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An Innovative Hybrid Multi-Criteria Decision-Making Approach under Picture Fuzzy Information

Ghous Ali ^{1,*} , Muhammad Zain Ul Abidin ¹ , Qin Xin ²  and Ferdous M. O. Tawfiq ³

¹ Department of Mathematics, Division of Science and Technology, University of Education, Lahore 54770, Pakistan

² Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100 Torshavn, Faroe Islands, Denmark

³ Department of Mathematics, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia

* Correspondence: ghous.ali@ue.edu.pk

Abstract: These days, multi-criteria decision-making (MCDM) approaches play a vital role in making decisions considering multiple criteria. Among these approaches, the picture fuzzy soft set model is emerging as a powerful mathematical tool for handling various kinds of uncertainties in complex real-life MCDM situations because it is a combination of two efficient mathematical tools, namely, picture fuzzy sets and soft sets. However, the picture fuzzy soft set model is deficient; that is, it fails to tackle information symmetrically in a bipolar soft environment. To overcome this difficulty, in this paper, a model named picture fuzzy bipolar soft sets (P_{RFBSS} s, for short) is proposed, which is a natural hybridization of two models, namely, picture fuzzy sets and bipolar soft sets. An example discussing the selection of students for a scholarship is added to illustrate the initiated model. Some novel properties of P_{RFBSS} s such as sub-set, super-set, equality, complement, relative null and absolute P_{RFBSS} s, extended intersection and union, and restricted intersection and union are investigated. Moreover, two fundamental operations of P_{RFBSS} s, namely, the AND and OR operations, are studied. Thereafter, some new results (De Morgan's law, commutativity, associativity, and distributivity) related to these proposed notions are investigated and explained through corresponding numerical examples. An algorithm is developed to deal with uncertain information in the P_{RFBSS} environment. To show the efficacy and applicability of the initiated technique, a descriptive numerical example regarding the selection of the best graphic designer is explored under P_{RFBSS} s. In the end, concerning both qualitative and quantitative perspectives, a detailed comparative analysis of the initiated model with certain existing models is provided.

Keywords: picture fuzzy soft set; bipolarity; score function; algorithm; multi-criteria decision-making



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1. Introduction

The development and improvement of technology have changed a lot of things in the past few decades. These changes include new trends and adoptions adding to the problems and complications emerging with these new possibilities. Conditions often emerge in which one must make decisions considering many different criteria and dependencies. This is where the multi-criteria decision-making (MCDM) techniques come in handy. The MCDM tools and methods allow the consideration of multiple criteria in one place, thus making the decision-making process smoother and easier.

For dealing with complicated decision problems and undeniable uncertainties in information, decision making is considered an important branch of science that provides tools and techniques to tackle such confusions and complications effectively. Early theories, e.g., the classical set theory, could only handle a limited set of problems and failed to deal with uncertain and incomplete information. This issue was of prime significance for decision makers keeping in view the wide area of problems facing uncertainty and

needing a solution. Following the 16th century works on the development of modern probability theory, Zadeh [1] solved this problem in 1965 by introducing his revolutionary fuzzy set theory. Zadeh's fuzzy sets (FSs) considered an extended range of memberships for the consideration of the truthfulness of an event, i.e., he mapped the set of objects to the close interval of $[0, 1]$ rather than the discrete set $\{0, 1\}$ as in classical sets. This interval-valued range allowed for the illustration of partial truthfulness in between the true and false bounds (1 and 0, respectively). This theory found many applications and great interest in the decision sciences. Some recent works include the application of FSs in risk assessment and excavation management by Lin et al. [2]; a discussion on the applications and contributions of FS theory in human reliability by Gholamizadeh et al. [3]; evidential fuzzy multi-criteria decision making (MCDM) based on belief entropy by Xiao [4]; and more.

