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Soft Sub-nearsemirings, Soft Ideals and Soft S-Subsemigroups of Nearsemirings

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Abstract

In this note, we introduce and discuss the notions of soft sub-nearsemirings, soft ideals and soft S-subsemigroups of nearsemirings. Some related properties and characterizations of these soft algebraic structures are discussed with illustrative examples. For the sake of investigations, we apply several operations of soft sets including soft intersection sum, soft product and soft uni-int product. Based on these operations, we also discuss few characterizations of distributively generated nearsemirings. In due course, we investigate few relationships among these soft algebraic structures and the classical nearsemirings. We also introduce the notion of soft ideals (left and right) of nearsemirings. Firstly, we investigate these ideals by applying few operations on them. Then, we present the relationship between these soft ideals of nearsemiring with the classical ideals of nearsemirings. Moreover, we introduce the notion of soft S-homomorphism between two soft S-subsemigroups and investigate that the homomorphic image of soft S-subsemigroup is a soft S-subsemigroup. Throughout, we shift several substructures of nearsemirings towards the soft algebraic substructures of nearsemirings by utilizing different algebraic methods. Consequently, we explore a linkage among the soft set theory, classical set theory and nearsemiring theory. Mainly, our study is the interplay between soft substructures of nearsemirings and classical substructures of nearsemirings.

Keywords: Nearsemirings, Soft sub-nearsemirings, Soft S-subsemigroups, Uni-int products

1. Introduction

Soft set theory was initiated in 1999 by Molodtsov [1] and the fuzzy set theory was initiated by Zadeh [2]. Actually, every hesitant fuzzy set introduced by Zadeh [2] can be considered a soft set on a common universe. Similarly, every soft set on a denumerable universe can be considered a fuzzy set. However, soft sets theory become an effective tool to deal with uncertainties in the given data. Numerous applications of soft sets have been explored in [3,4]. A lot of work has been done on the soft set theory and is developing rapidly. Many applications of soft sets have been explored towards data analysis, decision-making theory etc. A number of operations on soft sets have been introduced and discussed in the literature [5–8]. Among the others, the operation uni-int has its own importance due to its applications towards decision-making [9]. The generalized form of uni-int operation and its application in decision-making theory was presented in [10]. Moreover, Han and Geng [11] introduced some special method based on intm-intn and created a decision-making scheme. Number of algebraists introduced several soft algebraic structures such as soft groups [5], soft rings [12], soft fields [13], soft nearrings [14], soft semirings [15] and so on. Afterwards, a new view of soft intersection rings

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based on uni-int product has been explored in [16]. Soft subnearrings, soft ideals and soft N-subgroups of nearrings have been introduced in [17]. Notion of soft intersection h-ideals of hemirings have been introduced in [18]. Khan et al. [19] introduced the notion of soft nearsemirings. The concepts of soft intersection nearsemirings [20] and (M, N)-soft intersection nearsemirings were also introduced by Khan and Davvaz [21]. The notion of (α, β)- soft intersectional rings and ideals along with their applications have been explored in [22]. Recently, the notions of fuzzy sub-nearsemirings and fuzzy soft sub-nearsemirings [23] have been initiated by both the authors. In a sequel, we introduce and discuss the notions of soft sub-nearsemirings, soft ideals and soft S-subsemigroups of a nearsemiring S. We also provide few characterizations of distributively generated nearsemirings in the frame of the soft sets theory. Throughout our study, we interrelate several classical substructures of nearsemirings with the soft substructures of nearsemirings.

Near-semiring is the common generalization of nearrring and semiring. Basically, nearsemiring R is an algebraic structure equipped with two binary operations "$+$" and "$\cdot$" such that $(R, +)$ is a monoid, $(R, \cdot)$ is a semigroup, and both structures are joined through a single (left or right) distributive law with 0 is the one sided absorbing element. Moreover, if $0 \in R$ such that $a + 0 = 0 + a = a$, $a \cdot 0 = 0 \cdot a = 0$, then we say R is a zero-symmetric nearsemiring (or seminearring). Nearsemirings ascertained naturally from the mappings on monoid to itself under component-wise addition and composition of mappings. Due to having strong relationships of nearsemirings with different types of algebras, recently Chajda and Laenger [24] introduced the notion of balanced nearsemirings. Since the ideals have their own importance in algebra particularly when we study rings with two binary operations. In general, the ideals of nearsemirings are not similar to that of the standard ideals of rings (resp., nearrings, semirings) and hence many results in rings (resp., nearrings, semirings) theory have no similarities in nearsemirings using merely ideals. Hence the classified notion of ideal named S-ideal [25] has been initiated in seminearring (zero symmetric nearsemirings) theory. Some special types of prime ideals of seminearrings have been explored in [26].

On the other hands, Nearrings, semirings and nearsemirings have a number of applications in computer technology such as theory of automata, languages and machine learning. Variety of applications of semirings have been explored in [27,28]. In [28], the author explored various applications of nearrings, particularly, he extended the linear sequential machines of Eilenberg by using the theory of nearrings. Likewise, Krishna and Chatterjee [30] introduced the condition of minimality of generalized linear sequential machines at the base of nearsemirings.

This work comprises of six sections. In Section 2, we include necessary discussions about nearsemirings and useful terminologies about soft set theory. In Section 3, we introduce and discuss the notions of soft sub-nearsemirings and soft ideals of nearsemiring S. We apply a number of usual operations including soft uni-int products to these soft algebraic structures. Moreover, we investigate these soft structures by applying image, pre-image, α-inclusion mappings. We also provide few characterizations of distributively generated nearsemirings in the setting of soft set theory. In Section 4, we provide some relationships of newly established soft structures with the existing classical algebraic structures as applications. In section 5, we introduce the notions of soft S-subsemigroup and soft S-morphism between two soft S-subsemigroups of nearsemirings. Throughout our discussions, we examine the interplay of classical nearsemiring and its substructures with the soft nearsemirings and their soft substructures.

2. Preliminaries

Following [31], we call a system $(S, +, .)$ is a left(right) nearsemiring if $S$is a monoid with respect to addition, semigroup with respect to multiplication and obeying left (right) distributive law. Furthermore, if there exists $0 \in S$ such that $0.a = a.0 = 0$, then we call it a zero-symmetric nearsemiring. If S and S' are two nearsemirings, then we call a mapping $\phi : S \to S'$ a nearsemiring homomorphism if for all $a, b \in S$; $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$, $\phi(0) = 0$. Similarly, if S and S' are two nearsemirings, a mapping $\psi : S \to S'$ is the anti-homomorphism if for all $a, b \in S$; $\phi(a+b) = \phi(a)+ \phi(b)$; $\phi(ab) = \phi(b)\phi(a)$; $\phi(0) = 0$. The kernel of semigroup homomorphism is said to be an ideal of nearsemiring R. If S is a nearsemiring, then we call a semigroup $(\Gamma, +)$ having zero 0_Γ a left S-semigroup if there exists a composition $(x, \gamma) \rightarrowtail x\gamma$ of $S \times \Gamma$ satisfying: (i) $(x+y)\gamma = x\gamma + y\gamma$; (ii) $(xy)\gamma = x(y\gamma)$; and (iii) $0_S\gamma = 0_\Gamma$, for all $x, y \in S$ and $\gamma \in \Gamma$. It is obvious that Γ is an S-semigroup with $S = M(\Gamma)$, a nearsemiring consists of all mappings from Γ to itself. Also, a semigroup $(S, +)$ of a nearsemiring $(S, +, .)$ is also an S-semigroup. If Γ and Γ' are two S-semigroups of a nearsemiring S. A mapping $\phi : \Gamma \to \Gamma'$ is said to be an S-morphism between S-semigroups Γ and Γ' if it satisfies: (i) $\phi(x + y) = \phi(x)+ \phi(y)$, (ii) $\phi(rx) = r\phi(x)$, for all $x, y \in \Gamma$, $r \in S$ and (iii) $\phi(0_\Gamma) =$

$0_{\Gamma'}$. We refer [31–33] for further discussion on nearsemirings and S-semigroups of nearsemirings. Throughout, S represents a nearsemiring, U an initial universe, E the possible parameters associated with the elements in U, $\wp(U)$ represents the family of subsets of U, $S(U)$ the collection of soft sets and $A \neq \emptyset$ be a subset of E.

