}^{4} \tilde{E}^*_{A_gB_k}\tilde{x}_{gk} \\
\text{subject to} &\quad \sum_{k=1}^{4} \tilde{x}_{gk} \approx \tilde{s}_{A_g}, \quad g = 1,2,3, \\
&\quad \sum_{g=1}^{3} \tilde{x}_{gk} =\approx \tilde{d}_{B_k}, \quad k = 1,2,3,4, \\
&\quad \tilde{x}_{gk} \succeq \tilde{0}, \quad g = 1,2,3, \quad k = 1,2,3,4.
\end{aligned} \tag{38}$$

To solve the above model based on approach (Akram et al. 2023a), the steps are as follows:

Step 1 First, we solve the crisp model (39) (note that the transportation model (38) is balanced):

$$\begin{aligned}
E^*_{l'} = \max \quad & 0.48x^{l'}_{A_1B_1} + 0.43x^{l'}_{A_1B_2} + 0.40x^{l'}_{A_1B_3} \\
& + 0.43x^{l'}_{A_1B_4} + 0.44x^{l'}_{A_2B_1} + 0.27x^{l'}_{A_2B_2} \\
& + 0.39x^{l'}_{A_2B_3} + 0.38x^{l'}_{A_2B_4} + 0.57x^{l'}_{A_3B_1} \\
& + 0.49x^{l'}_{A_3B_2} + 0.51x^{l'}_{A_3B_3} + 0.33x^{l'}_{A_3B_4} \\
\text{subject to} & \\
& x^{l'}_{A_1B_1} + x^{l'}_{A_1B_2} + x^{l'}_{A_1B_3} + x^{l'}_{A_1B_4} = 100, \\
& x^{l'}_{A_2B_1} + x^{l'}_{A_2B_2} + x^{l'}_{A_2B_3} + x^{l'}_{A_2B_4} = 60, \\
& x^{l'}_{A_3B_1} + x^{l'}_{A_3B_2} + x^{l'}_{A_3B_3} + x^{l'}_{A_3B_4} = 65, \\
& x^{l'}_{A_1B_1} + x^{l'}_{A_2B_1} + x^{l'}_{A_3B_1} = 70, \\
& x^{l'}_{A_1B_2} + x^{l'}_{A_2B_2} + x^{l'}_{A_3B_2} = 55, \\
& x^{l'}_{A_1B_3} + x^{l'}_{A_2B_3} + x^{l'}_{A_3B_3} = 58, \\
& x^{l'}_{A_1B_4} + x^{l'}_{A_2B_4} + x^{l'}_{A_3B_4} = 42, \\
& x^{l'}_{A_gB_k} \geq 0, \quad g = 1,2,3, \quad k = 1,2,3,4.
\end{aligned} \tag{39}$$

The optimum solution of model (39) is listed in Table 6.

Table 6 Values of $x^{l'^*}_{A_gB_k}$ and $x^{l^*}_{A_gB_k}$

$x^{l'^*}_{A_1B_k}$	3	55	0	42	$x^{l^*}_{A_1B_k}$	4	59	42	0
$x^{l'^*}_{A_2B_k}$	2	0	58	0	$x^{l^*}_{A_2B_k}$	3	0	58	3
$x^{l'^*}_{A_3B_k}$	65	0	0	0	$x^{l^*}_{A_3B_k}$	67	0	2	0

Step 2 The bounded crisp TP should then be solved using the above model's solution:*

$$\begin{aligned} E_l^* = \max\ & 0.55x^l_{A_1B_1} + 0.49x^l_{A_1B_2} + 0.45x^l_{A_1B_3} \\ & + 0.47x^l_{A_1B_4} + 0.48x^l_{A_2B_1} + 0.30x^l_{A_2B_2} \\ & + 0.42x^l_{A_2B_3} + 0.42x^l_{A_2B_4} + 0.63x^l_{A_3B_1} \\ & + 0.56x^l_{A_3B_2} + 0.57x^l_{A_3B_3} + 0.37x^l_{A_3B_4} \\ \text{subject to}\ & x^l_{A_1B_1} + x^l_{A_1B_2} + x^l_{A_1B_3} + x^l_{A_1B_4} = 105, \\ & x^l_{A_1B_1} + x^l_{A_1B_2} + x^l_{A_1B_3} + x^l_{A_1B_4} = 64, \\ & x^l_{A_1B_1} + x^l_{A_1B_2} + x^l_{A_1B_3} + x^l_{A_1B_4} = 69, \\ & x^l_{A_1B_1} + x^l_{A_2B_1} + x^l_{A_3B_1} = 74, \\ & x^l_{A_1B_2} + x^l_{A_2B_2} + x^l_{A_3B_2} = 59, \\ & x^l_{A_1B_3} + x^l_{A_2B_3} + x^l_{A_3B_3} = 60, \\ & x^l_{A_1B_4} + x^l_{A_2B_4} + x^l_{A_3B_4} = 45, \\ & x^l_{A_gB_k} \geq x^{l'*}_{A_gB_k}, \quad g = 1,2,3, \\ & k = 1,2,3,4. \end{aligned} \tag{40}$$

The optimum solution of model (40) is represented in Table 6.

Step 3 The bounded crisp TP should then be solved using the model's output described above:

$$\begin{aligned} E_m^* = \max\ & 0.67x^m_{A_1B_1} + 0.59x^m_{A_1B_2} + 0.58x^m_{A_1B_3} \\ & + 0.64x^m_{A_1B_4} + 0.56x^m_{A_2B_1} + 0.36x^m_{A_2B_2} \\ & + 0.52x^m_{A_2B_3} + 0.49x^m_{A_2B_4} + 0.74x^m_{A_3B_1} \\ & + 0.66x^m_{A_3B_2} + 0.65x^m_{A_3B_3} + 0.45x^m_{A_3B_4} \\ \text{subject to}\ & x^m_{A_1B_1} + x^m_{A_1B_2} + x^m_{A_1B_3} + x^m_{A_1B_4} = 110, \\ & x^m_{A_1B_1} + x^m_{A_1B_2} + x^m_{A_1B_3} + x^m_{A_1B_4} = 70, \\ & x^m_{A_1B_1} + x^m_{A_1B_2} + x^m_{A_1B_3} + x^m_{A_1B_4} = 80, \\ & x^m_{A_1B_1} + x^m_{A_2B_1} + x^m_{A_3B_1} = 85, \\ & x^m_{A_1B_2} + x^m_{A_2B_2} + x^m_{A_3B_2} = 65, \\ & x^m_{A_1B_3} + x^m_{A_2B_3} + x^m_{A_3B_3} = 62, \\ & x^m_{A_1B_4} + x^m_{A_2B_4} + x^m_{A_3B_4} = 48, \\ & x^m_{A_gB_k} \geq x^{l*}_{A_gB_k}, \quad g = 1,2,3, \\ & k = 1,2,3,4. \end{aligned} \tag{41}$$

The optimum solution of model (41) is shown in Table 7.

Step 4 Now, the bounded crisp TP should be solved using the model (41)'s solution:

$$\begin{aligned} E_r^* = \max\ & 0.83x^r_{A_1B_1} + 0.74x^r_{A_1B_2} + 0.74x^r_{A_1B_3} \\ & + 0.75x^r_{A_1B_4} + 0.65x^r_{A_2B_1} + 0.49x^r_{A_2B_2} \\ & + 0.79x^r_{A_2B_3} + 0.57x^r_{A_2B_4} + 0.85x^r_{A_3B_1} \\ & + 0.80x^r_{A_3B_2} + 0.76x^r_{A_3B_3} + 0.53x^r_{T_3B_4} \\ \text{subject to}\ & x^r_{A_1B_1} + x^r_{A_1B_2} + x^r_{A_1B_3} + x^r_{A_1B_4} = 115, \\ & x^r_{A_1B_1} + x^r_{A_1B_2} + x^r_{A_1B_3} + x^r_{A_1B_4} = 75, \\ & x^r_{A_1B_1} + x^r_{A_1B_2} + x^r_{A_1B_3} + x^r_{A_1B_4} = 89, \\ & x^r_{A_1B_1} + x^r_{A_2B_1} + x^r_{A_3B_1} = 94, \\ & x^r_{A_1B_2} + x^r_{A_2B_2} + x^r_{A_3B_2} = 70, \\ & x^r_{A_1B_3} + x^r_{A_2B_3} + x^r_{A_3B_3} = 64, \\ & x^r_{A_1B_4} + x^r_{A_2B_4} + x^r_{A_3B_4} = 51, \\ & x^r_{A_gB_k} \geq x^{m*}_{A_gB_k}, \quad g = 1,2,3, \\ & k = 1,2,3,4. \end{aligned} \tag{42}$$

The optimum solution of model (42) is given in Table 7.