The FS theory considered only the memberships declaring how much an object satisfies some particular aspect. However, decision makers had to face issues considering problems requiring knowledge of the degree of dissatisfaction degrees as well. To overcome this issue, Atanassov [5] introduced the intuitionistic fuzzy sets (IFSs) (analogous to the type-I fuzzy sets) as a natural extension of FSs. An IFS offers two memberships, i.e., a membership η^+ declaring the degree of satisfaction and a non-membership η^- declaring the degree of dissatisfaction of an object concerning some particular parameter, along with the restriction that bounds the sum of these two degrees by unity. Later extensions of the IFS model include Pythagorean fuzzy sets (analogous to the type-II fuzzy sets) [6] that softened the restrictions of IFSs. These Pythagorean fuzzy sets allowed for decision making in situations where the sum of squares of the two degrees (membership and non-membership) are bounded by unity, i.e., $0 \leq (\eta^+)^2 + (\eta^-)^2 \leq 1$. Many researchers adopted this idea of non-membership degrees. In 2019, Xiao [7] provided a distance measure for IFSs and applied it to pattern classification problems. Recently, Wang et al. [8] discussed interval-valued intuitionistic fuzzy Jensen–Shannon divergence and its application in multi-attribute decision making. Yu et al. [9] discussed the evolution of IFSs with time through a deep exploration of the literature. Wang et al. [10] presented some uncertainty measures for Pythagorean fuzzy sets and discussed their MCDM applications. The discussed theories could not effectively deal with multi-parameterized data sets. However, while making decisions, one often encounters problems considering the alternatives regarding several criteria or attributes. To deal with this issue, Molodtsov [11] launched the theory of soft sets, which acted as parameterized families of sets. After the initiation of the soft set model, many researchers investigated different operations for soft sets and explored several applications related to soft sets [12]. Early hybridizations of models based on soft sets include the hybridization of soft sets and fuzzy sets known as fuzzy soft sets (FSSs) [13], which was an approach introduced by Maji et al. [14]. The further improvements and extensions of the soft sets contributed significantly to the the development of MCDM (see [15–17]).

Apart from these scenarios, there arise many problems and uncertain situations in daily life requiring the involvement of neutral behavior of alternatives in the data sets. For example, in an election, voters may be divided into three groups, those who vote for, those who vote against, and those who refuse to vote. To solve such problems, Cuong [18–20] introduced the concept of picture fuzzy sets (PrFSs). These PrFSs can deal with membership, non-membership, and neutral degrees of given alternatives. After this powerful invention of PrFS theory, several researchers were attracted to the notion of PrFS and proposed various new operations and hybrid soft set models under the picture fuzzy environment. For instance, Ganie et al. [21] developed certain correlation coefficients of PrFSs to verify how much PrFSs correlate with each other. The prominent applications of PrFSs include decision making, pattern recognition, clustering analysis, medical diagnosis, etc. Further extending PrFS theory, Tchier et al. [22] introduced the notion of picture fuzzy soft expert sets (PrFSEs) and discussed their effective applications towards group decision making. The literature of PrFS theory is very rich in solving complex MCDM problems. For instance, Saraji and Streimikiene [23] evaluated the adoption of circular supply chains in manufacturing sectors under the picture fuzzy approach. Rong et al. [24]

presented a novel MCDM P_RFS -based method for estimating emergency management schemes. Simic et al. [25] proposed a novel picture fuzzy extension of the CODAS method for the multiple criteria of a vehicle shredding facility location (see also Wang et al. [26]). Some more works related to P_RFS s include the fuzzy logic operators for P_RFS s [27], picture fuzzy soft sets (P_RFSS s) [28], correlation coefficients for P_RFS s [29,30], similarity measures for P_RFS s and their applications towards MCDM [31,32], distance measures for P_RFS s [33], bipolar P_RFS s [34], 2-tuple linguistic complex q -rung picture fuzzy sets [35], and picture fuzzy Aczel–Alsina average aggregation operators [36], etc. Some recently explored applications of P_RFS s include online-review-based sentiment analysis for the evaluation of new electric vehicles under P_RFS s by He and Wang [37]; the evaluation of citizens' satisfaction level regarding municipality services using the P_RF VIKOR method by Yildirim and Yildirim [38]; and the evaluation of pedagogic systems via P_RFS -based group decision making by Van Pham et al. [39]. Figure 1 elaborates on the differences between IFS s and P_RFS s pictorially, and Table 1 discusses the recent MCDM contributions under picture fuzzy information.

Table 1. Summary some recent contribution to picture fuzzy MCDM.

