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On the solution of the Cayley impulsive dynamic control systems

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Abstract. In this article, we state Cayley-type impulsive dynamic control systems on time scales. For such a system, we study a new fundamental matrix and matrix exponential for impulsive dynamic systems (IDS). We get some fundamental properties for the transition matrix for IDS. To illustrate our results, we have included some numerical examples.

Key words: impulsive dynamic systems; transition matrix; Cayley transform.

1. INTRODUCTION

We know well the importance of the theory of impulsive dynamic systems (IDS) from a theoretical point of view, and their applications in control theory, dynamical games, and also in optimal control and mathematical programming [1,2]. Wang *et al.* in [3] reviewed some recent developments in impulsive control theory. Lupulescu *et al.* [4] have addressed some very recent exciting results.

Wang *et al.* [3] describe an IDS by the following three components:

1. A differential system which governs the state of the system between impulses.
2. A jump criterion defines a set of jump events in which the impulse equation is active.
3. An impulse equation that models an impulsive jump defined by a jump function at the instant an impulse occurs.

In the last century, the time scale calculus was affirmed as a tool for an explanation of a lot of new phenomena in electricity, mechanics, biology, economics, etc. [5]. It confirms time scale calculus as a way of the treatment of discrete versions of the continuous scientific problems [6]. There are many books and papers where notation and notions concerning time scales calculus are explained [7, 8].

Lupulescu *et al.* [9], introduced the basic notions of theory for linear IDS, the transition matrix for linear IDS, and established some properties of transition matrix. Transition matrix for linear IDS plays an important role in control theory and its applications [4].

On the other hand, several stability results on IDS were obtained in recent years by [10, 11]. The models of impulsive control systems have been constantly expended, including stochastic IDS [12, 13], neural networks IDS [11, 14], IDS with chaotic behavior [15] functional IDS with fractional order [16], stability of switched IDS [17]. Besides the traditional exponential (asymptotic) stability theory [18], the dynamical properties of IDS were also extended to the stability of two measures [19], finite-time stability [20, 21], input-to-state stability [22, 23].

1.1. Motivation and research gaps

The impulsive dynamic systems can present a dynamical system more accurately compared to the non-impulsive dynamic systems where the sense of disturbance in nature is involved [24–26]. On the other hand, dynamic systems on time scale calculus combine discrete and continuous phenomena [27]. So, hybridization of two different phenomena where the disturbance in the mathematical model concerned and the mixed domain (continuous and discrete) in the related parameters claims for merging the notions of impulsive dynamic systems and time scale theory [28].

Recently, Cieslinski [29] proposed a new definition of the exponential function on time scales, based on the Cayley transformation. Complex-valued Cayley transformation transforms the imaginary axis into the unit circle. Therefore, it is possible to define trigonometric and hyperbolic complex-valued functions on time scales in a standard way. The following functions maintain most of the qualitative properties of the analogous continuous functions. In special, Pythagorean trigonometric identities hold exactly on any time scale. Dynamic equations satisfied by Cayley-type functions have a natural resemblance to the corresponding differential equations, Cayley h-difference equations [30], and Cayley quantum equations [31].

Matrix exponential and the fundamental matrix of homogeneous dynamic systems play a key role in the stability theory. However, up to the author's information, all these studies are performed on Hilger's exponential function. In the current literature, we have seen some articles like [4, 9] that addressed different solution properties of impulsive dynamic systems with Hilger's approach. Here, in the current article, we took an effort

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to show the solution of linear impulsive dynamic systems with a Cayley-type fundamental matrix.

1.2. Novelties of the work

In the last three decades, several thought-provoking articles have been published addressing various issues of IDS and their applications. Motivated with recent studies [9, 29], the current study claims its uniqueness from the following aspects:

(i) We define fundamental matrices of linear IDS on time scales that are based on Cayley transformation. Impulsive transition matrix and its properties are key ingredients for stability theory.
(ii) We propose an alternative approach to the exponential function of impulsive dynamic systems.
(iii) We establish the uniqueness of the solution of IDS.
(iv) We get variations of a constant formula for linear impulsive dynamic systems.

1.3. Structure of the paper

The outline of the paper is as follows. In Section 2, we introduce the definitions and properties of time scale calculus and qualitative properties of dynamic systems on time scales. In Section 3, we developed fundamental concepts of homogeneous impulsive dynamic systems on time scales. These properties are used to investigate linear non-homogeneous dynamic systems. In Section 4, we get variations of the constants formula and controllability criteria for linear dynamic systems. The Section 5 presents a brief summary of control IDS.

2. PRELIMINARIES

Let $\mathbb{X}$ be a finite dimensional Banach space with a norm $\|\cdot\|$. For simplicity we have denote all norms by the same symbol $\|\cdot\|$, without danger of confusion. For $M \in \mathbb{M}_n(\mathbb{R})$, let us denote its conjugate transpose by M^*.

2.1. Time scale calculus

To keep this article self–contained, this section contains some preliminary definitions of the associated notation [6, 8, 32]. For a nonempty closed subset of real numbers (time scale) $\mathbb{T}$, let us consider the following operators and step size function:

$$t^\sigma = \sigma(t) := \inf\{s \in \mathbb{T} : s > t\};$$
$$t^\rho = \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

and

$$\mu : \mathbb{T} \to \mathbb{R}^+ : \mu(t) := t^\sigma - t.$$

It is easy to see that for $t \in \mathbb{T}, t^\sigma \geq t$ and $t^\rho \leq t$.

Throughout this paper, we assume that $\sup \mathbb{T} = \infty$ and for a given $t \in \mathbb{T}$, let us define and denote $\mathbb{T}_{(t)} := [t, \infty) \cap \mathbb{T}$, and $\mathbb{T}_+ := \mathbb{T}_{(0)}$.

Let $f : \mathbb{T}_{(\tau)} \to \mathbb{R}$. f is called right-dense continuous (rd-continuous): if $\lim_{t\to t_0^-} f(t)$ exists (finite) at left-dense points and $\lim_{t\to t_0^+} f(t) = f(t_0)$ at right-dense points. The delta derivative $f^\Delta(t)$ of a real-valued function f, is a number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}_{(\tau)}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$.

For a finite dimensional Banach space $\mathbb{X}$ the following spaces appear mostly:

- $C_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X}) := \left\{f : \mathbb{T}_{(\tau)} \to \mathbb{X} : f \text{ is } rd\text{-continuous}\right\}.$
- $C^1_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X})$
  $:= \left\{f : \mathbb{T}_{(\tau)} \to \mathbb{X} : f \;\; \Delta\text{-derivative is in } C_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X})\right\}.$
- $\mathscr{R}(\mathbb{T}_{(\tau)}, \mathbb{X})$
  $:= \left\{f \in C_{rd}(\mathbb{T}_{(\tau)}, \mathbb{X}) : \; (I_\mathbb{X} + \mu(t) f(t))^{-1} \text{ exists } \forall\, t \in \mathbb{T}_{(\tau)}\right\}.$
- $\mathscr{R}^+(\mathbb{T}_{(\tau)}, \mathbb{R})$
  $:= \left\{f \in \mathscr{R}(\mathbb{T}_{(\tau)}, \mathbb{R}) : I_\mathbb{X} + \mu(t) f(t) > 0, \;\; \forall \;\, t \in \mathbb{T}_{(\tau)}\right\}.$

Where $\mathbb{X} = \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ or $\mathbb{M}_n(\mathbb{R})$ and $I_\mathbb{X}$ is corresponding multiplicative identity of $\mathbb{X}$.