Step 5 The bounded crisp TP should then be solved using the model (42)'s solution:

$$\begin{aligned} E_{r'}^* = \max\ & 0.94x^{r'}_{A_1B_1} + 0.89x^{r'}_{A_1B_2} + 0.84x^{r'}_{A_1B_3} \\ & + 0.88x^{r'}_{A_1B_4} + 0.74x^{r'}_{A_2B_1} + 0.57x^{r'}_{A_2B_2} \\ & + 0.95x^{r'}_{A_2B_3} + 0.66x^{r'}_{A_2B_4} + 1.00x^{r'}_{A_3B_1} \\ & + 0.90x^{r'}_{A_3B_2} + 0.81x^{r'}_{A_3B_3} + 0.61x^{r'}_{A_3B_4} \\ \text{subject to}\ & x^{r'}_{A_1B_1} + x^{r'}_{A_1B_2} + x^{r'}_{A_1B_3} + x^{r'}_{A_1B_4} = 120, \\ & x^{r'}_{A_2B_1} + x^{r'}_{A_2B_2} + x^{r'}_{A_2B_3} + x^{r'}_{A_2B_4} = 80, \\ & x^{r'}_{A_3B_1} + x^{r'}_{A_3B_2} + x^{r'}_{A_3B_3} + x^{r'}_{A_3B_4} = 97, \\ & x^{r'}_{A_1B_1} + x^{r'}_{A_2B_1} + x^{r'}_{A_3B_1} = 102, \\ & x^{r'}_{A_1B_2} + x^{r'}_{A_2B_2} + x^{r'}_{A_3A_2} = 75, \\ & x^{r'}_{A_1B_3} + x^{r'}_{A_2B_3} + x^{r'}_{A_3B_3} = 67, \\ & x^{r'}_{A_1B_4} + x^{r'}_{A_2B_4} + x^{r'}_{A_3B_4} = 53, \\ & x^{r'}_{A_gB_k} \geq x^{r*}_{A_gB_k}, \quad g = 1,2,3, \\ & k = 1,2,3,4. \end{aligned} \tag{43}$$

The optimum solution of model (43) is given in Table 7.

Step 6 Fermatean fuzzy optimum solution of the example is shown in Table 8.

The membership and non-membership of objective functions (transportation profit, time for delivery and transportation cost) related to the Fermatean fuzzy optimum solution are shown in Figs. 3, 4 and 5, respectively.

Table 7 Values of $x^{m^*}_{A_gB_k}$, $x^{r^*}_{A_gB_k}$ and $x^{r'^*}_{A_gB_k}$

$x^{m^*}_{A_1B_k}$	6	59	0	45	$x^{r^*}_{A_1B_k}$	11	59	0	45	$x^{r'^*}_{A_1B_k}$	11	64	0	45
$x^{m^*}_{A_2B_k}$	7	0	60	3	$x^{r^*}_{A_2B_k}$	7	0	62	6	$x^{r'^*}_{A_2B_k}$	7	0	65	8
$x^{m^*}_{A_3B_k}$	72	6	2	0	$x^{r^*}_{A_3B_k}$	76	11	2	0	$x^{r'^*}_{A_3B_k}$	84	11	2	0

5.1 Performance analysis

Consider an example in which a transport company wants to transport a specific brand of leather shoes from three warehouses to its three stores. $\{(74,85,94);(70,85,102)\}$, $\{(59,65,70);(55,65,75)\}$, $\{(60,62,64);(58,62,67)\}$ and $\{(75,78,81);(72,78,83)\}$, $\{(60,66,71);(56,66,76)\}$, $\{(58,68,76);(55,68,85)\}$ are estimated demand levels of stores and capacities of warehouses, respectively. The total transportation profit, transportation cost and time for delivery through the selected route are given in Table 9.

Model (16) may be used to calculate the FFESs $\tilde{E}^{1^*}_{A_gB_k}$s $(g=1,2,3,\ k=1,2,3)$ by taking the delivery time and transportation cost as input 1 and input 2, respectively, and transportation profit as output 1. Note that the values of $\tilde{E}^{1^*}_{A_gB_k}$ are determined by treating the source A_g as the target. For example, the model to find $\tilde{E}^{1^*}_{A_1B_k}$s $(k=1,2,3,4)$ is as follows (where, $\varepsilon = 10^{-6}$):

$$\begin{aligned} \max\quad & \phi_1 \\ \text{subject to}\quad & 110u_1 - 3v_1 - 8v_2 \ge \phi_1, \\ & 75u_1 - 4v_1 - 8v_2 \ge \phi_1, \\ & 30u_1 - 3.4v_1 - 12v_2 \ge \phi_1, \\ & 135u_1 - v_1 - 6v_2 \le 0, \\ & 95u_1 - 2v_1 - 5v_2 \le 0, \\ & 70u_1 - 1.5v_1 - 9v_2 \le 0, \\ & u_1 + v_1 + v_2 = 1, \\ & u_1, v_1, v_2 \ge \varepsilon, (\text{where}\varepsilon = 10^{-6}). \end{aligned} \tag{44}$$

Optimum value of above model is $\phi_1^* = -3.41912$. The MOLP model (45) can be solved using a lexicographic technique to find a unique optimum solution of the above model as

$$\begin{aligned} \min\quad & (Lex\{u_1, -v_1, -v_2\}) \\ \text{subject to}\quad & 110u_1 - 3v_1 - 8v_2 \ge \phi_1^*, \\ & 75u_1 - 4v_1 - 8v_2 \ge \phi_1^*, \\ & 30u_1 - 3.4v_1 - 12v_2 \ge \phi_1^*, \\ & 135u_1 - v_1 - 6v_2 \le 0, \\ & 95u_1 - 2v_1 - 5v_2 \le 0, \\ & 70u_1 - 1.5v_1 - 9v_2 \le 0, \\ & u_1 + v_1 + v_2 = 1, \\ & u_1, v_1, v_2 \ge \varepsilon. \end{aligned} \tag{45}$$

The above model yields a unique optimum solution $(u_1^*, v_1^*, v_2^*) = (0.007353, 0.99265, 10^{-6})$. Therefore, the FFES can be determined from the equation as follows:

$$\begin{aligned} \tilde{E}^{1^*}_{A_1B_k} &= \frac{0.007353\tilde{y}^1_{A_1B_k}}{0.99265\tilde{x}^1_{A_1B_k} + 10^{-6}\tilde{x}^2_{A_1B_k}} \\ &= \left\{ \left(\frac{0.007353\tilde{y}^{1,l}_{A_1B_k}}{0.99265\tilde{x}^{1,r}_{A_1B_k} + 10^{-6}\tilde{x}^{2,r}_{A_1B_k}}, \frac{0.007353\tilde{y}^{1,m}_{A_1B_k}}{0.99265\tilde{x}^{1,m}_{A_1B_k} + 10^{-6}\tilde{x}^{2,m}_{A_1B_k}}, \frac{0.007353\tilde{y}^{1,r}_{A_1B_k}}{0.99265\tilde{x}^{1,l}_{A_1B_k} + 10^{-6}\tilde{x}^{2,r}_{A_1B_k}} \right); \right. \\ &\quad \left. \left(\frac{0.007353\tilde{y}^{1,l'}_{A_1B_k}}{0.99265\tilde{x}^{1,r'}_{A_1B_k} + 10^{-6}\tilde{x}^{2,r'}_{A_1B_k}}, \frac{0.007353\tilde{y}^{1,m}_{A_1B_k}}{0.99265\tilde{x}^{1,m}_{A_1B_k} + 10^{-6}\tilde{x}^{2,m}_{A_1B_k}}, \frac{0.007353\tilde{y}^{1,r'}_{A_1B_k}}{0.99265\tilde{x}^{1,l'}_{A_1B_k} + 10^{-6}\tilde{x}^{2,l'}_{A_1B_k}} \right) \right\}. \end{aligned} \tag{46}$$

Similarly, the values of the $\tilde{E}^{1^*}_{A_gB_k}$ can be determined for the remaining arcs by altering the source A_g. The corresponding results are shown in Table 10.