Reference	Decision Model	Contribution
Simic et al. [25]	P_RFS -based CODAS method	MCDM CODAS method extended for picture fuzzy information and applied to a vehicle shredding facility location.
Sindhu et al. [34]	Bipolar P_RFS -based operators	Aggregation operator, TOPSIS, and VIKOR MCDM approaches based on bipolar picture fuzzy sets applied to an MCDM investment problem.
Senapati [36]	P_RFS -based Aczel–Alsina operators	MADM method based on picture fuzzy Aczel–Alsina aggregation operators applied to an investment problem.
Yildirim and Yildirim [38]	P_RFS -based VIKOR method	Picture fuzzy VIKOR method applied to evaluate the satisfaction level of citizens with municipality services.
Akram et al. [40]	q -Rung P_RFS -based Einstein operators	Einstein operators based aggregation operators applied to MADM problems for the selection of business sites and suppliers under q -rung picture fuzzy sets.
Akram [41]	q -Rung P_RFS -based VIKOR and TOPSIS methods	q -Rung picture fuzzy VIKOR and TOPSIS methods applied to the selection of housing society and industrial robots.
Haktanir and Kahraman [42]	P_RFS -based CRITIC and REGIME MCDM methods	Picture fuzzy CRITIC and REGIME MCDM methods applied to selection of wearable health technology.
Shit et al. [43]	Trapezoidal P_RFS -based harmonic operators	MADM technique based on trapezoidal picture fuzzy harmonic aggregation operators applied to site selection for a telecom tower.
Karamti et al. [44]	P_RFS -based divergence measure	Picture fuzzy divergence measure-based similarity MCDM applications to dengue sickness and pattern identification.
Rehman and Mahmood [45]	Picture fuzzy N -soft set	Picture fuzzy N -soft set-based MCDM applications for the selection of coronavirus vaccine and next-generation firewall.
Li et al. [46]	q -Rung picture linguistic set	MAGDM technique based on q -rung picture Heronian mean operators applied to choosing an enterprise resource planning system.
Mahmood et al. [47]	Complex picture fuzzy N -soft set	Complex picture fuzzy N -soft sets-based MADM algorithm applied to the performance assessment of e-waste recycling program and winner prediction for FIFA world cup 2022.

For other important results related to P_RFS s, the readers are referred to [48–50].

In addition, many real-life problems involve positive parameters (favoring parameters) and symmetrically negative parameters (opposing parameters). Because of this, Shabir and Naz [51] developed the idea of bipolar soft sets (B_RSS s). This structure considers two sets of parameters; one set contains the favorable parameters, while the other set contains the parameters opposite to those in the favorable set. In this way, B_RSS s effectively depict the

bipolarity of parameters in complicated decision-making problems. In addition, Naz and Shabir [52] combined FSs with B_RSSs to formulate a hybridized model called fuzzy B_RSSs. Currently, the theory of B_RSSs is playing a vital role in different domains to solve several MCDM problems in the form of various hybridized uncertainty theories (see [53–57]).

Considering the research gaps in the literature (i.e., limitations of models such as P_RFSSs failing to consider the bipolarity of decision attributes and inefficacy of B_RSSs in considering uncertainties in positive, negative and neutral degrees), this work introduces a model capable of combining the above-discussed qualities in one method.

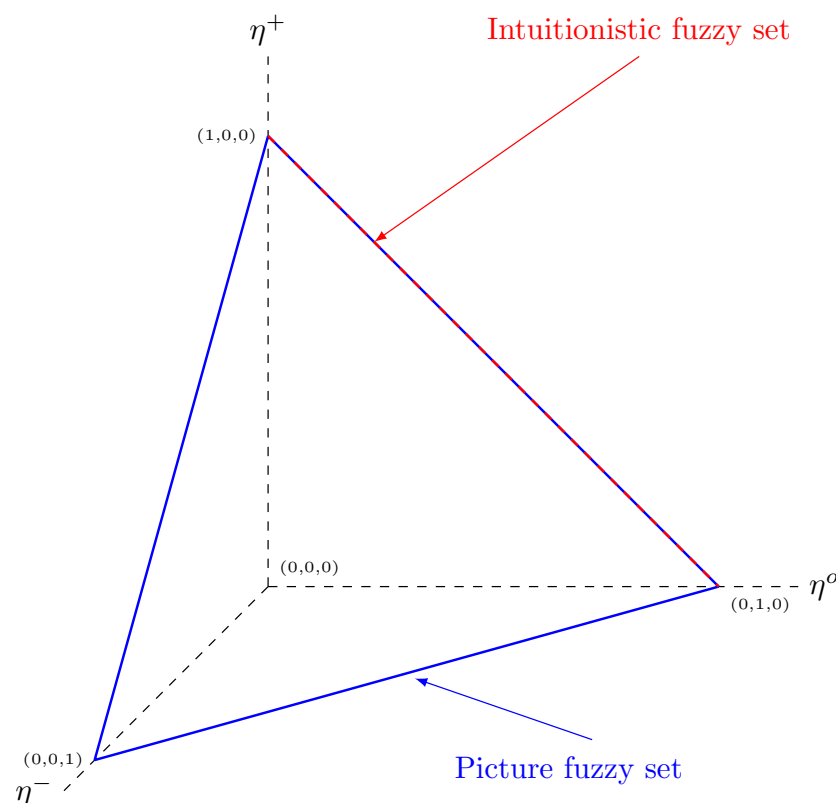


Figure 1. Comparison among the IFS [5] and PFS [20] models.