2.1.1. Hilger's exponential matrix and dynamic systems

For $M \in \mathscr{R}(\mathbb{T}_+, \mathbb{M}_n(\mathbb{R}))$, the Hilger exponential matrix $e_M(t,s)$ uniquely satisfies the following:

$$X^\Delta = MX, \quad X(s) = I_{\mathbb{M}_n(\mathbb{R})}$$

and $x(t) = e_M(t,s)x_0$, $t \geq s$, is the unique solution of IVP

$$x^\Delta = Mx, \;\; x(s) = x_0$$

and $x(t) = e_{\ominus M^T}(t,s)x_0$, $t \geq s$ is the unique solution of the following adjoint IVP

$$x^\Delta = -M^T x^\sigma, \;\; x(s) = x_0\,.$$

For $f \in \mathscr{R}(\mathbb{T}_+, \mathbb{C})$, Hilger defined the exponential function as follows:

$$e_f(t,s) = \exp\left(\int_s^t \frac{\log(1 + \mu(r) f(r))}{\mu(r)} \Delta r\right).$$

If $f, g \in \mathscr{R}(\mathbb{T}_+, \mathbb{C})$, then the following properties hold:

- $e_f(t^\sigma, s) = (1 + \mu(t) f(t))\, e_f(t,s)$,
- $\left(e_f(t,s)\right)^{-1} = e_{\ominus^\mu f}(t,s)$, where $\ominus^\mu f := \dfrac{-f}{1 + \mu f}$,
- $e_f(t,s) e_f(s,r) = e_f(t,r)$,
- $e_f(t,s) e_g(t,s) = e_{f \oplus^\mu g}(t,s)$, where $f \oplus^\mu g := f + g + \mu f g$.

2.1.2. Approach motivated by the Cayley transformation

Cieslinski [29, 33], introduced improved exponential function (or the Cayley-exponential function). To formulate a new definition, we need a new regressivity notion:

A complex-valued function $g : \mathbb{T}_+ \to \mathbb{C}$ is said to be regressive, if $\mu(t) g(t) \neq \pm 2$ for any $t \in \mathbb{T}^k_+$. The space of all complex-valued regressive function $g : \mathbb{T}_+ \to \mathbb{C}$ is denoted by $\mathscr{R}^C(\mathbb{T}_+, \mathbb{C})$.

For $f \in \mathscr{R}^C(\mathbb{T}_+,\mathbb{C})$, the Cayley-exponential function is defined by

$$E_f(t,s) := \exp\left(\int_s^t \frac{1}{\mu(r)} \log\left(\frac{1+\frac{1}{2}\mu(r)f(r)}{1-\frac{1}{2}\mu(r)f(r)}\right)\Delta r\right), \quad \mu(r)>0.$$

If $f,g \in \mathscr{R}^C(\mathbb{T}_+,\mathbb{C})$, then the following properties hold:

- $E_f(t^\sigma,s) = \left(\dfrac{1+\frac{1}{2}\mu(t)f(t)}{1-\frac{1}{2}\mu(t)f(t)}\right)E_f(t,s)$,
- $\big(E_f(t,s)\big)^{-1} = E_{-f}(t,s)$,
- $\overline{\big(E_f(t,s)\big)} = E_{\bar{f}}(t,s)$,
- $E_f(t,s)E_f(s,r) = E_f(t,r)$,
- $e_f(t,s)e_g(t,s) = e_{f\oplus g}(t,s)$, where $f\oplus g := \frac{f+g}{1+\frac{1}{4}\mu^2 fg}$.

A real-valued function $f:\mathbb{T}_+\to\mathbb{R}$ is said to be positively regressive, if for all $t\in\mathbb{T}_+^k$ we have $|f(t)\mu(t)|<2$. The set of all positively regressive function $f:\mathbb{T}_+\to\mathbb{R}$ is denoted by $\mathscr{R}_+^C(\mathbb{T}_+,\mathbb{C})$. $\mathscr{R}_+^C(\mathbb{T}_+,\mathbb{C})$ is an Abelian group regarding the addition $\oplus$. However, the set $\mathscr{R}^C(\mathbb{T}_+,\mathbb{C})$ is not closed under $\oplus$.

It is easy to prove that

$$x^\Delta(t) = \beta(t)x(t) \iff x^\Delta(t) = \alpha(t)\langle x(t)\rangle,$$

where $\beta(t) = \dfrac{\alpha(t)}{1-\frac{1}{2}\mu(t)\alpha(t)}$ and $\langle x(t)\rangle := \dfrac{x(t)+x(\sigma(t))}{2}$.

3. CAYLEY EXPONENTIAL MATRIX

Definition 1. (Regressivity) An $n\times n$ matrix-valued function M on a time scale $\mathbb{T}_+$ is called regressive, if

$$I \pm \frac{\mu(t)M(t)}{2} \quad \text{are non-singular for all } t\in\mathbb{T}_+^k. \tag{1}$$

The class of all such regressive and rd-continuous matrix-valued function is denoted by $\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$.

Definition 2. Assume M and $N \in \mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$. Then we define circle addition $\oplus$ of M and N by

$$(M\oplus N)(t) := (M+N)\left(I+\frac{1}{4}\mu^2 MN\right)^{-1} \quad \text{for all } t\in\mathbb{T}_+^k.$$

Example 1. Assume $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$ is constant $n\times n$-matrix. If $\mathbb{T}=\mathbb{R}$, then $E_M(t,t_0)=e^{M(t-t_0)}$.

In this section, we generalized Cayley exponential function corresponding to the following homogeneous dynamic systems:

$$x^\Delta(t) = M(t)\langle x(t)\rangle, \tag{2}$$

and the following non-homogeneous LDS is given by

$$x^\Delta(t) = M(t)\langle x(t)\rangle + h(t), \tag{3}$$

where $h\in C_{rd}(\mathbb{T}_+,(\mathbb{R}^n))$, and $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$.

A vector-valued function $x\in C_{rd}^1(\mathbb{T}_+,(\mathbb{R}^n))$ solves (3) on $\mathbb{T}_+$, if $x^\Delta(t) = M(t)\langle x(t)\rangle + h(t)$ $(\forall)$ $t\in\mathbb{T}_+$.

For $h\in C_{rd}(\mathbb{T}_+,\mathbb{R}^n))$, and $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, the following IVP

$$x^\Delta(t) = M(t)\langle x(t)\rangle + h(t), \quad x(\tau)=\eta$$

has a unique solution $x:\mathbb{T}_\tau\to\mathbb{R}^n$, for each $(\tau,\eta)\in\mathbb{T}_+\times\mathbb{R}^n$.

- A matrix $X_M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$ is called a *matrix solution* of (2) if each column of X_M satisfies (2).
- A matrix solution X_M of (2) is called a *fundamental matrix, if* $\det X_M(t)\neq 0$ $(\forall)$ $t\in\mathbb{T}_+$.
- A fundamental matrix of (2) becomes a *transition matrix*, if $X_M(\tau)=I$ at initial time $\tau\in\mathbb{T}_+$, and it is denoted by $E_M(t,\tau)$.

Therefore, it is easy to see that the transition matrix $E_M(t,\tau)$ of (2) uniquely satisfies the following IVP

$$X^\Delta(t) = M(t)\langle X(t)\rangle, \quad X(\tau)=I, \quad \text{at initial time } \tau\in\mathbb{T}_+,$$

and $x(t)=E_M(t,\tau)\eta$, $t\geq\tau$, is the only solution of the IVP

$$x^\Delta(t) = M(t)\langle x(t)\rangle, \quad x(\tau)=\eta.$$

Theorem 1. If $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$ is a matrix-valued functions on $\mathbb{T}_+$, then

1. $E_M(t,t)\equiv I$ and $E_0(t,s)\equiv I$, where 0 denotes the zero matrix;
2. $E_M(\sigma(t),s) = \left(I-\frac{\mu M}{2}\right)^{-1}\left(I+\dfrac{\mu M}{2}\right)E_M(t,s)$;
3. $E_M^{-1}(t,s) = E_{-M^*}(t,s)$;
4. $E_M(t,s) = E_M^{-1}(s,t)$;
5. $E_M(t,s)E_M(s,r) = E_M(t,r)$.