Definition 1 ([1]). A η be a mapping from A to $\wp(U)$. Then the pair (η, A) is said to be soft set over the universe U.

Definition 2 ([9]). Intersection of η_A and η_B represented by $\eta_A \tilde{\cap} \eta_B$, defined by $\eta_A \tilde{\cap} \eta_B = \eta_{A \tilde{\cap} B}$, where $\eta_{A \tilde{\cap} B}(x) = \eta_A(x) \cap \eta_B(x)$, for all $x \in E$.

Definition 3 ([9]). The $\wedge$-product of η_A and η_B of two soft sets over U i.e., $\eta_A \wedge \eta_B$ is the mapping$\eta_{A\wedge B} : E \times E \to \wp(U)$, and $\eta_{A\wedge B}(x, y) = \eta_A(x) \cap \eta_B(y)$.

Definition 4 ([9]). The $\vee$-product of two soft sets η_A and η_B is defined by $\eta_{A\vee B} : E \times E \to \wp(U)$ where $\eta_{A\vee B}(x, y) = \eta_A(x) \cup \eta_B(y)$.

Definition 5 ([34]). Let Ψ be a function from A to B and $\eta_A, \eta_B \in S(U)$. Then, the soft subsets $\Psi(\eta_A) \in S(U)$ and $\Psi^{-1}(\eta_B) \in S(U)$ defined as

$$\Psi(\eta_A)(y) = \begin{cases} \cup\{\eta_A(x) : x \in A,\ \Psi(x) = y\}, & \text{if } y \in \Psi(A), \\ \emptyset, & \text{if } y \notin \Psi(A). \end{cases}$$

for all $y \in B$ and $\Psi^{-1}(\eta_B)(x) = \eta_B(\Psi(x))$, for all $x \in A$. Then $\Psi(\eta_A)$ is the soft image of η_A and $\Psi^{-1}(\eta_B)$ is the soft pre-image (or soft inverse image) of η_B under the napping Ψ .

Definition 6 ([34]). Let η_A be a soft set over U and α be a subset of U. Then upper α-inclusion of soft set η_A is defined by $\eta_A^{\supseteq\alpha} = \{x \in A : \eta_A(x) \supseteq \alpha\}$.

Definition 7 ([19]). Let (ζ, A) be a non-null soft set over a nearsemiring R. Then, (ζ, A) is called a soft nearsemiring over R, if $\zeta(x)$ is a sub-nearsemiring of R for all $x \in Supp(\zeta, A)$.

Definition 8 ([19]). Let (η, A) be the soft right nearsemiring over right nearsemiring R_1 and (ζ, B) be the soft left nearsemiring over the left nearsemiring R_2. Let $f : R_1 \to R_2$ and $g : A \to B$ be the two mappings. Then, the pair (f, g) is said to be a soft nearsemiring anti-homomorphism if,

(i) f is an anti-epimorphism of nearsemirings.

(ii) g is Onto mapping.

(iii) $f(\eta(x)) = \zeta(g(x))$ for all $x \in A$.

Now, if f is an anti-isomorphism and$g$ is bijective then we say (f, g) a soft nearsemiring anti-isomorphism .

Table 1. Right nearsemiring

$+$	0	a	b	c	d	$.$	0	a	b	c	d
0	0	a	b	c	d	0	0	0	0	0	0
a	a	a	b	d	d	a	0	a	a	a	a
b	b	b	b	d	d	b	0	a	b	b	d
c	c	d	d	c	d	c	0	a	b	b	d
d	d	d	d	d	d	d	0	a	b	b	d

3. Soft Sub-nearsemirings and Soft Ideals

In this section, we initiate the notions of soft sub-nearsemirings and soft ideals of nearsemiring by using intersection operation of soft sets. For the sake of investigations, we apply several operations of soft set theory to them. We also provide few characterizations of distributively generated nearsemirings related to these soft substructures of nearsemirings.

Definition 9. Let (η, T) be a soft set over S, where T be a sub-nearsemiring of S. Then the soft set (η, T) is said to be a soft sub-nearsemiring of S if it satisfies the followings.

(i) $\eta(x + y) \supseteq \eta(x) \cap \eta(y)$

(ii) $\eta(xy) \supseteq \eta(x) \cap \eta(y)$

for all $x, y \in T$. We denote it by $(\eta, T) \tilde{<} S$ or $\eta_T \tilde{<} S$.

Corollary 1. It is easy to verify that if $\eta_T(x) = S$ for all $x \in T$, then η_T is a soft sub-nearsemiring of S. We will abbreviate such a soft sub-nearsemiring by $\eta_{\tilde{T}}$.

Example 1. Let $S = \{0, a, b, c, d\}$ be a (right) nearsemiring with the operations defined by the following Cayley tables.

Clearly, $T = \{0, a, b\}$ is a sub-nearsemiring of nearsemiring S. For $T = \{0, a, b\}$, let $\eta : T \to \wp(S)$ be a set-valued function defined by $\eta(0) = S$, $\eta(a) = \{0, b, c, d\}$, $\eta(b) = \{0, a\}$. It is easy to verify that (η, T) is a soft sub-nearsemiring of S. On the other hand, if we take $T_1 = \{0, b, c, d\}$ a sub-nearsemiring of S, and $\vartheta : T_1 \to \wp(S)$ be a set-valued function defined by $\vartheta(0) = \{0, a, b, d\}$, $\vartheta(b) = \vartheta(d) = \{0, b, d\}$, $\vartheta(c) = \{0, a\}$. Then $\vartheta(c.c) = \vartheta(b) = \{0, b, d\} \not\supseteq \vartheta(c) \cap \vartheta(c) = \{0, a\}$. Thus, (ϑ, T_1) is not a soft sub-nearsemiring of S.

If $\eta_T \tilde{<} S$ and $\eta_{T_1} \tilde{<} S$, then we can easily deduce the followings.

Proposition 1. (1) If $\eta_T \tilde{<} S$ and $\eta_{T_1} \tilde{<} S$, then $\eta_T \tilde{\cap} \eta_{T_1} \tilde{<} S$

(2) If $\eta_{T_1} \tilde{<} S_1$ and $\vartheta_{T_2} \tilde{<} S_2$, then $\eta_{T_1} \times \vartheta_{T_2} \tilde{<} S_1 \times S_2$.

(3) If $\eta_T \tilde{<} S$ and $\eta_{T_1} \tilde{<} S$, then $\eta_T \wedge \eta_{T_1} \tilde{<} S$.

Table 2. Right nearsemiring

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

$.$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	d
c	0	a	b	b	d
d	0	a	b	b	d

Proof. Easy to proof (1), (2) & (3) and hence omitted. □

On the other hand, if η_S and η_T are two soft sub-nearsemirings of a nearsemiring R, then $\eta_S \vee \eta_T$ need not be a soft sub-nearsemiring as we see in the below example.

Example 2. Let $S = \{0, a, b, c, d\}$ be a (right) nearsemiring with the operations defined by the following Cayley tables.