The following are the motivations of our proposed study:

1. Existing decision-making structures, including P_RFSSs, are unable to deal with the bipolarity of decision attributes efficiently.
2. Models such as B_RSSs and fuzzy B_RSSs which are capable of depicting the bipolarity of attributes fail to handle uncertainties effectively since they fail to consider positive, negative, and neutral degrees of opinion.
3. Problems such as selection of a fashion designer specific to the company's needs are complicated MCDM problems requiring a strong decision-making algorithm.

The contributions of this paper are provided below:

1. A model called picture fuzzy bipolar soft set (or P_RFBSS), a natural hybrid extension of P_RFS and B_RSS, is proposed.
2. Some novel properties and two fundamental operations of P_RFBSSs are presented and illustrated via corresponding numerical examples.
3. Important results including the commutative, associative, and distributive properties are presented. Furthermore, De Morgan's Laws for the proposed properties and operations are shown.
4. A P_RFBSS-based algorithm using score functions for picture fuzzy numbers is presented to deal with MCDM problems considering the decision attributes in symmetry.

5. An MCDM application of P_{RFBSS} s, i.e., the selection of a fashion designer for a studio, is presented and solved using the newly proposed algorithm based on P_{RFBSS} s.
6. Finally, a detailed comparative analysis concerning both qualitative and quantitative perspectives of the proposed model with certain existing models is provided.

Organization of Paper

The remaining formulation of the paper is given as follows: In Section 2, firstly, some basic notions, including B_{RSS} s and P_{RFS} s, and some of their fundamental results are recalled. In Section 3, the main notion of our study called picture fuzzy bipolar soft sets (or P_{RFBSS} s) is proposed. Some novel properties and fundamental operations of P_{RFBSS} s are presented and explained with numerical examples. In Section 4, an MCDM application related to the initiated model is explored, supported by a novel algorithm. In Section 5, a detailed comparative analysis of the proposed model with certain existing models is studied. Finally, in Section 6, the concluding remarks are discussed along with future directions.

2. Preliminaries

This section recalls some fundamental notions, including B_{RSS} s and P_{RFS} s, along with their score and accuracy functions, which will support the developments in coming sections. The following is the definition of B_{RSS} s.

Definition 1 ([51]). For a given universal set $\mathcal{P}$ and a universe of attributes $\mathcal{R}$, a triple $(\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg\mathcal{Q})$ is called a bipolar soft set, or B_{RSS} , over $\mathcal{P}$, where $\mathcal{Q} \subseteq \mathcal{R}$ if the functions $\mathcal{G}$ and $\mathcal{H}$ are provided by $\mathcal{G} : \mathcal{Q} \rightarrow \mathcal{F}(\mathcal{P})$ and $\mathcal{H} : \neg\mathcal{Q} \rightarrow \mathcal{F}(\mathcal{P})$, such that

$$\mathcal{G}(\mathfrak{p}) \cap \mathcal{H}(\neg\mathfrak{p}) = \emptyset, \quad \forall \mathfrak{p} \in \mathcal{Q}, \neg\mathfrak{p} \in \neg\mathcal{Q}.$$

Here $\mathcal{F}(\mathcal{P})$ represents the power set of the universal set $\mathcal{P}$, and $\neg\mathcal{Q}$ serves as the collection of opposite attributes as compared to the attributes in $\mathcal{Q}$.

Definition 2 ([20]). For a given universal set $\mathcal{P}$ and a universe of attributes $\mathcal{R}$, a picture fuzzy set (or P_{RFS}) $\mathcal{G}$ over the universal set $\mathcal{P}$ is provided as below:

$$\mathcal{G} = \{ \langle \mathfrak{p}, \eta^+(\mathfrak{p}), \eta^0(\mathfrak{p}), \eta^-(\mathfrak{p}) \rangle \mid \mathfrak{p} \in \mathcal{P} \},$$

which satisfies the following condition:

$$0 \leq \eta^+(\mathfrak{p}) + \eta^0(\mathfrak{p}) + \eta^-(\mathfrak{p}) \leq 1, \quad (1)$$

where $\eta^+(\mathfrak{p})$, $\eta^0(\mathfrak{p})$ and $\eta^-(\mathfrak{p})$ are the degrees of positive, neutral and negative memberships, respectively. For simplicity, we call the triple $\mathcal{F} = (\eta_{\mathcal{F}}^+, \eta_{\mathcal{F}}^0, \eta_{\mathcal{F}}^-)$ a picture fuzzy number (PFN).