Similarly, the values of $\tilde{E}^{2^*}_{A_gB_k}(g=1,2,3,k=1,2,3)$ are obtained by treating destinations k as target. To find $\tilde{E}^{2^*}_{A_gB_1}(g=1,2,3)$, we solve

Table 8 Fermatean fuzzy optimum solution

$x^{r'^{*}}_{A_1B_k}$	{(4, 6, 11); (3, 6, 14)}	{(59, 59, 59); (55, 59, 64)}	{(0, 0, 0); (0, 0, 0)}	{(42, 45, 45); (42, 45, 45)}
$x^{r'^{*}}_{A_2B_k}$	{(3, 7, 7); (2, 7, 7)}	{(0, 0, 0); (0, 0, 0)}	{(58, 60, 62); (58, 60, 65)}	{(3, 3, 6); (0, 3, 8)}
$x^{r'^{*}}_{A_3B_k}$	{(67, 72, 76); (65, 72, 84)}	{(0, 6, 11); (0, 6, 11)}	{(2, 2, 2); (0, 2, 2)}	{(0, 0, 0); (0, 0, 0)}

$$\begin{aligned}
\max\quad & \phi_2 \\
\text{subject to}\quad & 110u_1 - 3v_1 - 8v_2 \geq \phi_2, \\
& 130u_1 - 4.5v_1 - 9.9v_2 \geq \phi_2, \\
& 90u_1 - 5.4v_1 - 11.4v_2 \geq \phi_2, \\
& 135u_1 - v_1 - 6v_2 \leq 0, \\
& 180u_1 - 2v_1 - 6.5v_2 \leq 0, \\
& 110u_1 - 3v_1 - 8v_2 \leq 0, \\
& u_1 + v_1 + v_2 = 1, \\
& u_1, v_1, v_2 \geq \varepsilon.
\end{aligned} \tag{47}$$

Optimum value of above model is $\phi_2^* = -4.69853$. The MOLP model (48) can be solved using a lexicographic technique to find a unique optimum solution of above model as

$$\begin{aligned}
\min\quad & (Lex\{u_1, -v_1, -v_2\}) \\
\text{subject to}\quad & 110u_1 - 3v_1 - 8v_2 \geq \phi_2^*, \\
& 130u_1 - 4.5v_1 - 9.9v_2 \geq \phi_2^*, \\
& 90u_1 - 5.4v_1 - 11.4v_2 \geq \phi_2^*, \\
& 135u_1 - v_1 - 6v_2 \leq 0, \\
& 180u_1 - 2v_1 - 6.5v_2 \leq 0, \\
& 110u_1 - 3v_1 - 8v_2 \leq 0, \\
& u_1 + v_1 + v_2 = 1, \\
& u_1, v_1, v_2 \geq \varepsilon.
\end{aligned} \tag{48}$$

The above model yields a unique optimum solution $(u_1^*, v_1^*, v_2^*) = (0.007353, 0.99265, 10^{-6})$. Then, the FFES can be calculated from the equation as follows:

By changing the destinations B_k, the values of $\tilde{E}^{2^*}_{A_gB_k}$ are determined for remaining arcs represented in Table 11.

Now, by finding the mean of the FFESs $\tilde{E}^{1^*}_{A_gB_k}$s and $\tilde{E}^{2^*}_{A_gB_k}$s, the FFESs $\tilde{E}^{*}_{A_gB_k}$s can be calculated that are given in Table 12.

Therefore, the three Fermatean fuzzy objectives for every arc are transformed into unique positive Fermatean fuzzy attribute $\tilde{E}^{*}_{A_gB_k}$. Then, to determine the solution of the current fully FFMOTP, we solve the model:

$$\begin{aligned}
\tilde{E}^{*} = \max\quad & \sum_{g=1}^{3}\sum_{k=1}^{3} \tilde{E}^{*}_{A_gB_k}\tilde{x}_{gk} \\
\text{subject to}\quad & \sum_{k=1}^{3} \tilde{x}_{gk} \approx \tilde{s}_{B_p}, \quad g = 1, 2, 3, \\
& \sum_{k=1}^{3} \tilde{x}_{gk} \approx \tilde{d}_{C_q}, \quad k = 1, 2, 3, \\
& \tilde{x}_{gk}, \quad g = 1, 2, 3, \quad k = 1, 2, 3.
\end{aligned} \tag{50}$$

To solve the model (50) based on approach (Akram et al. 2023a), we solve

$$\begin{aligned}
\tilde{E}^{*}_{l'} = \max\quad & 0.27x^{l'}_{11} + 0.27x^{l'}_{12} + 0.16x^{l'}_{13} \\
& + 0.27x^{l'}_{21} + 0.22x^{l'}_{22} + 0.29x^{l'}_{23} \\
& + 0.29x^{l'}_{31} + 0.14x^{l'}_{32} + 0.40x^{l'}_{33} \\
\text{subject to}\quad & x^{l'}_{11} + x^{l'}_{12} + x^{l'}_{13} = 72, \quad x^{l'}_{21} + x^{l'}_{22} + x^{l'}_{23} = 56, \\
& x^{l'}_{31} + x^{l'}_{32} + x^{l'}_{33} = 55, \quad x^{l'}_{11} + x^{l'}_{21} + x^{l'}_{31} = 70, \\
& x^{l'}_{12} + x^{l'}_{22} + x^{l'}_{32} = 55, \quad x^{l'}_{13} + x^{l'}_{23} + x^{l'}_{33} = 58, \\
& x^{l'}_{gk} \geq 0, \quad g = 1, 2, 3, \quad k = 1, 2, 3.
\end{aligned} \tag{51}$$

$$\begin{aligned}
\tilde{E}^{2^*}_{A_gB_1} &= \frac{0.007353\tilde{y}^{1}_{A_gB_1}}{0.99265\tilde{x}^{1}_{A_gB_1} + 10^{-6}\tilde{x}^{2}_{A_gB_1}} \\
&= \left\{ \left(\frac{0.007353\tilde{y}^{1,l}_{A_gB_1}}{0.99265\tilde{x}^{1,r}_{A_gB_1} + 10^{-6}\tilde{x}^{2,r}_{A_gB_1}}, \frac{0.007353\tilde{y}^{1,m}_{A_gB_1}}{0.99265\tilde{x}^{1,m}_{A_gB_1} + 10^{-6}\tilde{x}^{2,m}_{A_gB_1}}, \frac{0.007353\tilde{y}^{1,r}_{A_gB_1}}{0.99265\tilde{x}^{1,l}_{A_gB_1} + 10^{-6}\tilde{x}^{2,l}_{A_gB_1}} \right); \right. \\
&\quad \left. \left(\frac{0.007353\tilde{y}^{1,l'}_{A_gB_1}}{0.99265\tilde{x}^{1,r'}_{A_gB_1} + 10^{-6}\tilde{x}^{2,r'}_{A_gB_1}}, \frac{0.007353\tilde{y}^{1,m}_{A_gB_1}}{0.99265\tilde{x}^{1,m}_{A_gB_1} + 10^{-6}\tilde{x}^{2,m}_{A_gB_1}}, \frac{0.007353\tilde{y}^{1,r'}_{A_gB_1}}{0.99265\tilde{x}^{1,l'}_{A_gB_1} + 10^{-6}\tilde{x}^{2,l'}_{A_gB_1}} \right) \right\}.
\end{aligned} \tag{49}$$