Proof.

1. We have to prove $E_0(t,s)\equiv I$ and $E_M(t,t)\equiv I$. Consider the IVP $Y^\Delta = 0$ and $Y(s)=I$, which has unique solution. Since $Y(t)\equiv I$ is the solution of this IVP, we have $E_0(t,s) = \langle Y(t)\rangle \equiv \langle I\rangle = I$.
2. We know that $\langle E_M(t,s)\rangle = \dfrac{E_M(t,s)+E_M(\sigma(t),s)}{2}$, therefore, we have

$$\begin{aligned} E_M(\sigma(t),s) &= E_M(t,s) + \mu M(t)\langle E_M(t,s)\rangle \\ &= E_M(t,s) + \mu(t)M(t)\left(\frac{E_M(t,s)+E_M(\sigma(t),s)}{2}\right) \\ &= E_M(t,s) + \frac{\mu(t)M(t)E_M(t,s)}{2} + \frac{\mu(t)M(t)E_M(\sigma(t),s)}{2}. \end{aligned}$$

It implies that

$$E_M(\sigma(t),s) - \frac{\mu(t)M(t)E_M(\sigma(t),s)}{2} = \left(I+\frac{\mu M}{2}\right)E_M(t,s).$$

Since $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, it implies that $\left(I-\dfrac{\mu M}{2}\right)^{-1}$ exists, therefore, we have

$$E_M(\sigma(t),s) = \left(I-\frac{\mu M}{2}\right)^{-1}\left(I+\frac{\mu M}{2}\right)E_M(t,s).$$

3. Let $Y(t) := E_M^{-1}(t,s)^*$. By differentiating Y, we have

$$\begin{aligned}
Y^{\Delta}(t) &= -(E_M^{-1}(\sigma(t),s))E_M^{\Delta}(t,s)E_M^{-1}(t,s))^* \\
&= -\bigg(E_M^{-1}(\sigma(t),s)M(t)\left[\frac{E_M(t,s)+E_M(\sigma(t),s)}{2}\right] \\
&\qquad\qquad (E_M^{-1}(t),s)\bigg)^* \\
&= -M^*(t)\left[\frac{-E_M^{-1}(\sigma(t),s)M(t)+M(t)E_M^{-1}(t,s)}{2}\right]^* \\
&= -M^*(t)\left[\frac{-E_M^{-1}(\sigma(t),s)M(t)+M(t)E_M^{-1}(t,s)}{2}\right]^* \\
&= -M^*(t)\left\langle E_M^{-1}(t,s)\right\rangle^* \\
&= -M^*(t)\left\langle E_M^{-1}(t,s)\right\rangle^* \\
&= -M^*(t)\left\langle Y(t)\right\rangle.
\end{aligned}$$

Also, $Y(s)=I$. Therefore, by uniqueness of the solution, we have $E_M^{-1}(t,s) = E_{-M^*}(t,s)$.

4. Now we have to prove that

$$E_M(t,s) = E_M^{-1}(s,t).$$

Let $Y(t) := E_M^{-1}(s,t)$, then we have

$$\begin{aligned}
Y^{\Delta}(t) &= E_M^{-1}(s,\sigma(t))E_M^{\Delta}(s,t)E_M^{-1}(s,t) \\
&= (E_M^{-1}(s,\sigma(t)M(t)\left\langle E_M(s,t)\right\rangle E_M^{-1}(s,t) \\
&= -E_M^{-1}(s,\sigma(t))M(t)\left[\frac{E_M(s,t)+E_M(s,\sigma(t)}{2}\right]E_M^{-1}(s,t) \\
&= \frac{-E_M^{-1}(s,\sigma(t))M(t)+M(t)(E_M^{-1}(s,t)}{2} \\
&= M(t)\left[\frac{-E_M^{-1}(s,\sigma(t))+E_M^{-1}(s,t)}{2}\right] \\
&= M(t)E_M^{-1}(s,t) \\
&= M(t)\left\langle Y(t)\right\rangle.
\end{aligned}$$

Also, $Y(s)=I$. Therefore, by uniqueness of the solution, we have $E_M(t,s) = E_M^{-1}(s,t)$.

5. Let $Y(t) := E_M(t,s)E_M(s,r)$

$$\begin{aligned}
Y^{\Delta}(t) &= E_M^{\Delta}(t,s)E_M(s,r) \\
&= M(t)\left\langle E_M(t,s)\right\rangle E_M(s,r) \\
&= M(t)\left\langle E_M(t,s)E_M(s,r)\right\rangle \\
&= M(t)\left\langle Y\right\rangle.
\end{aligned}$$

Also, $Y(r)=I$. Therefore, by uniqueness of the solution, we have $E_M(t,r) = E_M(t,s)E_M(s,r)$.

□

Theorem 2. Let $M \in \mathscr{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ and suppose that $f : \mathbb{T}_+ \to \mathbb{R}^n$ is *rd*-continuous. Let $t_0 \in \mathbb{T}_+$ and $x_0 \in \mathbb{R}^n$. Then the IVP

$$\begin{aligned}
&x^{\Delta}(t) = M(t)\left\langle x(t)\right\rangle + \left(I - \frac{M(t)}{2}\mu(t)\right)f(t), \\
&x(t_0) = x_0
\end{aligned} \tag{4}$$

satisfies

$$x(\cdot) = E_M(\cdot,t_0)x_0 + \int_{t_0}^{\cdot} E_M(\cdot,\sigma(\tau))f(\tau)\Delta\tau. \tag{5}$$

Proof. Firstly, x given by (5) is well-defined and can be rewritten as

$$x(\cdot) = E_M(\cdot,t_0)\left\{x_0 + \int_{t_0}^{\cdot} E_M(t_0,\sigma(\tau))f(\tau)\Delta\tau\right\}.$$

We use the product rule to differentiate x:

$$\begin{aligned}
x^{\Delta}(\cdot) &= M(\cdot)\left\langle E_M(\cdot,t_0)\right\rangle\left\{x_0 + \int_{t_0}^{\cdot} E_M(t_0,(\tau^{\sigma}))f(\tau)\Delta\tau\right\} \\
&\quad + E_M(\cdot^{\sigma},t_0)E_M(t_0,\cdot^{\sigma})f(\cdot) \\
&= M(\cdot)\left\langle E_M(\cdot,t_0)\right\rangle\left\{x_0 + \int_{t_0}^{\cdot} E_M(t_0,\tau^{\sigma})f(\tau)\Delta\tau\right\} + f(\cdot) \\
&= M(\cdot)\left(\frac{E_M(\cdot,t_0)+E_M(\cdot^{\sigma},t_0)}{2}\right) \\
&\qquad \left\{x_0 + \int_{t_0}^{\cdot} E_M(t_0,\tau^{\sigma})f(\tau)\Delta\tau\right\} + f(\cdot) \\
&= \frac{M(\cdot)}{2}\left[E_M(\cdot,t_0)x_0 + \int_{t_0}^{\cdot} E_M(\cdot,\tau^{\sigma})f(\tau)\Delta\tau\right] \\
&\quad + \frac{M(\cdot)}{2}\left[E_M(\cdot^{\sigma},t_0)x_0 + \int_{t_0}^{\cdot} E_M(\cdot^{\sigma},(\tau^{\sigma})f(\tau)\Delta\tau\right] + f(\cdot).
\end{aligned} \tag{6}$$

We know that $\int_t^{t^{\sigma}} f(t)\Delta t = \mu(t)f(t)$. Therefore, we have

$$\int_t^{t^{\sigma}} E_M(t^{\sigma},\tau^{\sigma})f(\tau)\Delta\tau = \mu(t)E_M(t^{\sigma},t^{\sigma})f(t) = \mu(t)f(t).$$