Clearly, $T = \{0, a, b\}$ is a sub-nearsemiring of nearsemiring S, and let $\eta : T \to \wp(S)$ be a set-valued function defined by $\eta(0) = S$, $\eta(a) = \{0, b, c, d\}$, $\eta(b) = \{0, a\}$. It is easy to verify that (η, T) is a soft sub-nearsemiring of S. Again by taking a sub-nearsemiring $T_1 = \{0, a, b, d\}$, and a set-valued function $\vartheta : T_1 \to \wp(S)$ defined by $\vartheta(0) = S$, $\vartheta(a) = \{0, a, b, d\}$, $\vartheta(b) = \vartheta(d) = \{0, b, d\}$. Then, one can easily show that (ϑ, T_1) is a soft sub-nearsemiring of S. Consider, $(\eta_T \vee \vartheta_{T_1})((b, 0) + (0, a)) = (\eta_T \vee \vartheta_{T_1})\,(b, a) = \{0, a, b, d\} \not\supseteq (\eta_T \vee \vartheta_{T_1})(b, 0) \cap (\eta_T \vee \vartheta_{T_1})(0, a) = \{0, a, b, c, d\} \cap \{0, a, b, c, d\} = \{0, a, b, c, d\}$. Hence, $\eta_T \vee \vartheta_{T_1}$ is not a soft sub-nearsemiring.

The following lemma is obvious.

Lemma 1. (1) If S is distributively generated nearsemiring and T be its sub-nearsemiring. If $t \in T$, such that $t = \sum_{i=1}^{n} a_i b_i$ and η_T be a soft sub-nearsemiring of S, then $\eta_T(t) = \eta_T(\sum_{i=1}^{n} a_i b_i) \supseteq \eta_T(a_i) \cap \eta_T(b_i)$ for all $1 \leq i \leq n$.

(2) If S is an additively commutative nearsemiring and $t = \sum_{i=1}^{n} a_i + b_i$ such that η_T soft sub-nearsemiring of S. Then, $\eta_T(t) = \eta_T\,(\sum_{i=1}^{n} a_i + b_i) \supseteq \eta_T(a_i) \cap \eta_T(b_i)$ for all $1 \leq i \leq n$.

Now we adjust few operations introduced in [35] by taking T a sub-nearsemiring of a nearsemiring S and $\eta_T, \vartheta_T \in S(S)$.

Definition 10. Let η_T and ϑ_T be a soft sets over the nearsemiring S. Then their soft intersection sum $\eta_T \oplus \vartheta_T$ is defined as

$$(\eta_T \oplus \vartheta_T)(x) = \begin{cases} \bigcup\limits_{x=y+z} \{\eta_T(y) \cap \vartheta_T(z)\}, & \\ \quad \text{if there exist } y, z \in T \text{ such that } x = y + z, & \\ \emptyset, \quad \text{otherwise,} & \end{cases}$$

for all $x \in T$.

Definition 11. If η_T and ϑ_T are two soft sets over the nearsemiring S. Then their soft product is defined by

$$(\eta_T \odot \vartheta_T)(x) = \begin{cases} \bigcup\limits_{x=y.z} \{\eta_T(y) \cap \vartheta_T(y)\}, & \\ \quad \text{if there exist } y, z \in T \text{ such that } x = y.z, & \\ \emptyset, \quad \text{otherwise,} & \end{cases}$$

for all $x \in T$.

The soft uni-int product may be define as follows.

Definition 12. Let η_T and ϑ_T be soft set over the nearsemiring S. Then the soft union intersection (uni-int) product $\eta_S \star \vartheta_S$ is defined as

$$(\eta_T \star \vartheta_T)(x) = \begin{cases} \bigcup\limits_{x=\sum_{i=1}^{n} a_i b_i} \{\eta_T(a_i) \cap \vartheta_T(b_i)\}, & \\ \quad \text{if } x = \sum_{i=1}^{n} a_i b_i \text{ and } a_i b_i \neq 0, & \\ \quad \text{for all } 1 \leq i \leq n, & \\ \emptyset, \quad \text{otherwise.} & \end{cases}$$

Theorem 1. Let T be a distributively generated sub-nearsemiring of nearsemiring S and η_T be the soft set over S. Then, η_T is a soft sub-nearsemiring of S iff $\eta_T(t_1 + t_2) \supseteq \eta_T(t_1) \cap \eta_T(t_2)$ and $\eta_T \star \eta_T \mathrel{\tilde{\subset}} \eta_T$, for all $t_1, t_2 \in T$.

Proof. We assume that η_T is a soft sub-nearsemiring over S. Then $\eta_T(t_1 + t_2) \supseteq \eta_T(t_1) \cap \eta_T(t_2)$. Let $t = \sum_{i=1}^{n} x_i y_i$ such that $(\eta_T \star \eta_T)(t) = \emptyset$, then nothing to prove. Let $(\eta_T \star \eta_T)(t) \neq \emptyset$. Then

$$\begin{aligned}(\eta_T \star \eta_T)(t) &= \bigcup\nolimits_{t=\sum_{i=1}^{n} x_i y_i} (\eta_T(x_i) \cap \eta_T(y_i)) \\ &\subseteq \bigcup\nolimits_{t=\sum_{i=1}^{n} x_i y_i} (\eta_T(x_i y_i)) \\ &= \bigcup\nolimits_{t=\sum_{i=1}^{n} x_i y_i} (\eta_T)(t) \\ &= (\eta_T)(t).\end{aligned}$$

Hence $\eta_T \star \eta_T \mathrel{\tilde{\subset}} \eta_T$. Conversely, let $\eta_T(t_1+t_2) \supseteq \eta_T(t_1) \cap \eta_T(t_2)$ and $\eta_T \star \eta_T \mathrel{\tilde{\subset}} \eta_T$. Then

$$\begin{aligned}\eta_T(t_1t_2) &\supseteq (\eta_T \star \eta_T)(t_1t_2)\\ &= \bigcup_{t_1t_2=\sum_{i=1}^{n} x_iy_i} (\eta_T(x_i) \cap \eta_T(y_i))\\ &\supseteq (\eta_T(t_1) \cap \eta_T(t_2)).\end{aligned}$$

Thus, η_T is a soft sub-nearsemiring of S. □

Let T be an additive commutative sub-nearsemiring of a nearsemiring S, then we have the following.

Proposition 2. Let η_T and ϑ_T be two soft sub-nearsemirings of nearsemiring S. Then $f_R \oplus h_{\tilde{R}} \subseteq f_R$ (resp., $f_{\tilde{R}} \oplus h_R = h_R$).

Proof. Let η_T and ϑ_T be two soft sub-nearsemirings. Let $t = x+y$ such that $(\eta_T \oplus \vartheta_T)(t) \neq \emptyset$ and assume that $\vartheta_T(x) = \vartheta_{\tilde{T}}(x)$, for all $x \in T$. Then,

$$\begin{aligned}(\eta_T \oplus \vartheta_T)(t) &= \bigcup_{t=y+z} \{\eta_T(y) \cap \vartheta_T(z)\}\\ &= \bigcup_{t=y+z} \{\eta_T(y) \cap \vartheta_{\tilde{T}}(z)\}\\ &\subseteq \eta_T(y+z) \cap S\\ &= \eta_T(t).\end{aligned}$$

Hence $\eta_T \oplus \vartheta_T \subseteq \eta_T$. Similarly, by assuming $\eta_T = \eta_{\tilde{T}}$ one can prove $\eta_T \oplus \vartheta_T \subseteq \vartheta_T$. □

Theorem 2. Let η_T and ϑ_{T_1} be soft sets over nearsemiring S and Ψ be a nearsemiring isomorphism from T to T_1. If η_T is a soft sub-nearsemiring of S, then $\Psi(\eta_T)$ is also a soft sub-nearsemiring of S.