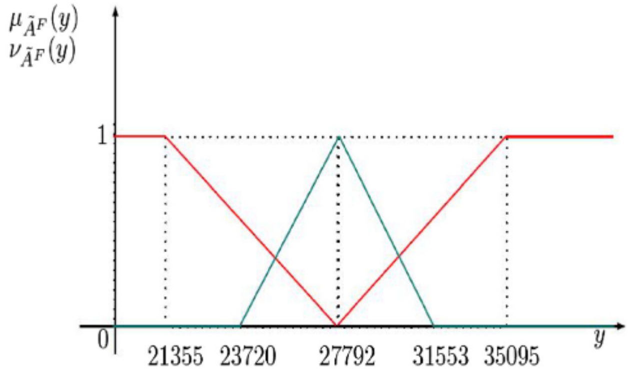

Fig. 3 Membership and non-membership functions of transportation profit

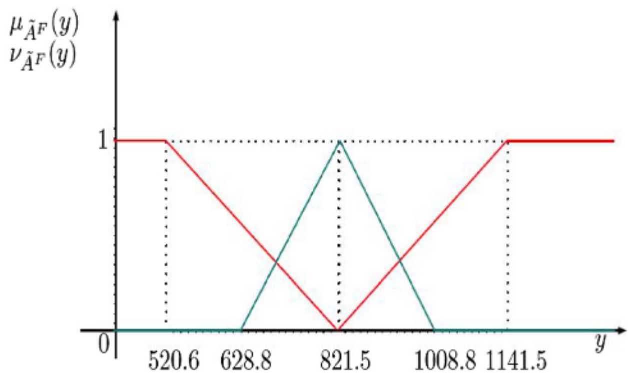

Fig. 4 Membership and non-membership functions of delivery time

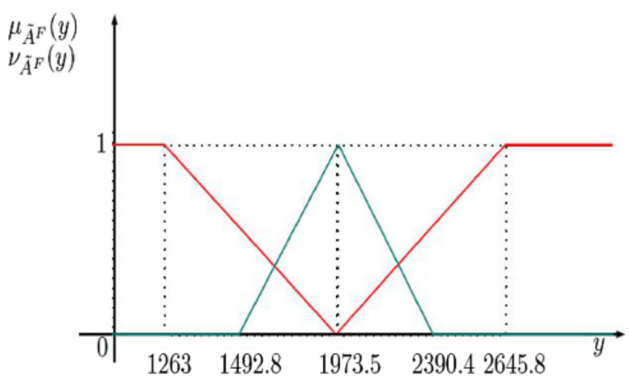

Fig. 5 Membership and non-membership functions of transportation cost

$$
\begin{aligned}
\tilde{E}^*_l = \max\ & 0.30x^l_{11} + 0.33x^l_{12} + 0.23x^l_{13} \\
& + 0.31x^l_{21} + 0.25x^l_{22} + 0.33x^l_{23} \\
& + 0.32x^l_{31} + 0.18x^l_{32} + 0.48x^l_{33} \\
\text{subject to}\ & x^l_{11} + x^l_{12} + x^l_{13} = 75, \quad x^l_{21} + x^l_{22} + x^l_{23} = 60, \\
& x^l_{31} + x^l_{32} + x^l_{33} = 58, \quad x^l_{11} + x^l_{21} + x^l_{31} = 74, \\
& x^l_{12} + x^l_{22} + x^l_{32} = 59, \quad x^l_{13} + x^l_{23} + x^l_{33} = 60, \\
& x^l_{11} \geq 17, \quad x^l_{12} \geq 55, \quad x^l_{21} \geq 53, \quad x^l_{23} \geq 3, \\
& x^l_{33} \geq 55, \\
& x^l_{gk} \geq x^{l'*}_{gk}, \quad g = 1,2,3, \quad k = 1,2,3.
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
\tilde{E}^*_m = \max\ & 0.46x^m_{11} + 0.41x^m_{12} + 0.30x^m_{13} + 0.61x^m_{21} \\
& + 0.37x^m_{22} + 0.47x^m_{23} + 0.44x^m_{31} \\
& + 0.24x^m_{32} + 0.60x^m_{33} \\
\text{subject to}\ & x^m_{11} + x^m_{12} + x^m_{13} = 78, \quad x^m_{21} + x^m_{22} + x^m_{23} = 66, \\
& x^m_{31} + x^m_{32} + x^m_{33} = 68, \quad x^m_{11} + x^m_{21} + x^m_{31} = 85, \\
& x^m_{12} + x^m_{22} + x^m_{32} = 65, \quad x^m_{13} + x^m_{23} + x^m_{33} = 62, \\
& x^m_{11} \geq 17, \quad x^m_{12} \geq 58, \quad x^m_{21} \geq 56, \quad x^m_{22} \geq 1, \\
& x^m_{23} \geq 3, \quad x^m_{31} \geq 1, \quad x^m_{33} \geq 57, \\
& x^m_{gk} \geq x^{l*}_{gk}, \quad g = 1,2,3, \quad k = 1,2,3.
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
\tilde{E}^*_r = \max\ & 0.64x^r_{11} + 0.52x^r_{12} + 0.48x^r_{13} \\
& + 0.63x^r_{21} + 0.58x^r_{22} + 0.51x^r_{23} \\
& + 0.54x^r_{31} + 0.34x^r_{32} + 0.68x^r_{33} \\
\text{subject to}\ & x^r_{11} + x^r_{12} + x^r_{13} = 81, \quad x^r_{21} + x^r_{22} + x^r_{23} = 71, \\
& x^r_{31} + x^r_{32} + x^r_{33} = 76, \quad x^r_{11} + x^r_{21} + x^r_{31} = 94, \\
& x^r_{12} + x^r_{22} + x^r_{32} = 70, \quad x^r_{13} + x^r_{23} + x^r_{33} = 64, \\
& x^r_{11} \geq 17, \quad x^r_{12} \geq 61, \quad x^r_{21} \geq 62, \quad x^r_{22} \geq 1, \\
& x^r_{23} \geq 3, \quad x^r_{31} \geq 6, \quad x^r_{32} \geq 3, \quad x^r_{33} \geq 59, \\
& x^r_{gk} \geq x^{m*}_{gk}, \quad g = 1,2,3, \quad k = 1,2,3.
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\tilde{E}^*_{r'} = \max\ & x^{r'}_{11} + 0.68x^{r'}_{12} + 0.68x^{r'}_{13} + 0.84x^{r'}_{21} + 0.77x^{r'}_{22} \\
& + 0.61x^{r'}_{23} + 0.64x^{r'}_{31} + 0.39x^{r'}_{32} + 0.80x^{r'}_{33} \\
\text{subject to}\ & x^{r'}_{11} + x^{r'}_{12} + x^{r'}_{13} = 83, \quad x^{r'}_{21} + x^{r'}_{22} + x^{r'}_{23} = 76, \\
& x^{r'}_{31} + x^{r'}_{32} + x^{r'}_{33} = 85, \quad x^{r'}_{11} + x^{r'}_{21} + x^{r'}_{31} = 102, \\
& x^{r'}_{12} + x^{r'}_{22} + x^{r'}_{32} = 75, \quad x^{r'}_{13} + x^{r'}_{23} + x^{r'}_{33} = 67, \\
& x^{r'}_{11} \geq 20, \quad x^{r'}_{12} \geq 61, \quad x^{r'}_{21} \geq 62, \quad x^{r'}_{22} \geq 6, \\
& x^{r'}_{23} \geq 3, \quad x^{r'}_{31} \geq 12, \quad x^{r'}_{32} \geq 3, \quad x^{r'}_{33} \geq 61, \\
& x^{r'}_{gk} \geq x^{r*}_{gk}, \quad g = 1,2,3, \quad k = 1,2,3.
\end{aligned} \tag{55}
$$

Table 9 Data of example

Warehouse	Stores			Supply
	B_1	B_2	B_3	
	Trans. p.$\{(115, 125, 130); (110, 125, 135)\}$	$\{(80, 85, 90); (75, 85, 95)\}$	$\{(40, 45, 60); (30, 45, 70)\}$	
A_1	Trans. cost$\{(1.5, 2, 2.8); (1, 2, 3)\}$	$\{(2.5, 3, 3.5); (2, 3, 4)\}$	$\{(2, 2.6, 3); (1.5, 2.6, 3.4)\}$	$\{(75, 78, 81); (72, 78, 83)\}$
	Delivery t.$\{(6.8, 7, 7.5); (6, 7, 8)\}$	$\{(5.5, 6, 7.5); (5, 6, 8)\}$	$\{(9.5, 10, 11.5); (9, 10, 12)\}$	
	Trans. p.$\{(135, 165, 170); (130, 165, 180)\}$	$\{(70, 79, 90); (65, 79, 95)\}$	$\{(80, 82, 85); (75, 82, 90)\}$	
A_2	Trans. cost$\{(2.5, 2.5, 4); (2, 2.5, 4.5)\}$	$\{(2.5, 3.5, 4.5); (2, 3.5, 4.9)\}$	$\{(4, 4.2, 6); (3.5, 4.2, 6.4)\}$	$\{(60, 66, 71); (56, 66, 76)\}$
	Delivery t.$\{(7, 9, 9.5); (6.5, 9, 9.9)\}$	$\{(7.6, 8, 9.5); (7, 8, 10)\}$	$\{(6.5, 7, 8); (6, 7, 8.5)\}$	
	Trans. p.$\{(92, 100, 108); (90, 100, 110)\}$	$\{(40, 45, 55); (35, 40, 65)\}$	$\{(60, 65, 70); (55, 65, 74)\}$	
A_3	Trans. cost$\{(3.5, 4, 5); (3, 4, 5.4)\}$	$\{(4, 4.5, 5.5); (4, 4.5, 6)\}$	$\{(2.8, 3, 3.5); (2.5, 3, 3.9)\}$	$\{(58, 68, 76); (55, 68, 85)\}$
	Delivery t.$\{(8.5, 10, 11); (8, 10, 11.4)\}$	$\{(8.6, 9, 10.5); (8.3, 9, 10.9)\}$	$\{(10.4, 11, 12); (10, 11, 12.4)\}$	
Demands	$\{(74, 85, 94); (70, 85, 102)\}$	$\{(59, 65, 70); (55, 65, 75)\}$	$\{(60, 62, 64); (58, 62, 67)\}$	$\{(193, 212, 228); (183, 212, 244)\}$