It implies that

$$\begin{aligned}
x^{\Delta}(t) &= \frac{M(t)}{2}\left[E_M(t,t_0)x_0 + \int_{t_0}^{t} E_M(t_0,\tau^{\sigma})f(\tau)\Delta\tau\right] \\
&\quad + \frac{M(t)}{2}\left[E_M(t^{\sigma},t_0)x_0 + \int_{t_0}^{t} E_M(t^{\sigma},\tau^{\sigma})f(\tau)\Delta\tau\right] \\
&\quad + f(t) + \frac{M}{2}\mu(t)f(t) - \frac{M}{2}\mu(t)f(t),
\end{aligned}$$

we have

$$
\begin{aligned}
x^{\Delta}(t) &= \frac{M(t)}{2}x(t) \\
&\quad + \frac{M(t)}{2}\left[E_M(t^{\sigma},t_0)x_0 + \int_{t_0}^{t} E_M(t^{\sigma},\tau^{\sigma})f(\tau)\Delta\tau\right] \\
&\quad + f(t) + \frac{M}{2}\mu(t)f(t) - \frac{M}{2}\mu(t)f(t) \\
&= \frac{M(t)}{2}(x(t)) + \frac{M(t)}{2}x(t^{\sigma}) + \left(I - \frac{M}{2}\mu(t)\right)f(t) \\
&= M\langle x(t)\rangle + \left(I - \frac{M}{2}\mu(t)\right)f(t).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&x^{\Delta}(t) = M\langle x(t)\rangle + \left(I - \frac{M}{2}\mu(t)\right)f(t) \\
&x(t_0) = x_0\,.
\end{aligned}
$$

□

Theorem 3. (Putzer Algorithm). Let $M \in \mathscr{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ be a constant $n\times n$-matrix. Suppose $t_0 \in \mathbb{T}_+$. If $\lambda_1, \lambda_2, \cdots \lambda_n$ are the eigenvalues of M, then

$$
E_M(t,t_0) = \sum_{i=0}^{n-1} r_{i+1}(t)P_i, \tag{7}
$$

where $r(t) := (r_1(t), r_2(t), \cdots, r_n(t))^T$ is the solution of IVP

$$
r^{\Delta} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \ddots & \vdots \\ 0 & 1 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_n \end{pmatrix} \langle r\rangle, \quad r(t_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{8}
$$

and the P matrices $P_0, P_1, \cdots, P_n$ are recursively defined by $P_0 = I$ and

$$
P_{k+1} = (M - \lambda_{k+1}I)P_k, \ \text{ for } \ 0 \le k \le n-1.
$$

Example 2. Let us consider the following dynamic system:

$$
x^{\Delta}(t) = M\langle x(t)\rangle, \ \text{ where } \ M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{9}
$$

on time scales $\mathbb{T}$ with constant step size function μ. We find the matrix exponential $E_M(t,t_0)$. M is regressive for $\mu \neq \frac{2}{a}, \frac{2}{b}$ that is

$$
\det\left(I + \frac{\mu}{2}M\right) \neq 0 \ \text{ and } \ \det\left(I - \frac{\mu}{2}M\right) \neq 0 \ \text{ for } \ \mu \neq \frac{2}{a}, \frac{2}{b}.
$$

Equation (9) is equivalently written as

$$
x^{\Delta}(t) = \left(I - \frac{\mu}{2}M\right)^{-1} Mx(t).
$$

It is easy to see that

$$
E_M(t,t_0) = \begin{pmatrix} e_{\frac{2a}{2-b\mu}} & 0 \\ 0 & e_{\frac{2b}{2-a\mu}} \end{pmatrix}
$$

and

$$
x(t) = \begin{pmatrix} e_{\frac{2a}{2-b\mu}} & 0 \\ 0 & e_{\frac{2b}{2-a\mu}} \end{pmatrix} x_0.
$$

Similar to Theorem 5.1 in [34], we have the following result.

Lemma 1. Let $M \in \mathscr{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ be a constant $n\times n$ matrix, and $t_0 \in \mathbb{T}_+$. Then there exist scalar $\mathscr{C}^{\infty}_{rd}(\mathbb{T}_+, \mathbb{R})$ functions $\gamma_0(t,t_0), \cdots, \gamma_{n-1}(t,t_0)$ such that the transition matrix $E_M(t,t_0)$ has the representation:

$$
E_M(t,t_0) = \sum_{k=0}^{n-1} \langle \gamma_k(t,t_0)\rangle M^k.
$$

Example 3. The regressive linear matrix dynamic system from [34]:

$$
X^{\Delta} = MX, \ \ X(t_0) = I \tag{10}
$$

has a unique solution of the following form:

$$
e_M(t,t_0) = \begin{pmatrix} \cos_1(t,t_0) & \sin_1(t,t_0) \\ -\sin_1(t,t_0) & \cos_1(t,t_0) \end{pmatrix}, \tag{11}
$$

where $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\cos_1(t,t_0) := \dfrac{e_i(t,t_0) + e_{-i}(t,t_0}{2}$, and $\sin_1(t,t_0) := \dfrac{e_i(t,t_0) - e_{-i}(t,t_0)}{2i}$.

Also, it is easy to see that $M \in \mathscr{R}^C(\mathbb{T}_+, M_2(\mathbb{R}))$, and Cayley dynamic IVP:

$$
X^{\Delta} = M\langle X\rangle, \quad X(t_0) = I,
$$

has the following solution:

$$
E_M(t,t_0) = \begin{pmatrix} \mathrm{Cos}_1(t,t_0) & \mathrm{Sin}_1(t,t_0) \\ -\mathrm{Sin}_1(t,t_0) & \mathrm{Cos}_1(t,t_0) \end{pmatrix},
$$

where $\mathrm{Cos}_1(t,t_0) := \dfrac{E_i(t,t_0) + E_{-i}(t,t_0)}{2}$, and $\mathrm{Sin}_1(t,t_0) := \dfrac{E_i(t,t_0) - E_{-i}(t,t_0)}{2i}$.

From Lemma 3.26 in [8], it implies that

$$
\|e_M(t,t_0)\| = \sqrt{2}e_{\mu}(t,t_0). \tag{12}
$$

On the other hand, from Theorem 3.20 in [33], we obtain

$$
\|e_M(t,t_0)\| = \sqrt{2}. \tag{13}
$$

Equation (13) implies that the norm of the solution of Cayley dynamic IVP behaves like a classical case. However, the norm of the solution of Hilger (or Bohner) dynamic IVP (10) increases as μ changes. In the case of unbounded μ, the norm of the solution of (10) becomes unbounded, too.

4. IMPULSIVE DYNAMIC SYSTEM

4.1. Homogeneous case

In this section, we consider the following homogeneous linear IDS

$$\begin{cases} x^{\Delta}(t) = M(t)\langle x(t)\rangle, & t\in\mathbb{T}_+,\ t\neq t_k \\ x(t_k^+) = x(t_k^-)+B_k x(t_k), & k=1,2,\cdots, \end{cases} \tag{14}$$

where

(i) $B_k\in M_n(\mathbb{R})$, $k=1,2,\cdots$, $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$,

(ii) $0=t_0<t_1<t_2<\cdots<t_k<\cdots$, with $\lim_{k\to\infty}t_k=\infty$,

(iii) $x(t_k^-)$ denotes the left hand limit of the state vector $x(t)$ at $t=t_k$ (in case of left-scattered points $x(t_k^-):=x(t)$) and similar for $x(t_k^+)$.

(iv) $t_k^{\sigma}=t_k$ for $k=1,2,\cdots$.