Proof. Let $t_1', t_2' \in T_1$. Since Ψ is surjective, there exists t_1, t_2 $\in T$ such that $\Psi(t_1) = t_1'$ and $\Psi(t_2) = t_2'$. Then,

$$\begin{aligned}&\Psi(\eta_T)(t_1'+t_2')\\ &= \bigcup\{\eta(t) : t \in T,\ \Psi(t) = t_1'+t_2'\}\\ &= \bigcup\{\eta(t) : t \in T,\ t = \Psi^{-1}(t_1'+t_2')\}\\ &= \bigcup\{\eta(t) : t \in T,\ t = \Psi^{-1}(\Psi(t_1+t_2))\}\\ &= \bigcup\{\eta(t_1+t_2) : t_i \in T,\ \Psi(t_i) = t_i',\ i = 1,\ 2\}\\ &\supseteq \{\eta(t_1) \cap \eta(t_2) : t_i \in T,\ \Psi(t_i) = t_i',\ i = 1,\ 2\}\\ &= \Psi(\eta_T)(t_1') \cap \Psi(\eta_T)(t_2').\end{aligned}$$

Table 3. Right nearsemiring

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

$.$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	b	b
d	0	a	d	d	d

Similarly, one can easily prove that $\Psi(\eta_T)(t_1't_2') \supseteq \Psi(\eta_T)(t_1')$ $\cap$ $\Psi(\eta_T)(t_2')$. Hence, $\Psi(\eta_T)$ is a soft sub-nearsemiring of S. □

Theorem 3. Let T and T_1 be the two sub-nearsemirings of S. Let η_T and ϑ_{T_1} be soft sets over nearsemiring S and Ψ be a nearsemiring homomorphism from T to T_1. If ϑ_{T_1} is a soft sub-nearsemiring of S, then so is $\Psi^{-1}(\vartheta_{T_1})$.

Proof. For each t_1, $t_2 \in T$,

$$\begin{aligned}(\Psi^{-1}(\vartheta_{T_1})(t_1+t_2) &= \vartheta_{T_1}(\Psi(t_1+t_2))\\ &= \vartheta_{T_1}(\Psi(t_1)+\Psi(t_2))\\ &\supseteq \vartheta_{T_1}(\Psi(t_1)) \cap \vartheta_{T_1}(\Psi(t_2))\\ &= (\Psi^{-1}(\vartheta_{T_1}))(t_1) \cap (\Psi^{-1}(\vartheta_{T_1}))(t_2).\end{aligned}$$

Similarly, we can prove that $(\Psi^{-1}(\vartheta_{T_1}))(t_1t_2) \supseteq (\Psi^{-1}(\vartheta_{T_1}))$ $(t_1)\cap$ $(\Psi^{-1}(\vartheta_{T_1}))(t_2)$. Thus $\Psi^{-1}(\vartheta_{T_1})$ is a soft subnearsemiring of S. □

Soft ideals of a nearsemirings

Definition 13. Let I be an ideal of nearsemiring S (i.e., $I \mathrel{\tilde{\lhd}} S$) and (η, I) be a soft set over S. Then, (η, I) is said to be soft right (resp., left) ideal of S, if it satisfies the following conditions.

(i) $\eta(x+y) \supseteq \eta(x) \cap \eta(y)$

(ii) $\eta(xr) \supseteq \eta(x)$ (resp., $\eta(rx) \supseteq \eta(x)$) for all x, $y \in I$, $r \in S$.

We will abbreviate the soft right (left) ideal of S by $\eta_I \mathrel{\tilde{\lhd}_r} S$ (resp., $\eta_I \mathrel{\tilde{\lhd}_l} S$). We call (η, I) a soft ideal if it is two sided soft ideal and will be abbreviated as $\eta_I \mathrel{\tilde{\lhd}} S$.

Corollary 2. It is easy to verify that if $\eta_I(x) = S$, for all $x \in I$, then η_I is a soft (two sided) ideal of S. This type of soft ideal will be abbreviated by $\eta_{\tilde{I}}$.

Example 3. Let $S = \{0, a, b, c, d\}$ be a (right) nearsemiring with the operations defined by the following Cayley tables, and $I = \{0, a, b, d\}$ be an ideal of S.

Let $\eta : I \to \wp(S)$ defined by $\eta(0) = S = \{0, a, b, c, d\}$, $\eta(a) = \{0, b, c, d\}$, $\eta(b) = \eta(d) = \{0, b, d\}$. It is easy to verify that (η, I) is a soft ideal over S.

Remark 1. Every soft ideal of a nearsemiring S is a soft sub-nearsemiring of S.

Lemma 2. Let $x = \sum_{i=1}^{n} a_i b_i$ and η_I be a soft left (resp., right) ideal of S. Then, $\eta_I(x) = \eta_I(\sum_{i=1}^{n} a_i b_i) \supseteq \eta_I(b_i)$ (resp., $\eta_I(x) = \eta_I(\sum_{i=1}^{n} a_i b_i) \supseteq \eta_I(a_i)$), for all $1 \leq i \leq n$.

Theorem 4. If S is a left seminearring (i.e., zero symmetric nearsemiring) and η_I be a soft right ideal (i.e., $\eta_I \mathrel{\tilde{\lhd}_r} S$) and ϑ_J be a soft left ideal (i.e., $\vartheta_J \mathrel{\tilde{\lhd}_l} S$). Then, $\eta_I \odot \vartheta_J \subseteq \eta_I \tilde{\cap} \vartheta_J$.

Proof. Let S be a seminearring. Suppose $(\eta_I \odot \vartheta_J) \neq \emptyset$. Since ϑ_J is a soft left ideal ($\vartheta_J \mathrel{\tilde{\lhd}_l} S$) of S, $\vartheta_J(x) = \vartheta_J(yz) \supseteq \vartheta_J(z)$. Also, η_I is a soft right ideal and S is a zero-symmetric nearsemiring, then $\eta_I(x) = \eta_I(yz) = \eta_I((0 + y)z + 0z) \supseteq \eta_I(y)$. Therefore,

$$\begin{aligned}(\eta_I \odot \vartheta_J)(x) &= \bigcup_{x=y.z} \{\eta_T(y) \cap \vartheta_{T_1}(z)\}\\ &\subseteq \eta_I(y) \cap (\vartheta_J)(z)\\ &\subseteq \eta_I(x) \cap \vartheta_J(x)\\ &= (\eta_I \tilde{\cap} \vartheta_J)(x).\end{aligned}$$

Hence, $\eta_I \odot \vartheta_J \subseteq \eta_I \tilde{\cap} \vartheta_J$. ☐

Theorem 5. Let I be a left ideal of a nearsemiring S, and η_I be the soft set over S. Then, η_I is a soft left ideal of S iff $\eta_I(x + y) \supseteq \eta_I(x) \cap \eta_I(y)$ and $\eta_{\tilde{I}} \star \eta_I \mathrel{\tilde{\subset}} \eta_I$.