Then, the Fermatean fuzzy optimum solution of models (51-55) can be seen in Table 13.

The membership and non-membership of objective functions (transportation profit, time for delivery and transportation cost) related to the Fermatean fuzzy optimum solution are shown in Figs. 6, 7 and 8, respectively.

Now, the proposed Fermatean fuzzy DEA approach to solve the model (1) is compared with an extended goal programming (GP) approach (Ehrgott 2005). GP is a popular strategy for reducing a TP with multiple objective functions into a single objective function. The goal of GP is just to minimize the distance between objective functions and an aspiration level vector, which is either determined by the decision maker or equal to $\tilde{f} = (\tilde{f}_1^*, \tilde{f}_2^*, \ldots, \tilde{f}_h^*)$, where

$$\begin{aligned} \tilde{f}_a^* &\approx Optimize \quad \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{c}_{gk}^a \tilde{x}_{gk} \\ \text{subject to} \quad & \sum_{k=1}^{q} \tilde{x}_{gk} \approx \tilde{s}_g, \quad g = 1, \ldots, p, \\ & \sum_{g=1}^{p} \tilde{x}_{gk} \approx \tilde{d}_k, \quad k = 1, \ldots, q, \\ & \tilde{x}_{gk} \succeq \tilde{0}, \quad g = 1, \ldots, p, \quad k = 1, \ldots, q. \end{aligned} \tag{56}$$

The model (56) can be solved using approach (Akram et al. 2023a). Suppose $\tilde{n}_d = \{(n_d^l, n_d^m, n_d^r); (n_d^{l'}, n_d^m, n_d^{r'})\}$ and $\tilde{p}_d = \{(p_d^l, p_d^m, p_d^r); (p_d^{l'}, p_d^m, p_d^{r'})\}$ are the under deviations and over deviations of the objectives $\tilde{f}_a = Optimize \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{c}_{gk}^a \tilde{x}_{gk}$ from their aspiration levels $\tilde{f}_a^*$, respectively. In addition assume that, $\tilde{f}_a = \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{c}_{gk}^a \tilde{x}_{gk} (a = 1, \ldots, i)$ and $\tilde{f}_a = \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{c}_{gk}^a \tilde{x}_{gk} (a = i+1, \ldots, i+j)$ are those objective functions of model (1) which need to be minimized and maximized, respectively. In this manner, using the GP approach, the model (1) is converted into a deviational parameter minimization problem that minimizes the sum of deviational parameters as follows:

Table 10 Values of $\tilde{E}^{1^*}_{A_g B_k}$

Warehouse	Stores		
	B_1	B_2	B_3
A_1	$\{(0.30, 0.46, 0.64); (0.27, 0.46, 1.00)\}$	$\{(0.48, 0.60, 0.76); (0.39, 0.60, 1.00)\}$	$\{(0.36, 0.47, 0.73); (0.25, 0.47, 1.00)\}$
A_2	$\{(0.25, 0.49, 0.50); (0.21, 0.49, 0.67)\}$	$\{(0.33, 0.48, 0.76); (0.28, 0.48, 1.00)\}$	$\{(0.51, 0.71, 0.78); (0.45, 0.71, 0.93)\}$
A_3	$\{(0.14, 0.19, 0.23); (0.12, 0.19, 0.27)\}$	$\{(0.15, 0.21, 0.29); (0.12, 0.21, 0.34)\}$	$\{(0.49, 0.60, 0.68); (0.41, 0.60, 0.78)\}$

Table 11 Values of $\tilde{E}^{2^*}_{A_g B_k}$

Warehouse	Stores		
	B_1	B_2	B_3
A_1	$\{(0.30, 0.46, 0.64); (0.27, 0.46, 1.00)\}$	$\{(0.17, 0.21, 0.27); (0.14, 0.21, 0.35)\}$	$\{(0.10, 0.13, 0.22); (0.07)0.13, 0.35)\}$
A_2	$\{(0.37, 0.73, 0.76); (0.32, 0.73, 1.00)\}$	$\{(0.17, 0.25, 0.40); (0.15, 0.25, 0.53)\}$	$\{(0.15, 0.22, 0.24); (0.13, 0.22, 0.29)\}$
A_3	$\{(0.50, 0.68, 0.84); (0.45, 0.68, 1.00)\}$	$\{(0.20, 0.27, 0.38); (0.16, 0.27, 0.44)\}$	$\{(0.47, 0.59, 0.68); (0.38, 0.59, 0.81)\}$

$$\begin{aligned}
\min \quad & \sum_{a=1}^{i} \tilde{p}_a + \sum_{a=i+1}^{i+j} \tilde{n}_a \\
\text{subject to} \quad & \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{c}^a_{gk}\tilde{x}_{gk} \preceq \tilde{p}_a + \tilde{f}^*_a, \quad a = 1, \ldots, i, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} \tilde{c}^a_{gk}\tilde{x}_{gk} + \tilde{n}_a \succeq \tilde{f}^*_a, \quad a = i+1, \ldots, i+j, \\
& \tilde{p}_a \succeq \tilde{0}, \quad a = 1, \ldots, i, \\
& \tilde{n}_a \succeq \tilde{0}, \quad a = i+1, \ldots, i+j, \\
& \sum_{k=1}^{q} \tilde{x}_{gk} = \approx \tilde{s}_g, \quad g = 1, \ldots, p, \\
& \sum_{g=1}^{p} \tilde{x}_{gk} \approx \tilde{d}_k, \quad k = 1, \ldots, q, \\
& \tilde{x}_{gk} \succeq \tilde{0}, \quad g = 1, \ldots, p, \quad k = 1, \ldots, q.
\end{aligned} \tag{57}$$

To solve the model (57), suppose

$$\begin{aligned}
\tilde{n}_a &= \{(n^l_a, n^m_a, n^r_a); (n^{l'}_a, n^m_a, n^{r'}_a)\}, \\
\tilde{p}_a &= \{(p^l_a, p^m_a, p^r_a); (p^{l'}_a, p^m_a, p^{r'}_a)\}, \\
\tilde{c}^a_{gk} &= \{(c^{a,l}_{gk}, c^{a,m}_{gk}, c^{a,r}_{gk}); (c^{a,l'}_{gk}, c^{a,m}_{gk}, c^{a,r'}_{gk})\} \quad \text{and} \\
\tilde{f}^*_a &= \{(f^{l^*}_a, f^{m^*}_a, f^{r^*}_a); (f^{l'^*}_a, f^{m^*}_a, f^{r'^*}_a)\}.
\end{aligned}$$