For further discussion, suppose the following IVP corresponding to (14):

$$\begin{cases} x^{\Delta}(t) = M(t)\langle x(t)\rangle, & t\in\mathbb{T}_{(\tau)},\ t\neq t_k \\ x(t_k^+) = x(t_k^-)+B_k x(t_k), & k=1,2,\cdots \\ x(\tau^+)=\eta, & \tau\geq 0. \end{cases} \tag{15}$$

For solution of (15), let us define:

$$\Omega := \left\{ \begin{aligned} &x\colon \mathbb{T}_+\to\mathbb{R}^n;\ x\in C((t_k,t_{k+1},\mathbb{R}^n), \\ &\qquad k=0,1,2,\cdots,x(t_k^+) \\ &\qquad\text{and } x(t_k^-) \\ &\text{exists with } x(t_k^-)=x(t_k),\ k=1,2,\cdots, \end{aligned} \right\}$$

and

$$\Omega^1 := \left\{x\in\Omega;\ x\in C^1((t_k,t_{k+1}),\ \mathbb{R}^n),\ k=0,1,2,\cdots\right\},$$

A function $x\in\Omega^1$ is called the solution of (14) if it satisfies $x^{\Delta}(t)=M(t)\langle x(t)\rangle$ everywhere on $\mathbb{T}_{(\tau)}\backslash\{\tau,t_{k(\tau)},t_{k(\tau)+1},\cdots\}$ and for each $j=k(\tau),k(\tau)+1,\cdots$ satisfies the impulsive and initial conditions $x(t_j^+)=x(t_j)+B_j x(t_j)$, $x(\tau^+)=\eta$, where $k(\tau):=\min\{\ k=0,1,2,\cdots;\tau<t_k\}$.

By using similar method [9, Theorem 3.1], we can obtain the following integral equation:

Theorem 4. Let $B_k\in M_n(\mathbb{R})$, and $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, $k=1,2,\cdots$, then the vector-valued function $x(\cdot)$ solves the IVP (14) if and only if $x(\cdot)$ solves

$$x(t) = x(\tau)+\int_{\tau}^{t} M(s)\langle x(s)\rangle\,\Delta s + \sum_{\tau<t_j<t} B_j x(t_j),\quad t\in\mathbb{T}. \tag{16}$$

By using [35, Lemma 3.1] and [36, Lemma 3.1], we can obtain the following impulsive dynamic inequality:

Lemma 2. Let $\tau\in\mathbb{T}_+$, $p\in\mathscr{R}_+^C(\mathbb{T}_+,\mathbb{R})$, and $c_k,d_k\in\mathbb{R}_+$, for all $k=1,2,\cdots$, then

$$\begin{cases} x^{\Delta}(t)\leq p(t)\langle x(t)\rangle + b(t), & t\in\mathbb{T}_{(\tau)},\ t\neq t_k, \\ x(t_k^+)\leq c_k x(t_k)+d_k, & k=1,2,\cdots \end{cases}$$

implies

$$x(t)\leq x(\tau)\prod_{\tau<t_k<t} c_k E_p(t,\tau) + \sum_{\tau<t_k<t}\left(\prod_{t_k<t_j<t} c_j E_p(t,t_k)\right) d_k + \int_{\tau}^{t}\prod_{s<t_k<t} c_k\left\langle E_{-p}(s,t)\right\rangle b(s)\Delta s.$$

Lemma 3. Let $\tau\in\mathbb{T}_+,a,b\in\mathscr{R}^C(\mathbb{T}_+,\mathbb{R})$, $p\in\mathscr{R}_+^C(\mathbb{T}_+,\mathbb{R})$ and $c,b_k\in\mathbb{R}_+$, for all $k=1,2,\cdots$, then

$$x(t)\leq c+\int_{\tau}^{t} p(s)\langle x(s)\rangle\,\Delta s + \sum_{\tau<t_k<t} b_k x(t_k),\quad t\in\mathbb{T}_{(\tau)} \tag{17}$$

implies

$$x(t)\leq c\prod_{\tau<t_k<t}(1+b_k)E_p(t,\tau),\quad t\geq\tau. \tag{18}$$

Proof. Let

$$v(t):=c+\int_{\tau}^{t} P(s)\langle x(s)\rangle\,\Delta s + \sum_{\tau<t_k<t} b_k x(t_k),\ \ t\geq\tau$$

then

$$\begin{cases} v^{\Delta}(t)=p(t)\langle x(t)\rangle, & t\neq t_k,\ \ v(\tau)=c, \\ v(t_k^+)=v(t_k)+b_k x(t_k), & k=1,2,\cdots \end{cases}$$

since $\langle x(t)\rangle\leq\langle v(t)\rangle$, $t\geq\tau$, we then have

$$\begin{cases} v^{\Delta}(t)\leq p(t)\langle v(t)\rangle, & t\neq t_k,\ \ v(\tau)=c, \\ v(t_k^+)=(1+b_k)v(t_k), & k=1,2,\cdots. \end{cases}$$

Lemma 2 yields

$$v(t)\leq c\prod_{\tau<t_k<t}(1+b_k)E_p(t,\tau),\quad t\geq\tau,$$

which implies (18). □

Theorem 5. If $B_k\in M_n(\mathbb{R})$, and $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, for all $k=1,2,\cdots$, then any vector-valued solution $x(\cdot)$ of (15) satisfies the following estimate

$$\|x(t)\|\leq\|x(\tau)\|\prod_{\tau<t_k\leq t}(1+\|B_k\|) \times\exp\left(\int_{\tau}^{t}\frac{\|M(s)\|}{1-\mu(s)\|M(s)\|}\Delta s\right),\quad \mu(s)>0 \tag{19}$$

for $\tau,t\in\mathbb{T}_+$ with $t\geq\tau$.

Proof. From (16) we obtain that

$$\|x(t)\| \leq \|x(\tau)\| + \int_{\tau}^{t} \|M(s)\| \langle \|x(s)\| \rangle \Delta s + \sum_{\tau<t_j\leq t} \left\|B_j\right\| \left\|x(t_j)\right\|, \ t \geq \tau.$$

Then, by Lemma 3, we have

$$\|x(t)\| \leq \|x(\tau)\| \prod_{\tau<t_k\leq t} (1+\|B_k\|) E_{\|M(\cdot)\|}(t,\tau), \ \ t \geq \tau.$$

Since for any $a \geq 0$,

$$\lim_{u\to\mu(s)} \frac{\ln\left(\dfrac{1+\dfrac{au}{2}}{1-\dfrac{au}{2}}\right)}{u} = \begin{cases} a & \text{if } \ \mu(s)=0, \\ \dfrac{\ln\left(\dfrac{1+\dfrac{a\mu(s)}{2}}{1-\dfrac{a\mu(s)}{2}}\right)}{\mu(s)} \leq \dfrac{a}{1-\dfrac{a\mu(s)}{2}} & \text{if } \ \mu(s)>0. \end{cases}$$

It implies that

$$E_{\|M(\cdot)\|}(t,\tau) = \exp\left(\int_{\tau}^{t} \lim_{u\to\mu(s)} \frac{\ln\left(\dfrac{1+\dfrac{\|M(s)\|\,\mu(s)}{2}}{1-\dfrac{\|M(s)\|\,\mu(s)}{2}}\right) y}{\mu(s)} \Delta s \right) \leq \exp\left(\int_{\tau}^{t} \frac{\|M(s)\|}{1-\mu(s)\,\|M(s)\|} \Delta s \Delta s \right), \quad t \geq \tau.$$

Thus we obtain (19). □

Along with (14), let us define the following transition matrix $G_M(t,s)$, $0 \leq s \leq t$, regarding the sequence $\{B_k,t_k\}_{k=1}^{\infty}$:

$$G_M(t,s) = \begin{cases} E_M(t,s), \quad \text{if } \ t_{k-1} \leq s \leq t \leq t_k; \\ E_M(t,t_k^+)(I+B_k)E_M(t_k,s), \\ \qquad \text{if } \ t_{k-1} \leq s \leq t \leq t_k < t < t_{k+1}; \\ E_M(t,t_k^+) \left[\prod\limits_{s<t_j\leq t} E_M(t_j,t_{j-1}^+) \right] \\ \qquad (I+B_i)E_M(t_i,s), \\ \qquad \text{if } \ t_{i-1} \leq s < t_i < \cdots < t_k < t < t_{k+1}, \end{cases} \tag{20}$$

where $E_M(t,s)$, $0 \leq s \leq t$ represents the transition matrix of (2) at initial time $s \in \mathbb{T}_+$.