Proof. We assume that η_I is a soft left ideal of S. Then, $\eta_I(x + y) \supseteq \eta_I(x) \cap \eta_I(y)$. Let $z = \sum_{i=1}^{n} x_i y_i$ such that $(\eta_{\tilde{I}} \star \eta_I)(z) = \emptyset$, then it is easy to show that $\eta_{\tilde{I}} \star \eta_I \mathrel{\tilde{\subset}} \eta_I$. For this, let $(\eta_{\tilde{I}} \star \eta_I)(z) \neq \emptyset$, then

$$\begin{aligned}(\eta_{\tilde{I}} \star \eta_I)(z) &= \bigcup_{z=\sum_{i=1}^{n} x_i y_i} (\eta_{\tilde{I}}(x_i) \cap \eta_I(y_i))\\ &\subseteq \bigcup_{z=\sum_{i=1}^{n} x_i y_i} (S \cap \eta_I(\sum_{i=1}^{n} x_i y_i))\\ &= \bigcup_{z=\sum_{i=1}^{n} x_i y_i} (\eta_I)(z)\\ &= (\eta_I)(z).\end{aligned}$$

Thus, $\eta_{\tilde{I}} \star \eta_I \mathrel{\tilde{\subset}} \eta_I$. Conversely, let $\eta_I(x + y) \supseteq \eta_I(x) \cap \eta_I(x)$ and $\eta_{\tilde{I}} \star \eta_I \mathrel{\tilde{\subset}} \eta_I$. Then

$$\begin{aligned}\eta_I(xy) &\supseteq (\eta_{\tilde{I}} \star \eta_T)(xy)\\ &= (\bigcup_{xy=\sum_{i=1}^{n} x_i y_i} (\eta_{\tilde{I}}(x_i) \cap \eta_I(y_i))\\ &\supseteq (\eta_{\tilde{I}}(x) \cap \eta_I(y))\\ &= S \cap \eta_I(y)\\ &= \eta_I(y).\end{aligned}$$

Hence, η_T is a soft left ideal of a nearsemiring S. ☐

Corollary 3. Let I be a right ideal of a nearsemiring S and η_I be the soft set over S. Then, η_I is a soft right ideal of S iff $\eta_I(x + y) \supseteq \eta_I(x) \cap \eta_I(y)$ and $\eta_I \star \eta_{\tilde{I}} \mathrel{\tilde{\subset}} \eta_I$.

Remark 2. Let I be an ideal of a nearsemiring S and η_I be the soft set over S. Then, η_I is a soft ideal of S iff $\eta_I(x + y) \supseteq \eta_I(x) \cap \eta_I(y)$ and $\eta_I \star \eta_{\tilde{I}} \mathrel{\tilde{\subset}} \eta_I$,$\eta_{\tilde{I}} \star \eta_I \mathrel{\tilde{\subset}} \eta_I$.

4. Relationships between Soft Substructures and Classical Substructures of Nearsemirings

In this section, we provide few interrelationships among some soft substructures and classical substructures of nearsemirings. For this, we continue our study about the soft ideals of nearsemirings under α-inclusion, images, pre-images, nearsemiring homomorphisms and nearsemiring anti-homomorphisms.

Theorem 6. Let I be an ideal of a nearsemiring S. Let η_I be the soft set over S. Then η_I is a soft left (right) ideal of nearsemiring S iff $\eta_I^{\supseteq\alpha}$is a left (right) ideal of S for any $\alpha \in \wp(S)$ such that $\eta_I^{\supseteq\alpha} \neq \emptyset$.

Proof. Let η_I be the soft left ideal of S. Let $a, b \in \eta_I^{\supseteq\alpha} \Rightarrow \eta_I(a) \supseteq \alpha$ and $\eta_I(b) \supseteq \alpha$. It means $\eta_I(a + b) \supseteq \eta_I(a) \cap \eta_I(b) \supseteq \alpha$. Hence $a + b \in \eta_I^{\supseteq\alpha}$. We consider $a \in \eta_I^{\supseteq\alpha}$ which implies $\eta_I(a) \supseteq \alpha$ and $r \in S$. Hence $\eta_I(ra) \supseteq \eta_I(a) \supseteq \alpha$. Thus $ra \in \eta_I^{\supseteq\alpha}$ and hence $\eta_I^{\supseteq\alpha}$ is a left ideal of S for any $\alpha \in \wp(S)$ such that $\eta_I^{\supseteq\alpha} \neq \emptyset$.

Conversely, suppose that $\eta_I^{\supseteq\alpha}$ is a left ideal of S such that $\eta_I^{\supseteq\alpha} \neq \emptyset$, for any $\alpha \in \wp(S)$. Let $a, b \in \eta_I^{\supseteq\beta}$ such that $\beta = \eta_I(a) \cap \eta_I^{\supseteq\alpha}(b)$ for each $a, b \in S \Rightarrow a + b \in \eta_I^{\supseteq\beta}$. Thus, $\eta_I(a + b) \supseteq \beta = \eta_I(a) \cap \eta_I(b)$. Also, for each $a \in \eta_I^{\supseteq\gamma}$ such that $\gamma = \eta_I(a)$, we get $ra \in \eta_I^{\supseteq\gamma} \Rightarrow \eta_I(ra) \supseteq \eta_I(a)$. Hence η_I is a soft left ideal of S. ☐

Example 4. Let $S = \{0, a, b, c, d\}$ be a (right) nearsemiring with the operations defined by the following Cayley tables.

Let $U = S$, $I = \{0, a, b, d\}$ be an ideal of S and $\alpha = \{0, b\}$ be a subset of S. We define the soft set η_I as $\eta_I(0) = S$, $\eta_I(a) = \{0, b\}$, $\eta_I(b) = \{0, a, b, d\}$, $\eta_I(d) = \{0, a, b, d\}$. Clearly, η_I is a soft ideal of nearsemiring S, one can easily verify that $\eta_I^{\supseteq\alpha} \neq \emptyset$. is an ideal of S.

Theorem 7. Let η_R and ϑ_T be soft sets over nearsemirings R and T, respectively. Let Ψ be a nearsemiring anti-isomorphism from R to T. If η_R is a soft left ideal of R, then $\Psi(\eta_R)$ is a soft right ideal of T.

Proof. Let $t_1, t_2 \in T$. Since Ψ is surjective, then there exist r_1, $r_2 \in R$ such that $\Psi(r_1) = t_1$, $\Psi(r_2) = t_2$. Then, we have

$$\begin{aligned}
&(\Psi(\eta_R))(t_1 + t_2)\\
&= \bigcup\{\eta_R(r) : r \in R,\ \Psi(r) = t_1 + t_2\}\\
&= \bigcup\{\eta_R(r) : r \in R,\ r = \Psi^{-1}(t_1 + t_2)\}\\
&= \bigcup\{\eta_R(r) : r \in R,\ r = \Psi^{-1}(\Psi(r_1 + r_2)) = r_1 + r_2\}\\
&= \bigcup\{\eta_R(r_1 + r_2) : r_i \in R,\ \Psi(r_i) = t_i,\ i = 1,\ 2\}\\
&\supseteq \bigcup\{\eta_R(r_1) \cap f_R(r_2) : r_i \in R,\ \Psi(r_i) = t_i,\ i = 1,\ 2\}\\
&= (\bigcup\{\eta_R(r_1) : r_1 \in R,\ \Psi(r_1) = t_1\})\\
&\quad \cap (\bigcup\{\eta_R(r_2) : r_2 \in R,\ \Psi(r_2) = t_2\})\\
&= (\Psi(\eta_R))(t_1)) \cap (\Psi(\eta_R))(t_2).
\end{aligned}$$

Similarly, one case easily verify that
$(\Psi(\eta_R))(t_1t_2) \supseteq \Psi(\eta_R))(t_1) \cap \Psi(\eta_R))(t_2)$. Moreover,

$$\begin{aligned}
&(\Psi(\eta_R))(t_1t_2)\\
&= \bigcup\{\eta_R(r) : r \in R,\ \Psi(r) = t_1t_2\}\\
&= \bigcup\{\eta_R(r) : r \in R,\ r = \Psi^{-1}(t_1t_2)\}\\
&= \bigcup\{\eta_R(r) : r \in R,\ r = \Psi^{-1}(\Psi(r_2r_1)) = r_2r_1\}\\
&= \bigcup\{\eta_R(r_2r_1) : r_i \in R,\ \Psi(r_i) = t_i,\ i = 1,\ 2\}\\
&\supseteq \bigcup\{\eta_R(r_1) : r_1 \in R,\ \Psi(r_1) = t_1\}\\
&= (\Psi(f_R))(t_1)).
\end{aligned}$$

Thus, $\Psi(f_R)$ is a soft right ideal of T. □

Table 4. Right nearsemiring

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	d	d
d	d	d	d	d	d

$.$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	b	b	b	b
c	0	a	a	a	a
d	0	a	a	a	a

Table 5. Right nearsemiring

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

$.$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	b	b	b	d
c	0	c	c	c	c
d	0	d	d	d	d

Theorem 8. Let η_R and η_T be soft sets over a nearsemirings R and T, respectively. Let Ψ be a nearsemiring anti-homomorphism from R to T. If η_T is a soft left ideal of T, then $\Psi^{-1}(\eta_T)$ is a soft right ideal of R.