Using Definitions 8 and 9, the model (57) is converted as follows:

$$\begin{aligned}
\min \quad & \sum_{a=1}^{i}(p^{l'}_a + p^l_a + p^m_a + p^r_a + p^{r'}_a) + \sum_{a=i+1}^{i+j}(n^{l'}_a + n^l_a + n^m_a + n^r_a + n^{r'}_a) \\
\text{subject to} \quad & \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,l'}_{gk} x^{l'}_{gk} \le p^{l'}_a + f^{l'^*}_a, \quad a = 1, \ldots, i, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,l}_{gk} x^{l}_{gk} \le p^{l}_a + f^{l^*}_a, \quad a = 1, \ldots, i, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,m}_{gk} x^{m}_{gk} \le p^{m}_a + f^{m^*}_a, \quad a = 1, \ldots, i, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,r}_{gk} x^{r}_{gk} \le p^{r}_a + f^{r^*}_a, \quad a = 1, \ldots, i, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,r'}_{gk} x^{r'}_{gk} \le p^{r'}_a + f^{r'^*}_a, \quad a = 1, \ldots, i, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,l'}_{ij} x_{gk} + n^{l'}_a \ge f^{l'^*}_a, \quad a = i+1, \ldots, i+j, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,l}_{gk} x_{gk} + n^{l}_a \ge f^{l^*}_a, \quad a = i+1, \ldots, i+j, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,m}_{gk} x_{gk} + n^{m}_a \ge f^{m^*}_a, \quad a = i+1, \ldots, i+j, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,r}_{gk} x_{gk} + n^{r}_a \ge f^{r^*}_a, \quad a = i+1, \ldots, i+j, \\
& \sum_{g=1}^{p}\sum_{k=1}^{q} c^{a,r'}_{gk} x_{gk} + n^{r'}_a \ge f^{r'^*}_a, \quad a = i+1, \ldots, i+j, \\
& p^{l'}_a \ge 0, \quad a = 1, \ldots, i, \\
& n^{l'}_a \ge 0, a = i+1, \ldots, i+j, \\
& \text{constraints of model (26).}
\end{aligned} \tag{58}$$

The model (58) is a LPP which can be solved by utilizing the simplex method.

The next three FFTPs are solved similarly to the model (56) to find the solution to this example using the GP approach:

$$
\begin{aligned}
f_1^* \approx \max \quad & \{(115,125,130);(110,125,135)\}x_{11} + \{(80,85,90);(75,85,95)\}x_{12} + \{(40,45,60);(30,45,70)\}x_{13} \\
& + \{(135,165,170);(130,165,180)\}x_{21} + \{(70,79,90);(65,79,95)\}x_{22} + \{(80,82,85);(75,82,90)\}x_{23} \\
& + \{(92,100,108);(90,100,110)\}x_{31} + \{(40,45,55);(35,40,65)\}x_{32} + \{(60,65,70);(55,65,74)\}x_{33} \\
\text{subject to} \quad & \sum_{k=1}^{3} \tilde{x}_{1q} \approx \{(75,78,81);(72,78,83)\}, \quad \sum_{k=1}^{3} \tilde{x}_{2q} \approx \{(60,66,71);(56,66,76)\}, \\
& \sum_{k=1}^{3} \tilde{x}_{3q} \approx \{(58,68,76);(55,68,85)\}, \quad \sum_{g=1}^{3} \tilde{x}_{p1} \approx \{(74,85,94);(70,85,102)\}, \\
& \sum_{g=1}^{3} \tilde{x}_{p2} \approx \{(59,65,70);(55,65,75)\}, \quad \sum_{g=1}^{3} \tilde{x}_{p1} \approx \{(60,62,64);(58,62,67)\}, \\
& \tilde{x}_{gk} \succeq \tilde{0}, \quad g=1,2,3, \quad k=1,2,3.
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
f_2^* \approx \min \quad & \{(1.5,2,2.8);(1,2,3)\}x_{11} + \{(2.5,3,3.5);(2,3,4)\}x_{12} + \{(2,2.6,3);(1.5,2.6,3.4)\}x_{13} \\
& + \{(2.5,2.5,4);(2,2.5,4.5)\}x_{21} + \{(2.5,3.5,4.5);(2,3.5,4.9)\}x_{22} + \{(4,4.2,6);(3.5,4.2,6.4)\}x_{23} \\
& + \{(3.5,4,5);(3,4,5.4)\}x_{31} + \{(4,4.5,5.5);(4,4.5,6)\}x_{32} + \{(2.8,3,3.5);(2.5,3,3.9)\}x_{33} \\
\text{subject to} \quad & \sum_{k=1}^{3} \tilde{x}_{1q} \approx \{(75,78,81);(72,78,83)\}, \quad \sum_{k=1}^{3} \tilde{x}_{2q} \approx \{(60,66,71);(56,66,76)\}, \\
& \sum_{k=1}^{3} \tilde{x}_{3q} \approx \{(58,68,76);(55,68,85)\}, \quad \sum_{g=1}^{3} \tilde{x}_{p1} \approx \{(74,85,94);(70,85,102)\}, \\
& \sum_{g=1}^{3} \tilde{x}_{p2} \approx \{(59,65,70);(55,65,75)\}, \quad \sum_{g=1}^{3} \tilde{x}_{p1} \approx \{(60,62,64);(58,62,67)\}, \\
& \tilde{x}_{gk} \succeq \tilde{0}, \quad g=1,2,3, \quad k=1,2,3.
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
f_3^* \approx \min \quad & \{(6.8,7,7.5);(6,7,8)\}x_{11} + \{(5.5,6,7.5);(5,6,8)\}x_{12} \\
& + \{(9.5,10,11.5);(9,10,12)\}x_{13} + \{(7,9,9.5);(6.5,9,9.9)\}x_{21} \\
& + \{(7.6,8,9.5);(7,8,10)\}x_{22} + \{(6.5,7,8);(6,7,8.5)\}x_{23} \\
& + \{(8.5,10,11);(8,10,11.4)\}x_{31} + \{(8.6,9,10.5);(8.3,9,10.9)\}x_{32} \\
& + \{(10.4,11,12);(10,11,12.4)\}x_{33} \\
\text{subject to} \quad & \sum_{k=1}^{3} \tilde{x}_{1q} \approx \{(75,78,81);(72,78,83)\}, \quad \sum_{k=1}^{3} \tilde{x}_{2q} \approx \{(60,66,71);(56,66,76)\}, \\
& \sum_{k=1}^{3} \tilde{x}_{3q} \approx \{(58,68,76);(55,68,85)\}, \quad \sum_{g=1}^{3} \tilde{x}_{p1} \approx \{(74,85,94);(70,85,102)\}, \\
& \sum_{g=1}^{3} \tilde{x}_{p2} \approx \{(59,65,70);(55,65,75)\}, \quad \sum_{g=1}^{3} \tilde{x}_{p1} \approx \{(60,62,64);(58,62,67)\}, \\
& \tilde{x}_{gk} \succeq \tilde{0}, \quad g=1,2,3, \quad k=1,2,3.
\end{aligned} \tag{61}
$$

Table 12 Values of $\tilde{E}^{*}_{A_g B_k}$

Warehouse	Stores		
	B_1	B_2	B_3
A_1	$\{(0.30, 0.46, 0.64); (0.27, 0.46, 1.00)\}$	$\{(0.33, 0.41, 0.52); (0.27, 0.41, 0.68)\}$	$\{(0.23, 0.30, 0.48); (0.16, 0.30, 0.68)\}$
A_2	$\{(0.31, 0.61, 0.63); (0.27, 0.61, 0.84)\}$	$\{(0.25, 0.37, 0.58); (0.22, 0.37, 0.77)\}$	$\{(0.33, 0.47, 0.51); (0.29, 0.47, 0.61)\}$
A_3	$\{(0.32, 0.44, 0.54); (0.29, 0.44, 0.64)\}$	$\{(0.18, 0.24, 0.34); (0.14, 0.24, 0.39)\}$	$\{(0.48, 0.60, 0.68); (0.40, 0.60, 0.80)\}$

Table 13 Fermatean Fuzzy optimum solution

Warehouse	Stores		
	B_1	B_2	B_3
A_1	$\{(17, 17, 20); (17, 17, 22)\}$	$\{(58, 61, 61); (55, 61, 61)\}$	$\{(0, 0, 0); (0, 0, 0)\}$
A_2	$\{(56, 62, 62); (53, 62, 62)\}$	$\{(1, 1, 6); (0, 1, 11)\}$	$\{(3, 3, 3); (3, 3, 3)\}$
A_3	$\{(1, 6, 12); (0, 6, 18)\}$	$\{(0, 3, 3); (0, 3, 3)\}$	$\{(57, 59, 61); (55, 59, 64)\}$