Remark 1. By definition

$$G_M(t,s) = E_M(t,t_k^+)(I+B_k)E_M(t_k,t_{k-1}^+) \left[\prod_{s<t_j<t_k} (I+B_j)E_M(t_j,t_{j-1}^+) \right] (I+B_i)\,E_M(t_i,s),$$

it follows that

$$G_M(t,s) = E_M(t,t_k^+)(I+B_k)G_M(t_k,s)$$

for $t_{i-1} \leq s < t_i < \cdots < t_k \leq t < t_{k+1}$.

(H): $\det(I+B_k) \neq 0$ for all $k = 1,2,\cdots$.

Theorem 6. Let (H) holds. If $M \in \mathscr{R}^C(\mathbb{T}_+, M_n(\mathbb{R}))$ and $B_k \in M_n(\mathbb{R})$, for all $k = 1,2,\cdots$, then the unique solution of (15) is given by

$$x(t) = G_M(t,\tau)\eta, \quad t \geq \tau.$$

Proof. For $(\tau,\eta) \in \mathbb{T}_+ \times \mathbb{R}^n$, there exists $i \in \{1,2,\cdots\}$ such that $\tau \in [t_{i-1},t_i)$. Then by definition of the solution, we have

$$x(t) = E_M(t,\tau)\eta = G_M(t,\tau)\eta, \quad t \in [\tau,t_i].$$

Further we consider the IVP

$$\begin{cases} x^{\Delta} = M(t)\,\langle x(t) \rangle, \quad t \in (t_i,t_{i+1}], \\ x(t_i^+) = x(t_i) + B_i x(t_i). \end{cases}$$

This IVP has the unique solution given by

$$x(t) = E_M(t,t_i^+)x(t_i^+), \quad t \in (t_i,t_{i+1}].$$

It follows that

$$\begin{aligned} x(t) &= E_M(t,t_i^+)x(t_i^+), \\ &= E_M(t,t_i^+)(I+B_i)x(t_i) \\ &= E_M(t,t_i^+)(I+B_i)E_M(t_i,\tau)\eta \\ &= G_M(t,\tau)\eta, \end{aligned}$$

and so $x(t) = G_M(t,\tau)\eta$, $t \in [t_i,t_{i+1})$.

Now, we can let the unique solution of (15) on $(t_{k-1},t_k]$, for any $k > i+2$:

$$x(t) = G_M(t,\tau)\eta = E_M(t,t_{k-1}^+) \times \left[\prod_{s<t_j<t_k} (I+B_j)E_M(t_j,t_{j-1}^+) \right] (I+B_i)E_M(t_i,\tau)\eta.$$

Then the IVP

$$\begin{cases} x^{\Delta} = M(t)\,\langle x(t) \rangle, \quad t \in (t_k,t_{k+1}], \\ x(t_k^+) = x(t_k) + B_k x(t_k), \end{cases}$$

has the unique solution solution

$$x(t) = E_M(t,t_k^+)x(t_k^+), \quad t \in (t_k,t_{k+1}].$$

It follows that

$$\begin{aligned}
x(t) &= E_M(t,t_k^+)x(t_i^+) \\
&= E_M(t,t_k^+)(I+B_k)x(t_k^+) \\
&= E_M(t,t_k^+)(I+B_k)E_M(t,t_{k-1}^+) \\
&\quad \times \left[\prod_{s<t_j\le t}(I+B_j)E_M(t_j,t_{j-1}^+)\right](I+B_i)E_M(t_i,\tau)\eta \\
&= E_M(t,t_k^+)\left[\prod_{s<t_j\le t}(I+B_j)E_M(t_j,t_{j-1}^+)\right] \\
&\quad \times (I+B_i)E_M(t_i,\tau)\eta \\
&= G_M(t,\tau)\eta,
\end{aligned}$$

and so $x(t) = G_M(t,\tau)\eta$, $t\in(t_k,t_{k+1}]$. It completes our proof. □

Corollary 1. Let (H) holds. If $B_k\in M_n(\mathbb{R})$, and $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$ for all $k=1,2,\cdots$, then $G_M(t,s)$, $0\le s\le t$, uniquely solves the following IVP

$$\begin{cases} Y^\Delta(t) = M(t)\langle Y(t)\rangle, & t\in\mathbb{T}_{(s)},\ t\neq t_k; \\ Y(t_k^+) = (I+B_k)Y(t_k), & k=1,2,\cdots; \\ Y(\tau^+) = I, & s\ge 0. \end{cases} \tag{21}$$

Moreover, the following properties hold:

1. $G_M(t_k^+,s) = (I+B_k)G_M(t_k,s)$, $t_k\ge s$, $k=1,2,\cdots$;
2. $G_M(t,t_k^+) = G_M(t,t_k)(I+B_k)^{-1}$, $t_k\le t$, $k=1,2,\cdots$;
3. $G_M(t,t_k^+)G_M(t_k^+,s) = G_M(t,s)$, $0\le s\le t_k\le t$, $k=1,2,\cdots$.

Corollary 2. Let (H) holds. If $B_k\in M_n(\mathbb{R})$, and $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, for all $k=1,2,\cdots$, then

$$\begin{aligned}
&\|G_M(t,\tau)\| \\
&\le \prod_{\tau<t_k\le t}(I+\|B_k\|\exp\left(\int_\tau^t \frac{\|M(s)\|}{1-\mu(s)\|M(s)\|}\Delta s\right), \\
&\qquad \mu(s)>0,
\end{aligned} \tag{22}$$

for $\tau,t\in\mathbb{T}_+$ with $t\ge\tau$. Moreover, for any $\tau,t\in\mathbb{T}_+$ the mapping $(r,s)\to G_M(r,s)$ is bounded on set $\{(r,s)\in\mathbb{T}_+\times\mathbb{T}_+;\tau\le s\le r\le t\}$.

Proof. For all for all $\tau,t\in\mathbb{T}_+$ with $t\in\mathbb{T}_{(\tau)}$, Theorems 4 and 5 yields

$$\begin{aligned}
\|x(t)\| &= \|G_M(t,\tau)x(\tau)\| \\
&\le \|x(\tau)\|\prod_{\tau<t_k\le t}(I+\|B_k\|\exp\left(\int_\tau^t \frac{\|M(s)\|}{1-\mu(s)\|M(s)\|}\Delta s\right), \\
&\qquad \mu(s)>0,
\end{aligned}$$

which implies (22). □

For our next result, let us define $X_M(t) := G_M(t,0)$, $t\in\mathbb{T}_+$.

Theorem 7. Let (H) holds. If $B_k\in M_n(\mathbb{R})$, and $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, for all $k=1,2,\cdots$, then:

1. $G_M(t,s) = X_M(t)X_M^{-1}(s)$, $0\le s\le t$;
2. $G_M(t,t) = I$, $t\ge 0$;
3. $G_M(t,s) = G_M^{-1}(s,t)$, $0\le s\le t$;
4. $G_M(\sigma(t),s) = \left(I-\frac{\mu}{2}M\right)^{-1}\left(I+\frac{\mu}{2}M\right)G_M(t,s)$, $0\le s\le t$;
5. $G_M(t,s)G_M(s,r) = G_M(t,r)$, $0\le r\le s\le t$.

Proof.