Proof. Let $r_1, r_2 \in R$. Then

$$\begin{aligned}
(\Psi^{-1}(\eta_T))(r_1 + r_2) &= \eta_T(\Psi(r_1 + r_2)\\
&= \eta_T((\Psi(r_1) + (\Psi(r_1))\\
&\supseteq \eta_T((\Psi(r_1) \cap (\Psi(r_1))\\
&= (\Psi^{-1}(\eta_T))(r_1) \cap (\Psi^{-1}(\eta_T))(r_2).
\end{aligned}$$

Similarly, one can easily verify that $(\Psi^{-1}(\eta_T))(r_1r_2) \supseteq \Psi^{-1}(\eta_T))(r_1) \cap \Psi^{-1}(\eta_T))(r_2)$. Also,

$$\begin{aligned}
(\Psi^{-1}(\eta_T))(r_1r_2) &= \eta_T(\Psi(r_1r_2)\\
&= \eta_T(\Psi(r_2)\Psi(r_1))\\
&\supseteq \eta_T(\Psi(r_1))\\
&= (\Psi^{-1}(\eta_T))(r_1).
\end{aligned}$$

Hence $\Psi^{-1}(\eta_T)$ is a soft right ideal of R. □

Example 5. Let $S = \{0, a, b, c, d\}$ be a nearsemiring with the operations defined by the following Cayley tables.

$I = \{0, b, d\}$ be an ideal of S and η_I be a soft right ideal of S defined as $\eta_I(0) = S$, $\eta_I(b) = \{0, b, c, d\}$ and $\eta_I(b) = \{0, b, c, d\}$.

Also, let $T = \{0, a, b, c, d\}$ be a nearsemiring with the operations defined by the following Cayley tables.

$J = \{0, b, d\}$ be an ideal of R and η_J be a soft left ideal of S defined as $\eta_J(0) = T$, $\eta_J(b) = \{0, b, c, d\}$ and $\eta_J(b) =$

Table 6. Right nearsemiring

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	b	d	d
b	b	b	b	d	d
c	c	d	d	c	d
d	d	d	d	d	d

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	b	b
d	0	a	d	d	d

$\{0, b, c, d\}$. Let $\psi : S \to T$ be an anti-homomorphism defined by $\psi(x) = x$. Then, for a soft left ideal η_J of T, the inverse map $\psi^{-1}(\eta_J) = \eta_I$ is a soft right ideal of S.

5. Soft S-Subsemigroup of Nearsemirings and Soft S-Morphisms

In this section, we introduce the notion of soft S-subsemigroup of a nearsemiring S. We shift few terms of S-morphism of classical nearsemiring theory towards soft set theory. A subsemigroup H of S is called a left S-subsemigroup of S if $SH \subseteq H$ and H is called a right S-subsemigroup of S if $HS \subseteq H$.

Definition 14. Let H be a left (resp., right) S-subsemigroup of a nearsemiring S and (η, H) be a soft set over S. If for all $x, y \in H$ and $r \in S$,

(i) $\eta(x + y) \supseteq \eta(x) \cap \eta(y)$

(ii) $\eta(rx) \supseteq \eta(x)$ (resp., $\eta(xr) \supseteq \eta(x)$)

then the soft set (η, H) is the soft left (resp., right) S-subsemigroup of S and is denoted by $(\eta, H) <^l_S S$ (resp., $(\eta, H) <^r_S S$) or simply $\eta_H <^l_S S$ (resp., $\eta_H <^r_S S$).

Remark 3. Every soft left (resp., right) ideal of a nearsemiring S is also a soft left (resp., right) S-subsemigroup of a nearsemiring S.

Theorem 9. If $\eta_H <^l_S S$ (resp., $\eta_H <^r_S S$) and $\vartheta_K <^l_S S$ (resp., $\vartheta_K <^r_S S$), then $\eta_H \tilde{\cap} \vartheta_K <^l_S S$ (resp., $\eta_H \tilde{\cap} \vartheta_K <^r_S S$), if it is non-null.

Proof. Let $\eta_H <^l_S S$ and $\vartheta_K <^l_S S$. Let H and K are left S-subsemigroups of S such that $H \cap K \neq \emptyset$, then it can easily seen that $H \cap K$ is a left S-subsemigroup of S. Since, $\eta_H \tilde{\cap} \vartheta_K = \xi_{H \tilde{\cap} K}$, where $\xi_{H \tilde{\cap} K}(x) = \eta(x) \cap \vartheta(x)$, for all $x \in H \cap K$. Then, for all $x, y \in H \cap K$ and for all $r \in S$,

$$\begin{aligned}(i)\ \xi_{H \tilde{\cap} K}(x + y) &= \eta_H(x + y) \cap \vartheta_K(x + y)\\ &\supseteq (\eta(x) \cap \eta(y)) \cap (\vartheta(x) \cap \vartheta(y))\\ &= (\eta(x) \cap \vartheta(x)) \cap (\eta(y) \cap \vartheta(y))\\ &= \xi_{H \tilde{\cap} K}(x) \cap \xi_{H \tilde{\cap} K}(y)\end{aligned}$$

$$\begin{aligned}(ii)\ \xi_{H \tilde{\cap} K}(rx) &= \eta(rx) \cap \vartheta(rx)\\ &\supseteq (\eta(x) \cap \vartheta(x)) = \xi_{H \tilde{\cap} K}(x).\end{aligned}$$

Hence, $\eta_H \tilde{\cap} \vartheta_K <^l_S S$. Similarly, one can easily prove that $\eta_H \tilde{\cap} \vartheta_K <^r_S S$. □

Definition 15. Let S_1 and S_2 are nearsemirings and let $\eta_H <^l_{S_1} S_1$ (resp., $\eta_H <^r_{S_1} S_1$) and $\vartheta_K <^l_{S_2} S_2$ (resp., $\vartheta_K <^r_{S_2} S_2$). The product of two soft left (resp., right) S-subsemigroups (η, H) and (ϑ, K) is defined as $(\eta, H) \times (\vartheta, K) = \xi_{H\times K}$, where $\xi_{H\times K}(x, y) = \eta_H(x) \times \vartheta_K(y)$, for all $(x, y) \in H\times K$.

Theorem 10. If $\eta_H <^l_{S_1} S_1$(resp., $\eta_H <^r_{S_1} S_1$) and $\vartheta_K <^l_{S_2} S_2$ (resp., $\vartheta_K <^r_{S_2} S_2$), then $\eta_H \times \vartheta_K <^l_{S_1\times S_2} S_1\times S_2$ (resp., $\eta_H \times \vartheta_K <^r_{S_1\times S_2} S_1\times S_2$).