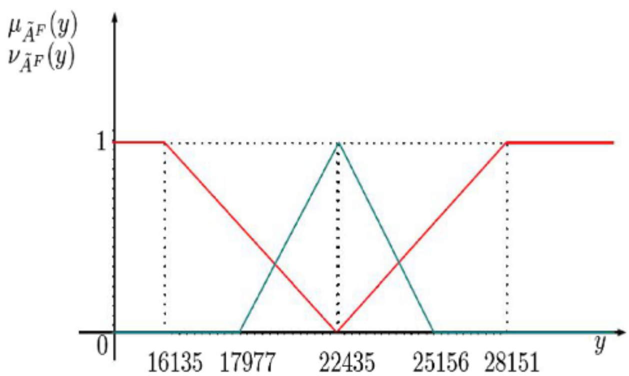

Fig. 6 Membership and non-membership functions of transportation profit

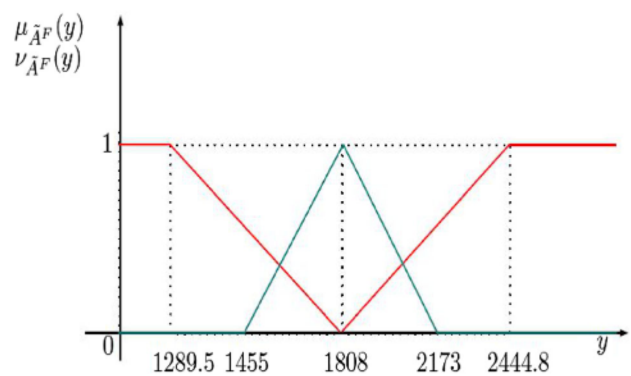

Fig. 8 Membership and non-membership functions of transportation cost

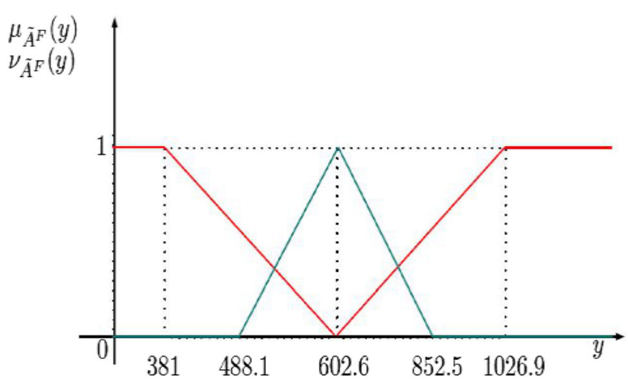

Fig. 7 Membership and non-membership functions of delivery time

The models (59-61) can be solved using approach (Akram et al. 2023a). Therefore, we obtained aspiration level vector as $\tilde{f} = (\tilde{f_1}, \tilde{f_2}, \tilde{f_3}) = (\{(17990, 22446, 25357); (16135, 22446, 28616)\}, \{(427.1, 597.8, 834.9); (323, 597.8, 987.4)\}, \{(1322.5, 1637, 2021); (1157, 1637, 2288.1)\})$. The next integer LPP is now solved using the model (58):

$$\min \sum_{a=1}^{2} = (p_a^{l'} + p_a^{l} + p_a^{m} + p_a^{r} + p_a^{r'}) + (n_1^{l'} + n_1^{l} + n_1^{m} + n_1^{r} + n_1^{r'})$$

subject to

$$\begin{aligned}
& x_{11}^{l'} + 2x_{12}^{l'} + 1.5x_{13}^{l'} + 2x_{21}^{l'} + 2x_{22}^{l'} + 3.5x_{23}^{l'} + 3x_{31}^{l'} + 4x_{32}^{l'} + 2.5x_{33}^{l'} \le p_1^{l'} + 323, \\
& 6x_{11}^{l'} + 5x_{12}^{l'} + 9x_{13}^{l'} + 6.5x_{21}^{l'} + 7x_{22}^{l'} + 6x_{23}^{l'} + 8x_{31}^{l'} + 8.3x_{32}^{l'} + 10x_{33}^{l'} \le p_2^{l'} + 1157, \\
& 1.5x_{11}^{l} + 2.5x_{12}^{l} + 2x_{13}^{l} + 2.5x_{21}^{l} + 2.5x_{22}^{l} + 4x_{23}^{l} + 3.5x_{31}^{l} + 4x_{32}^{l} + 2.8x_{33}^{l} \le p_1^{l} + 427.1, \\
& 6.8x_{11}^{l} + 5.5x_{12}^{l} + 9.5x_{13}^{l} + 7x_{21}^{l} + 7.6x_{22}^{l} + 6.5x_{23}^{l} + 8.5x_{31}^{l} + 8.6x_{32}^{l} + 10.4x_{33}^{l} \le p_2^{l'} + 1322.5, \\
& 2x_{11}^{m} + 3x_{12}^{m} + 2.6x_{13}^{m} + 2.5x_{21}^{m} + 3.5x_{22}^{m} + 4.2x_{23}^{m} + 4.2x_{31}^{m} + 4x_{32}^{m} + 4.5x_{33}^{m} \le p_1^{m} + 597.8, \\
& 7x_{11}^{m} + 6x_{12}^{m} + 10x_{13}^{m} + 9x_{21}^{m} + 8x_{22}^{m} + 7x_{23}^{m} + 10x_{31}^{m} + 9x_{32}^{m} + 11x_{33}^{m} \le p_2^{m} + 1637, \\
& 2.8x_{11}^{r} + 3.5x_{12}^{r} + 3x_{13}^{r} + 4x_{21}^{r} + 4.5x_{22}^{r} + 6x_{23}^{r} + 5x_{31}^{r} + 5.5x_{32}^{r} + 3.5x_{33}^{r} \le p_1^{r} + 834.9, \\
& 7.5x_{11}^{r} + 7.5x_{12}^{r} + 11.5x_{13}^{r} + 9.5x_{21}^{r} + 9.5x_{22}^{r} + 8x_{23}^{r} + 11x_{31}^{r} + 10.5x_{32}^{r} + 12x_{33}^{r} \le p_2^{r} + 2021, \\
& 3x_{11}^{r'} + 4x_{12}^{r'} + 3.4x_{13}^{r'} + 4.5x_{21}^{r'} + 4.9x_{22}^{r'} + 6.4x_{23}^{r'} + 5.4x_{31}^{r'} + 6x_{32}^{r'} + 3.9x_{33}^{r'} \le p_1^{r'} + 987.4, \\
& 8x_{11}^{r'} + 8x_{12}^{r'} + 12x_{13}^{r'} + 9.9x_{21}^{r'} + 10x_{22}^{r'} + 8.5x_{23}^{r'} + 11.4x_{31}^{r'} + 10.9x_{32}^{r'} + 12.5x_{33}^{r'} \le p_2^{r'} + 2288.1, \\
& 110x_{11}^{l'} + 75x_{12}^{l'} + 30x_{13}^{l'} + 130x_{21}^{l'} + 65x_{22}^{l'} + 75x_{23}^{l'} + 90x_{31}^{l'} + 35x_{32}^{l'} + 55x_{33}^{l'} + n_1^{l'} \ge 16135, \\
& 115x_{11}^{l} + 80x_{12}^{l} + 40x_{13}^{l} + 135x_{21}^{l} + 70x_{22}^{l} + 80x_{23}^{l} + 92x_{31}^{l} + 40x_{32}^{l} + 60x_{33}^{l} + n_1^{l} \ge 17990, \\
& 125x_{11}^{m} + 85x_{12}^{m} + 45x_{13}^{m} + 165x_{21}^{m} + 79x_{22}^{m} + 82x_{23}^{m} + 100x_{31}^{m} + 40x_{32}^{m} + 65x_{33}^{m} + n_1^{m} \ge 22446, \\
& 130x_{11}^{r} + 90x_{12}^{r} + 60x_{13}^{r} + 170x_{21}^{r} + 90x_{22}^{r} + 85x_{23}^{r} + 108x_{31}^{r} + 55x_{32}^{r} + 70x_{33}^{r} + n_1^{r} \ge 25357, \\
& 135x_{11}^{r'} + 95x_{12}^{r'} + 70x_{13}^{r'} + 180x_{21}^{r'} + 95x_{22}^{r'} + 90x_{23}^{r'} + 110x_{31}^{r'} + 65x_{32}^{r'} + 74x_{33}^{r'} + n_1^{r'} \ge 28616, \\
& p_d^{r'} \ge p_d^{r} \ge p_d^{m} \ge p_d^{l} \ge p_d^{l'}, \quad n_1^{r'} \ge n_1^{r} \ge n_1^{m} \ge n_1^{l} \ge n_1^{l'}, \\
& x_{11}^{l'} + x_{12}^{l'} + x_{13}^{l'} = 72, \quad x_{21}^{l'} + x_{22}^{l'} + x_{23}^{l'} = 56, \quad x_{31}^{l'} + x_{32}^{l'} + x_{33}^{l'} = 55, \\
& x_{11}^{l'} + x_{21}^{l'} + x_{31}^{l'} = 70, \quad x_{12}^{l'} + x_{22}^{l'} + x_{32}^{l'} = 55, \quad x_{13}^{l'} + x_{23}^{l'} + x_{33}^{l'} = 58, \\
& x_{11}^{l} + x_{12}^{l} + x_{13}^{l} = 75, \quad x_{21}^{l} + x_{22}^{l} + x_{23}^{l} = 60, \quad x_{31}^{l} + x_{32}^{l} + x_{33}^{l} = 58, \\
& x_{11}^{l} + x_{21}^{l} + x_{31}^{l} = 74, \quad x_{12}^{l} + x_{22}^{l} + x_{32}^{l} = 59, \quad x_{13}^{l} + x_{23}^{l} + x_{33}^{l} = 60, \\
& x_{11}^{m} + x_{12}^{m} + x_{13}^{m} = 78, \quad x_{21}^{m} + x_{22}^{m} + x_{23}^{m} = 66, \quad x_{31}^{m} + x_{32}^{m} + x_{33}^{m} = 68, \\
& x_{11}^{m} + x_{21}^{m} + x_{31}^{m} = 85, \quad x_{12}^{m} + x_{22}^{m} + x_{32}^{m} = 65, \quad x_{13}^{m} + x_{23}^{m} + x_{33}^{m} = 62, \\
& x_{11}^{r} + x_{12}^{r} + x_{13}^{r} = 81, \quad x_{21}^{r} + x_{22}^{r} + x_{23}^{r} = 71, \quad x_{31}^{r} + x_{32}^{r} + x_{33}^{r} = 76, \\
& x_{11}^{r} + x_{21}^{r} + x_{31}^{r} = 94, \quad x_{12}^{r} + x_{22}^{r} + x_{32}^{r} = 70, \quad x_{13}^{r} + x_{23}^{r} + x_{33}^{r} = 64, \\
& x_{11}^{r'} + x_{12}^{r'} + x_{13}^{r'} = 83, \quad x_{21}^{r'} + x_{22}^{r'} + x_{23}^{r'} = 76, \quad x_{31}^{r'} + x_{32}^{r'} + x_{33}^{r'} = 85, \\
& x_{11}^{r'} + x_{21}^{r'} + x_{31}^{r'} = 102, \quad x_{12}^{r'} + x_{22}^{r'} + x_{32}^{r'} = 75, \quad x_{13}^{r'} + x_{23}^{r'} + x_{33}^{r'} = 67, \\
& x_{gk}^{r'} \ge x_{gk}^{r} \ge x_{gk}^{m} \ge x_{gk}^{l} \ge x_{gk}^{l'} \ge 0 \\
& ,x_{gk}^{r'}, x_{gk}^{r}, x_{gk}^{m}, x_{gk}^{l}, x_{gk}^{l'} : \text{Integer} \quad g = 1,2,3, \quad k = 1,2,3.
\end{aligned} \tag{62}$$