1. Let
$$Y(t) := X_M(t)X_M^{-1}(s),\quad 0\le s\le t.$$
Then we have
$$\begin{aligned}
Y^\Delta(t) &= X_M^\Delta(t)X_M^{-1}(s) \\
&= M(t)\langle X_M(t)\rangle X_M^{-1}(s) \\
&= M(t)\langle X_M(t)X_M^{-1}(s)\rangle \\
&= M(t)\langle Y(t)\rangle,\ \ t\neq t_k.
\end{aligned}$$
Also
$$Y(s) = X_M(s)X_M^{-1}(s) = I,$$
and
$$\begin{aligned}
Y\left(t_k^+\right) - Y(t_k) &= X_M(t_k^+)X_M^{-1}(s) - X_M(t_k)X_M^{-1}(s) \\
&= \left[X_M(t_k^+) - X_M(t_k)\right]X_M^{-1}(s) \\
&= B_kX_M(t_k)X_M^{-1}(s) \\
&= B_kY(t_k) \text{ for each } t_k\ge s.
\end{aligned}$$
Therefore, $Y(t) = X_M(t)X_M^{-1}(s)$ solves the IVP (21), which has exactly one solution. Therefore
$$G_M(t,s) = X_M(t)X_M^{-1}(s)\quad 0\le s\le t.$$
2. From part 1, It is easy to see that $G_M(t,t) = X_M(t)X_M^{-1}(t) = I$.
3. By using part 1, we have $G_M(t,s) = X_M(t)X_M^{-1}(s) = \left(X_M(s)X_M^{-1}(t)\right)^{-1} = G_M^{-1}(s,t)$.
4. A well-known relation implies that
$$\begin{aligned}
G_M(\sigma(t),s) &= G_M(t,s) + \mu(t)G_M^\Delta(t,s) \\
&= G_M(t,s) + \mu(t)M(t)\langle G_M(t,s)\rangle.
\end{aligned}$$
It follows that
$$\begin{aligned}
&G_M(\sigma(t),s) = G_M(t,s) + \mu(t)X(t)\langle G_M(t,s)\rangle \\
&= G_M(t,s) + \mu(t)M(t)\left(\frac{G_M(t,s)+G_M(\sigma(t),s)}{2}\right) \\
&= G_M(t,s) + \frac{\mu(t)M(t)G_M(t,s)}{2} + \frac{\mu(t)M(t)G_M(\sigma(t),s)}{2} \\
&= G_M(\sigma(t),s) - \frac{\mu(t)M(t)G_M(\sigma(t),s)}{2} \\
&\quad -\left(I+\frac{\mu}{2}M\right)G_M(t,s);
\end{aligned}$$

Therefore, we have

$$\left(I-\frac{\mu}{2}M\right)G_M(\sigma(t),s)=\left(I+\frac{\mu}{2}M\right)G_M(t,s).$$

Since $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, it implies that

$$G_M(\sigma(t),s)=\left(I-\frac{\mu}{2}M\right)^{-1}\left(I+\frac{\mu}{2}M\right)G_M(t,s).$$

5. Let $Y(t)=G_M(t,s)G_M(s,r)$, $0\le r\le s\le t$, it follows that

$$\begin{aligned}Y^\Delta(t)&=G_M^\Delta(t,s)G_M(s,r)\\&=M(t)\langle G_M(t,s)\rangle G_M(s,r)\\&=M(t)\langle G_M(t,s)G_M(s,r)\rangle\\&=M(t)\langle Y(t)\rangle \quad\text{for } t\ne t_k,\end{aligned}$$

also

$$Y(r^+)=G_M(r^+,s)G_M(s,r^+)=G_M(r^+,s)G_M^{-1}(s,r^+)=I.$$

And

$$Y(t_k^+)=G_M(t_k^+,s)G_M(s,r)=(I+B_k)Y(t_k)$$

for each $t_k\ge s$. The conclusion now follows from uniqueness of the solution.

□

Theorem 8. Let (H) holds. If $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$ and $B_k\in M_n(\mathbb{R})$, for all $k=1,2,\cdots$, then

$$\frac{\partial}{\Delta s}G_M(t,s)=-G_M(t,\sigma(s))M(s)\langle G_M(s,t)\rangle G_M^{-1}(s,t),\quad s\ne t_k.$$

Proof. By Theorem 7, we have

$$\begin{aligned}\frac{\partial}{\Delta s}G_M(t,s)&=\frac{\partial}{\Delta s}G_M^{-1}(s,t)\\&=-G_M^{-1}(s,\sigma(t))\frac{\partial}{\Delta s}G_M(s,t)G_M^{-1}(s,t)\\&=-G_M^{-1}(s,\sigma(t))M(t)\langle G_M(s,t)\rangle G_M^{-1}(s,t).\end{aligned}$$

Therefore, $\frac{\partial}{\Delta s}G_M(t,s)=-G_M(t,\sigma(s))M(s)\langle G_M(s,t)\rangle G_M^{-1}(s,t)$ for all $s\in\mathbb{T}_+$, $s\ne t_k$, $k=1,2\cdots$. □

4.2. Non-homogeneous case

We know that $l^\infty(\mathbb{R}^n)$, the space of all vector sequences $s:=\{c_k\}_{k=1}^\infty, c_k\in\mathbb{R}^n$ $k=1,2,\cdots$, such that $\sup_{k\ge1}\|c_k\|<\infty$ is an infinite dimensional Banach space regarding the norm $\|s\|:=\sup_{k\ge1}\|c_k\|$.

In this section, we consider the following initial value problem:

$$\begin{cases}x^\Delta(t)=M(t)\langle x(t)\rangle+\left(I-\dfrac{\mu(t)}{2}M\right)f(t),\\ \qquad\qquad t\in\mathbb{T}_{(\tau)},\ t\ne t_k\\ x(t_k^+)=x(t_k)+B_kx(t_k)+c_k,\ k=1,2,\cdots\\ x(\tau^+)=\eta,\ \tau\ge0.\end{cases}\tag{23}$$

with

(a) $B_k\in M_n(\mathbb{R})$, $k=1,2,\cdots$;
(b) $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$;
(c) $c:=\{c_k\}_{k=1}^\infty,\in l^\infty(\mathbb{R}^n)$;
(d) $f\in\mathscr{R}(\mathbb{T}_+,\mathbb{R}^n)$is a given function.

Theorem 9. Let (H) holds. If $(a),(b),(c)$ and (d) hold. Then the solution of IVP (23) is

$$\begin{aligned}x(t)=G_M(t,\tau)\eta+\int_\tau^t G_M(t,\sigma(s))\left(I-\frac{\mu(s)}{2}M\right)f(s)\Delta s\\+\sum_{\tau<t_j<t}G_M(t,t_j^+)c_j,\quad t\ge\tau.\end{aligned}$$

Proof. Let $(\tau,\eta)\in\mathbb{T}_+\times R^n$. Then there exists $i\in\{1,2,\cdots\}$ such that $\tau\in[t_{i-1},t_i)$. Then unique solution of (23) on $[\tau,t_i]$ is given by

$$\begin{aligned}x(t)&=E_M(t,\tau)\eta+\int_\tau^t E_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s\\&=G_M(t,\tau)\eta+\int_\tau^t G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s,\ \ t\in[\tau,t_i].\end{aligned}$$

For $t\in(t_i,t_{i+1}]$ the Cauchy problem

$$\begin{cases}x^\Delta=M(t)\langle x(t)\rangle+\left(I-\dfrac{\mu}{2}M\right)f(t),\quad t\in(t_i,t_{i+1}],\\ x(t_i^+)=x(t_i)+B_ix(t_i)+c_i,\end{cases}$$

has unique solution

$$\begin{aligned}x(t)&=E_M(t,t_i^+)x(t_i^+)\\&\quad+\int_{t_i}^t E_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s,\quad t\in(t_i,t_{i+1}].\end{aligned}$$