Proof. Let $\eta_H <^l_{S_1} S_1$ and $\vartheta_K <^l_{S_2} S_2$. Since H is a left S_1-subsemigroup of S_1 and K is a left S_2-subsemigroup of S_2, it can be easily seen that $H\times K$ is a left $S_1\times S_2$-subsemigroup of $S_1\times S_2$. Since, $\eta_H \times \vartheta_K = \xi_{H\times K}$, where $\xi_{H\times K}(x) = \eta_H(x) \times \vartheta_K(y)$ for all $(x, y) \in H \times K$. Then, for all $(x_1, y_1), (x_2, y_2) \in H \times K$, $(r_1, r_2) \in S_1\times S_2$ we have

$$\begin{aligned}(i)\ &\xi_{H\times K}((x_1, y_1) + (x_2, y_2))\\ &= \xi_{H\times K}(x_1 + x_2, y_1 + y_2)\\ &= \eta_H(x_1 + x_2) \times \vartheta_K(y_1 + y_2)\\ &\supseteq (\eta_H(x_1) \cap \eta_H(x_2)) \times (\vartheta_K(y_1) \cap \vartheta_K(y_2))\\ &= (\eta_H(x_1) \times \vartheta_K(y_1)) \cap (\eta_H(x_2) \times \vartheta_K(y_2))\\ &= \xi_{H\times K}(x_1, y_1) \cap \xi_{H\times K}(x_2, y_2)\end{aligned}$$

$$\begin{aligned}(ii)\ &\xi_{H\times K}((r_1, r_2)(x_1, y_1))\\ &= \xi_{H\times K}(r_1x_1, r_2y_1)\\ &= \eta_H(r_1x_1) \times \vartheta_K(r_2y_1)\\ &\supseteq \eta_H(x_1) \times \vartheta_K(y_1)\\ &= \xi_{H\times K}(x_1, y_1).\end{aligned}$$

Hence, $\eta_H \times \vartheta_K <^l_{S_1\times S_2} S_1\times S_2$. Similarly, one can prove that $\eta_H \times \vartheta_K <^r_{S_1\times S_2} S_1\times S_2$. □

Definition 16. Let (η, H) be a soft left (resp., right) S_1-sub-semigroup of a nearsemiring S_1 and (ϑ, K) be a soft left (resp., right) S_2-subsemigroup of S_2. Suppose $f : S_1 \to S_2$ and $g : H \to K$ be the two mappings. Then the pair (f, g)

is called a soft S-homomorphism if it satisfies the following conditions.

(i) f is an S-epimorphism

(ii) g is a surjective mapping

(iii) $f(\eta(x)) = \vartheta(g(x))$ for all $x \in H$.

If there exists a soft S-homomorphism between (η, H) and (ϑ, K), we call (η, H) is soft S-homomorphic to (ϑ, K). Similarly, if f is an S-isomorphism of S-semigroups and g is a bijective mapping, then we call (f, g) a soft S-isomorphism, and say that (η, H) is soft S-isomorphic to (ϑ, K) i.e., $(\eta, H) \simeq (\vartheta, K)$.

Example 6. Let us take a nearsemiring $S = \{0, a, b, c, d\}$ given in the Example2. And let $H = \{0, a, b, d\}$ be a S-subsemigroup of an S-semigroup $(S, +)$. We also assume that $H = K$ and $S_1 = S_2 = S$. Then, let $f : S \to S$ be the map defined by $f(x) = x$. One can easily verify that f is an S-isomorphism. Let $g : H \to K$ be the map defined by $g(x) = x$, which is clearly a bijective mapping. Let $\eta : H \to \wp(R)$ be the map defined by $\eta(x) = \{y \in S : x\Re y \Leftrightarrow xy \in \{0, a\}\}$. Then, $\eta(0) = \eta(a) = S$, $\eta(b) = \eta(d) = \{0, a\}$. By doing a simple calculations, one can prove that (η, H) is a soft S-subsemigroup of S. Similarly, we define the map $\vartheta : K \to \wp(S)$ by $\vartheta(x) = \{y \in S : x\Re y \Leftrightarrow xy \in \{0, a\}\}$. Then, $\vartheta(0) = \vartheta(a) = S$, $\vartheta(b) = \vartheta(d) = \{0, a\}$. Evidently, (ϑ, K) is a soft S-subsemigroup of S. Finally, consider $f(\eta(0)) = S$, and $\vartheta(g(0)) = \vartheta(0) = S$, so $f(\eta(0)) = \vartheta(g(0))$. Again, consider $f(\eta(a)) = f(S) = S$ and $\vartheta(g(a)) = \vartheta(\{0, a\}) = S$, so $f(\eta(a)) = \vartheta(g(a))$. Similarly, we can show that for all $x \in H$, $f(\eta(x))= \vartheta(g(x))$. Hence $(\eta, H) \simeq (\vartheta, K)$.

Theorem 11. Let H, K be the two S-subsemigroups of S. Let η_H and ϑ_K be the soft sets over S and Ψ be an S-isomorphism from H to K. If η_H is a soft S-subsemigroup of S, then so is $\Psi(\eta_H)$.

Proof. Let $k_1, k_2 \in K$. Since Ψ is surjective, there exists h_1, $h_2 \in H$ such that $\Psi(h_1) = k_1$ and $\Psi(h_2) = k_2$. Thus

$$\begin{aligned}
&\Psi(\eta_H)(k_1 + k_2)\\
&= \bigcup\{\eta_H(h) : h \in H,\ \Psi(h) = (k_1 + k_2)\}\\
&= \bigcup\{\eta_H(h) : h \in H,\ h = \Psi^{-1}(k_1 + k_2\}\\
&= \bigcup\{\eta_H(h) : h \in H,\ h = \Psi^{-1}(\Psi(h_1 + h_2)) = h_1 + h_2\}\\
&= \bigcup\{\eta_H(h_1 + h_2) :\ k_i \in K,\ \Psi(h_i) = k_i,\ i = 1,\ 2\}\\
&\supseteq \bigcup\{\eta(h_1) \cap \eta(h_2) :\ h_i \in H,\ \Psi(h_i) = k_i,\ i = 1,\ 2\}\\
&= (\bigcup\{\eta(h_1) : k_1 \in K,\ \Psi(h_1) = k_1\})\\
&\quad \cap (\bigcup\{\eta(h_2) : k_2 \in K,\ \Psi(h_2) = k_2\})\\
&= (\Psi(\eta_H))(h_1) \cap (\Psi(\eta_H))(h_2).
\end{aligned}$$

Now, let $r \in S$ and $k \in K$. Since Ψ is surjective, there exists $h' \in H$ such that $\Psi(h') = k$. Then,

$$\begin{aligned}
&(\Psi(\eta_H)(rk)\\
&= \bigcup\{\eta_H(h) : h \in H,\ \Psi(h) = rk\}\\
&= \bigcup\{\eta_H(h) : h \in H,\ h = \Psi^{-1}(rk)\}\\
&= \bigcup\{\eta_H(h) : h \in H,\ h = \Psi^{-1}(r\Psi(h'))\}\\
&= \bigcup\{\eta_H(h) : h \in H,\ h = \Psi^{-1}(\Psi(rh')) = rh'\}\\
&= \bigcup\{\eta_H(rh') : h' \in H,\ \Psi(h') = k\}\\
&\supseteq \bigcup\{\eta_H(h') : h' \in H, \Psi(h') = k\}\\
&= (\Psi(\eta_H))(k).
\end{aligned}$$

Hence $\Psi(\eta_H)$ is a soft S-subsemigroup of S. □

6. Conclusion

In this study, we have introduced soft sub-nearsemirings, soft ideals of nearsemirings and soft S-subsemigroups of nearsemirings by using the definition of soft sets introduced by Molodtsov [1]. We have applied a number of soft set operations to our newly established soft algebraic structures. Moreover, we investigated these soft algebraic structures through soft nearsemiring homomorphism and soft nearsemiring anti-homorphism. Throughout our study, we provide bunch of illustrative examples. In addition, we have provided few relationships among these soft substructures and classical substructures of nearsemirings. Through these relationships, we established the connection among the soft set theory, classical set theory and nearsemiring theory. To extend this study, one can discuss some other classical algebraic structures such as balanced nearsemirings, affine near-semirings etc on the same pattern. Classical nearsemiring is used in the theory of automata and other computer languages thus we expect that the work presented in this manuscript would be useful in the theory of fuzzy (soft) automata.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

References

[1] D. Molodtsov, “Soft set theory—first results,” *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19-31, 1999. https://doi.org/10.1016/S0898-1221(99)00056-5

[2] L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, no. 3, pp. 338-353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X

[3] D. Molodtsov, V. Y. Leonov, and D. Kovkov, “Soft sets technique and its application,” *Nechetkie Sistemy I Myagkie Vychisleniya*, vol. 1, no. 1, pp. 8-39, 2006.