By solving the model (62), the Fermatean fuzzy optimum solution is represented in Table 14.

Now, the results of fully FFMOTP with data given in Table 9 determined by applying the proposed and GP technique are compared.

1. The FFESs of efficient transportation plans determined using the proposed and GP technique are $\{(70.52, 111.19, 137.57); (56.62, 111.19, 189.57)\}$ and $\{(70.44, 111.05, 136.91); (56.62, 111.05, 187.42)\}$, respectively. Both the optimal solutions produced by applying the GP methodology (Table 14) and the suggested method (Table 13) are pareto optimal. The suggested approach yields a higher FFES for the transportation plan than the GP technique. Therefore, the proposed method to solve fully FFMOTP is

Table 14 Optimum solution by GP technique (Ehrgott 2005)

Warehouse	Stores		
	B_1	B_2	B_3
A_1	$\{(17,17,17);(17,17,17)\}$	$\{(58,61,64);(55,61,66)\}$	$\{(0,0,0);(0,0,0)\}$
A_2	$\{(57,63,68);(53,63,73)\}$	$\{(0,0,0,);(0,0,0)\}$	$\{(3,3,3,);(3,3,3,)\}$
A_3	$\{(0,5,9);(0,5,12)\}$	$\{(1,4,6);(0,4,9)\}$	$\{(57,59,61);(55,59.64)\}$

preferable to obtain higher level of FFESs than the GP technique.

2. Model (50) is an LP problem, whereas Model (62) is an integer LP problem. An integer LP problem is more difficult to solve as compared to traditional LP problem.
3. The model (51) has seven constraints, while the model (62) has 45 constraints, therefore, the proposed methodology has less computational complexity as compared to the GP technique.
4. The proposed methodology maintains the structure of TP after converting the fully FFMOTP, whereas the GP technique does not.

6 Conclusion

Transportation problems are a subclass of LPP that appear in many situations. The main goal of a classic TP is to find the lowest total transportation cost to move goods from multiple sources to multiple destinations. However, in many practical applications, more than one feature must be evaluated simultaneously in the same TP. These problems are called MOTPs. In this paper, a new Fermatean fuzzy DEA approach has been proposed to handle the fully Fermatean fuzzy MOTP, in which all the parameters and variables are represented by triangular Fermatean fuzzy numbers. To achieve this goal, a new method has been introduced based on the CSW in DEA that converts the given fully FFMOTP into a single-objective FFTP. For this purpose, every arc in a fully FFMOTP has been treated as a DMU with numerous Fermatean fuzzy inputs and outputs. The arc characteristics that correspond to Fermatean fuzzy objectives that need to be minimized are defined as Fermatean fuzzy input, whereas the arc characteristics that correspond to Fermatean fuzzy objectives that need to be maximized are defined as Fermatean fuzzy output. Two different Fermatean fuzzy efficiency scores are determined for every arc. Next, a unique Fermatean fuzzy score is determined by combining these two Fermatean fuzzy efficiency scores. Then, by considering the unique Fermatean fuzzy relative efficiency for every arc as its unique characteristic, a single-objective FFTP has been determined, which can be solved using existing approaches. We have resorted to Akram et al.'s (2023a) method to find a solution to the single-objective FFTP that results from the aforementioned transformation. In addition, to support the proposed approach, a numerical example has been fully developed. Finally, we have compared the results of this technique with state-of-the-art results.

Among the key advantages of the present Fermatean fuzzy DEA, we underline its computational simplicity, which allows its implementation in complicated network structures. The best technique to solve a FFMOTP, which is a subclass of a Fermatean fuzzy multi-objective LPP, is to maximize the decision maker's Fermatean fuzzy utility function. In many cases, however, obtaining a mathematical description of the decision maker's Fermatean fuzzy utility function is impossible. On the other hand, the Fermatean fuzzy DEA technique presented in this research provides an efficient plan for the fully FFMOTP without explicit knowledge of the decision makers' Fermatean fuzzy utility function. In fact, existing methodologies transform all objective functions into one of two types: either maximizing or minimizing, and then determine efficient solutions using approaches for multi-objective problems. By contrast, the technique presented in this work solves the multi-objective problem efficiently without transforming all objective functions to the same type.

To summarize, the Fermatean fuzzy DEA technique presented in this work not only provides an efficient transportation plan without the requirement of precise knowledge of the decision maker, but it also preserves the problem's transportation structure. Unfortunately, this technique cannot be used to find Fermatean fuzzy efficient solutions for unbalanced fully FFMOTPs. Hence, further research on strengthening the proposed approach to address these limitations is an attractive topic for research. Furthermore, while the Fermatean fuzzy DEA technique has less computational complexity than current solution strategies that solve fully FFMOTPs, it is unavoidable that its computational difficulty increases as the quantity of sources and destinations grows. In the future, we will try to improve the Fermatean fuzzy DEA technique for solving fully FFMOTP in order to minimize its computational complexity.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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