It follows that

$$\begin{aligned}x(t)&=E_M(t,t_i^+)\left[(I+B_i)x(t_i)+c_i\right]\\&\quad+\int_{t_i}^t E_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s\\&=E_M(t,t_i^+)(I+B_i)E_M(t_i,\tau)\eta\\&\quad+\int_\tau^{t_i}G_M(t_i,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s+E_M(t,t_i^+)c_i\\&\quad+\int_{t_i}^t E_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s\end{aligned}$$

$$= E_M(t,t_i^+)(I+B_i)E_M(t_i,\tau)\eta + \int_{\tau}^{t_i} E_M(t,t_i^+)(I+B_i)E_M(t_i,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + E_M(t,t_i^+)c_i + \int_{t_i}^{t} E_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s.$$

Using (20) we get that

$$x(t) = G_M(t,\tau)\eta + \int_{\tau}^{t_k} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + \int_{t_k}^{t} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + G_M(t,t_i^+)c_i\,,$$

and so

$$x(t) = G_M(t,\tau)\eta + \int_{\tau}^{t} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + G_M(t,t_i^+)c_i,\ \ t\in(t_i,t_{i+1}].$$

Now, we suppose that the unique solution of (23) on $(t_{k-1},t_k]$, for any $k>i+2$, is

$$x(t) = G_M(t,\tau)\eta + \int_{\tau}^{t} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + \sum_{\tau<t_j<t_k} G_M(t,t_j^+)c_j,\ \ t\in(t_k,t_{k+1}].$$

Then the IVP

$$\begin{cases} x^{\Delta} = M(t)\langle x(t)\rangle + \left(I-\frac{\mu}{2}M\right)f(t),\ \ t\in(t_k,t_{k+1}], \\ x(t_k^+) = x(t_k) + B_k x(t_k) + c_k \end{cases}$$

has the unique solution given by

$$x(t) = E_M(t,t_k^+)x(t_k^+) + \int_{t_k}^{t} E_M(t,\sigma(s)M(s)\left(I-\frac{\mu}{2}M\right)f(s)\Delta s,\ \ t\in(t_k,t_{k+1}].$$

It follows that

$$\begin{aligned} x(t) &= E_M(t,t_k^+)\left[(I+B_k)x(t_k)+c_k\right] + \int_{t_k}^{t} E_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s \\ &= E_M(t,t_k^+)(I+B_k)G_M(t_k,\tau)\eta + \int_{\tau}^{t_k} G_M(t_k,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s \\ &\quad + \sum_{\tau<t_j<t_k} G_M(t_k,t_j^+)c_j + E_M(t,t_k^+)c_k + \int_{t_k}^{t} E_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s, \end{aligned}$$

hence

$$\begin{aligned} x(t) &= E_M(t,t_k^+)(I+B_k)G_M(t_k,\tau)\eta + \int_{\tau}^{t_k} E_M(t,t_k^+)(I+B_k)G_M(t_k,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s \\ &\quad + \sum_{\tau<t_j<t_k} E_M(t,t_k^+)(I+B_k)G_M(t_k,t_j^+)c_j + G_M(t,t_k^+)c_k + \int_{t_k}^{t} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s. \end{aligned}$$

Using Remark 1, we get

$$x(t) = G_M(t,\tau)\eta + \int_{\tau}^{t_k} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + \int_{t_k}^{t} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + \sum_{\tau<t_j<t_k} G_M(t,t_j^+)c_j,$$

and so

$$x(t) = G_M(t,\tau)\eta + \int_{\tau}^{t} G_M(t,\sigma(s))\left(I-\frac{\mu}{2}M\right)f(s)\Delta s + \sum_{\tau<t_j<t_k} G_M(t,t_j^+)c_j.$$

□

Corollary 3. If $B_k\in M_n(\mathbb{R})$, $M\in\mathscr{R}^C(\mathbb{T}_+,M_n(\mathbb{R}))$, $k=1,2,\cdots$ and $f\in\mathscr{R}^C(\mathbb{T}_+,\mathbb{R}^n)$ then, for each $(\tau,\eta)\in\mathbb{T}_+\times\mathbb{R}^n$, the IVP

$$\begin{cases} x^{\Delta}(t) = M(t)\langle x(t)\rangle + \left(I-\frac{\mu}{2}M\right)f(t),\ \ t\in\mathbb{T}_{(\tau)},\ \ t\neq t_k \\ x(t_k^+) = x(t_k^-) + B_k x(t_k),\ \ k=1,2,\cdots \\ x(\tau^+) = \eta,\ \ \tau\geq 0 \end{cases}$$

has a unique solution given by

$$x(t) = G_M(t,\tau)\eta + \int_{\tau}^{t} G_M(t,\sigma(s))\left(I - \frac{\mu}{2}M\right) f(s)\Delta s, \quad t \in \mathbb{T}_{(\tau)}.$$

Let us consider the following Cayley impulsive time-invariant control system with constant graininess function μ:

$$\begin{cases} x^{\Delta}(t) = M\langle x(t)\rangle + \left(I + \frac{\mu}{2}M\right)Bu(t) \\ x\left(t_k^+\right) = (I + B_k)\,x(t_k) \\ x(\tau^+) = \eta, \end{cases} \tag{24}$$

where $x(\cdot)$ is state vector, and $u(\cdot)$ is an input.

Definition 3. [4] The impulsive system (24) is called controllable on the interval $\left[t_0, t_f\right]$, if any initial state x_0 there exists a piece-wise rd-continuous input signal u such that the corresponding solution satisfied $x\left(t_f\right) = 0$.

Similar to Theorem 2.7 in [37], by using Lemma 2.6 in [37], and Lemma 1, we can obtain the following Kalman-type controllability condition.

Theorem 10. Let (H) holds. The time-invariant Cayley impulsive control system (24) is controllable on $[t_0, t_f]$ if and only if the $n \times nm$ block controllable matrix

$$C = \left[\left(I + \frac{\mu}{2}M\right)B : M\left(I + \frac{\mu}{2}M\right)B : \cdots : M^{n-1}\left(I + \frac{\mu}{2}M\right)B\right] \tag{25}$$

has rank n.

Remark 2. Observe that the block controllable matrix is dependent on graininess function μ. For each time scale with constant graininess, we will obtain different conditions.

Example 4. Let us consider the following Cayley impulsive time-invariant control system on arbitrary time scale with constant graininess function μ:

$$\begin{aligned} x^{\Delta}(t) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\langle x(t)\rangle + \begin{pmatrix} 1 & \frac{\mu}{2} \\ -\frac{\mu}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t), \\ x(t_k^+) &= \begin{pmatrix} 1 & \frac{\mu}{k} \\ -\frac{\mu}{k} & 1 \end{pmatrix} x(t_k), \\ x(\tau^+) &= \eta, \end{aligned} \tag{26}$$

The block controllable matrix (25) for the system (26) is as follows:

$$C = \begin{pmatrix} 1 + \frac{\mu}{2} & 1 - \frac{\mu}{2} \\ 1 - \frac{\mu}{2} & -1 - \frac{\mu}{2} \end{pmatrix}.$$

Determinant of $C = -\frac{1}{2}\mu^2 - 2 \neq 0$ for each μ.

Therefore, Theorem 10 implies that the system (26) is controllable for each choice of μ.

5. CONCLUSIONS AND FUTURE DIRECTIONS

We proposed the solution of linear time-varying Cayley impulsive dynamic system on time scales. We have introduced Cayley regressive matrices. The basic properties of the transition matrix and impulsive transition matrix have been proved. We have also given some estimates for the solution of the Cayley impulsive dynamic systems. Sufficient and necessary conditions for the controllability of time-invariant linear Cayley impulsive control systems on time scales with constant graininess have been derived. This criterion includes those results for time scale/continuous linear impulsive systems previously reported in the literature.

Moreover, these results will be very useful for the analysis and synthesis of impulsive control systems on time scales. Stability, and Hyers–Ulam stability for the Cayley impulsive dynamic systems can be discussed in the future. We can use the results for the study of the transfer function for linear Cayley dynamic systems.

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