[4] Y. Zou and Z. Xiao, “Data analysis approaches of soft sets under incomplete information,” *Knowledge-Based Systems*, vol. 21, no. 8, pp. 941-945, 2008. https://doi.org/10.1016/j.knosys.2008.04.004

[5] H. Aktas and N. Cagman, “Soft sets and soft groups,” *Information Sciences*, vol. 177, no 13, pp. 2726-2735, 2007. https://doi.org/10.1016/j.ins.2006.12.008

[6] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, “On some new operations in soft set theory,” *Computers & Mathematics with Applications*, vol. 57, no. 9, pp. 1547-1553, 2009. https://doi.org/10.1016/j.camwa.2008.11.009

[7] P. K. Maji, R. Biswas, and A. R. Roy, “Soft set theory,” *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555-562, 2003. https://doi.org/10.1016/S0898-1221(03)00016-6

[8] P. Majumdar and S. K. Samanta, “On soft mappings,” *Computers & Mathematics with Applications*, vol. 60, no. 9, pp. 2666-2672, 2010. https://doi.org/10.1016/j.camwa.2010.09.004

[9] N. Cagman and S. Enginoglu, “Soft set theory and uni–int decision making,” *European Journal of Operational Research*, vol. 207, no. 2, pp. 848-855, 2010. https://doi.org/10.1016/j.ejor.2010.05.004

[10] F. Feng, Y. Li, and N. Cagman, “Generalized uni–int decision making schemes based on choice value soft sets,” *European Journal of Operational Research*, vol. 220, no. 1, pp. 162-170, 2012. https://doi.org/10.1016/j.ejor.2012.01.015

[11] B. H. Han and S. L. Geng, “Pruning method for optimal solutions of intm–intn decision making scheme,” *European Journal of Operational Research*, vol. 231, no. 3, pp. 779-783, 2013. https://doi.org/10.1016/j.ejor.2013.06.044

[12] U. Acar, F. Koyuncu, and B. Tanay, “Soft sets and soft rings,” *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3458-3463, 2010. https://doi.org/10.1016/j.camwa.2010.03.034

[13] A. O. Atagun and A. Sezgin, “Soft substructures of rings, fields and modules,” *Computers & Mathematics with Applications*, vol. 61, no. 3, pp. 592-601, 2011. https://doi.org/10.1016/j.camwa.2010.12.005

[14] A. Sezgin, A. O. Atagun, and E. Aygun, “A note on soft near-rings and idealistic soft near-rings,” *Filomat*, vol. 25, no. 1, pp. 53-68, 2011.

[15] F. Feng, Y. B. Jun, and X. Zhao, “Soft semirings,” *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2621-2628, 2008. https://doi.org/10.1016/j.camwa.2008.05.011

[16] A. Sezgin, N. Cagman, and A. O. Atagun, “A completely new view to soft intersection rings via soft uni-int product,” *Applied Soft Computing*, vol. 54, pp. 366-392, 2017. https://doi.org/10.1016/j.asoc.2016.10.004

[17] A. O. Atagun and A. Sezgin, “Soft subnear-rings, soft ideals and soft N-subgroups of near-rings,” *Mathematical Sciences Letters*, vol. 7, no. 1, pp. 37-42, 2018. https://doi.org/10.18576/msl/070106

[18] X. Ma and J. Zhan, “Soft intersection h-ideals of hemirings and its applications,” *Italian Journal of Pure and Applied Mathematics*, vol. 32, pp. 301-308, 2014.

[19] W. A. Khan, A. Rehman, and A. Taouti, “Soft near-semirings,” *AIMS Mathematics*, vol. 5, no. 6, pp. 6464-6478, 2020. https://doi.org/10.3934/math.2020417

[20] W. A. Khan, B. Davvaz, and A. Muhammad, “(M, N)-Soft intersection nearsemirings and (M, N)-α-inclusion along with its algebraic applications,” *Lobachevskii Journal of Mathematics*, vol. 40, no. 1, pp. 67-78, 2019. https://doi.org/10.1134/S1995080219010098

[21] W. A. Khan and B. Davvaz, “Soft intersection nearsemirings and its algebraic applications,” *Lobachevskii Journal of Mathematics*, vol. 41, no. 3, pp. 362-372, 2020. https://doi.org/10.1134/S1995080220030105

[22] C. Jana, M. Pal, F. Karaaslan, and A. Sezgin, "(α, β)-Soft intersectional rings and ideals with their applications," *New Mathematics and Natural Computation*, vol. 15, no. 2, pp. 333-350, 2019. https://doi.org/10.1142/S1793005719500182

[23] A. Taouti and W. A. Khan, "Fuzzy subnear-semirings and fuzzy soft subnear-semirings," *AIMS Mathematics*, vol. 6, no. 3, pp. 2268-2286, 2021. https://doi.org/10.3934/math.2021137

[24] I. Chajda and H. Laenger, "Near semirings and semirings with involution," *Miskolc Mathematical Notes*, vol. 17, no. 2, pp. 801-810, 2016. https://doi.org/10.18514/mmn.2017.1936

[25] J. Ahsan, "Seminear-rings characterized by their S-ideals, I," *Proceedings of the Japan Academy Series A Mathematical Sciences*, vol. 71, no. 5, pp. 101-103, 1995.

[26] K. Koppula, K. B. Srinivas, and K. S. Prasad, "On prime strong ideals of a seminearring," *Matematicki Vesnik*, vol. 72, no. 3, pp. 243-256, 2020.

[27] K. Glazek, *A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences with Complete Bibliography*. Dordrecht, Netherlands: Kluwer Academic Publishers, 2002.

[28] J. S. Golan, *Semirings and their Applications*. Dordrecht, Netherlands: Kluwer Academic Publishers, 1999.

[29] M. Holcombe, "The syntactic near-ring of a linear sequential machine," *Proceedings of the Edinburgh Mathematical Society*, vol. 26, no. 1, pp. 15-24, 1983. https://doi.org/10.1017/S0013091500028030

[30] K. V. Krishna and N. Chatterjee, "A necessary condition to test the minimality of generalized linear sequential machines using the theory of near-semirings," *Algebra and Discrete Mathematics*, vol. 4, no. 3, pp. 30-45, 2005.

[31] K. V. Krishna, "Near semirings: theory and applications," Ph.D. dissertation, Indian Institute of Technology Delhi, New Delhi, India, 2005.

[32] W. G. Van Hoorn and B. Van Rootselaar, "Fundamental notions in the theory of seminearrings," *Compositio Mathematica*, vol. 18, no. 1-2, pp. 65-78, 1967.

[33] H. J. Weinert, "Seminearrings, seminearfields and their semigroup-theoretical background," *Semigroup Forum*, vol. 24, no. 1, pp. 231-254, 1982. https://doi.org/10.1007/BF02572770

[34] N. Cagman, F. Citak, and H. Aktas, "Soft int-group and its applications to group theory," *Neural Computing and Applications*, vol. 21, no. 1, pp. 151-158, 2012. https://doi.org/10.1007/s00521-011-0752-x

[35] F. Citak and N. Cagman, "Soft int-rings and its algebraic applications," *Journal of Intelligent & Fuzzy Systems*, vol. 28, no. 3, pp. 1225-1233, 2015. https://doi.org/10.3233/IFS-141406

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