From now on, we represent the collection of all P_{RFS} s over $\mathcal{P}$ as $\mathcal{F}^{\mathcal{P}}$.

Definition 3 ([48]). For a PFN $\mathcal{F} = (\eta_{\mathcal{F}}^+, \eta_{\mathcal{F}}^0, \eta_{\mathcal{F}}^-)$, we define its score function $s(\mathcal{F})$ as below:

$$s(\mathcal{F}) = \frac{1}{2}(1 + (2\eta_{\mathcal{F}}^+) - (\frac{\eta_{\mathcal{F}}^0}{2}) - (\eta_{\mathcal{F}}^-)). \quad (2)$$

Similarly, for a PFN $\mathcal{F} = (\eta_{\mathcal{F}}^+, \eta_{\mathcal{F}}^0, \eta_{\mathcal{F}}^-)$, we define its accuracy function of $\mathcal{A}(\mathcal{F})$ as follows:

$$\mathcal{A}(\mathcal{F}) = (\eta_{\mathcal{F}}^+) + (\eta_{\mathcal{F}}^0) + (\eta_{\mathcal{F}}^-). \quad (3)$$

Definition 4 ([48]). Let $s(\mathcal{F}_1)$, $s(\mathcal{F}_2)$ and $\mathcal{A}(\mathcal{F}_1)$, $\mathcal{A}(\mathcal{F}_2)$ be the score and accuracy functions of two PFNs $\mathcal{F}_1 = (\eta_{\mathcal{F}_1}^+, \eta_{\mathcal{F}_1}^0, \eta_{\mathcal{F}_1}^-)$ and $\mathcal{F}_2 = (\eta_{\mathcal{F}_2}^+, \eta_{\mathcal{F}_2}^0, \eta_{\mathcal{F}_2}^-)$, respectively. Then, $\mathcal{F}_1 < \mathcal{F}_2$ if and only if:

- (i) $s(\mathcal{F}_1) < s(\mathcal{F}_2)$; or
- (ii) $s(\mathcal{F}_1) = s(\mathcal{F}_2)$ and $\mathcal{A}(\mathcal{F}_1) < \mathcal{A}(\mathcal{F}_2)$.

3. Picture Fuzzy Bipolar Soft Sets

In this section, we first present a new hybrid MCDM model, namely, picture fuzzy bipolar soft sets (or P_RFBSS). Then, we discuss some basic properties and operations along with examples. We start with the definition of P_RFBSS as below:

Definition 5. For a given universal set $\mathcal{P}$ and a universe of attributes $\mathcal{R}$, a picture fuzzy bipolar soft set or P_RFBSS over $\mathcal{P}$, denoted by $\Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg\mathcal{Q})$ where $\mathcal{Q} \subseteq \mathcal{R}$ and $\neg\mathcal{Q} \subseteq \neg\mathcal{R}$, is a mixture of two mappings $\mathcal{G} : \mathcal{Q} \rightarrow \mathcal{F}^{\mathcal{P}}$ and $\mathcal{H} : \neg\mathcal{Q} \rightarrow \mathcal{F}^{\mathcal{P}}$, which are respectively defined as:

$$\begin{aligned}\mathcal{G}(\mathbf{q}) &= \{ \langle \mathbf{p}, \eta_{\mathcal{G}}^+(\mathbf{q})(\mathbf{p}), \eta_{\mathcal{G}}^0(\mathbf{q})(\mathbf{p}), \eta_{\mathcal{G}}^-(\mathbf{q})(\mathbf{p}) \rangle \mid \mathbf{p} \in \mathcal{P} \}, \\ \mathcal{H}(\neg\mathbf{q}) &= \{ \langle \mathbf{p}, \zeta_{\mathcal{H}}^+(\neg\mathbf{q})(\mathbf{p}), \zeta_{\mathcal{H}}^0(\neg\mathbf{q})(\mathbf{p}), \zeta_{\mathcal{H}}^-(\neg\mathbf{q})(\mathbf{p}) \rangle \mid \mathbf{p} \in \mathcal{P} \},\end{aligned}$$

for all $\mathbf{q} \in \mathcal{Q}$ and $\neg\mathbf{q} \in \neg\mathcal{Q}$ with the following conditions:

$$0 \leq \eta_{\mathcal{G}}^+(\mathbf{q})(\mathbf{p}) + \eta_{\mathcal{G}}^0(\mathbf{q})(\mathbf{p}) + \eta_{\mathcal{G}}^-(\mathbf{q})(\mathbf{p}) \leq 1, \quad (4)$$

$$0 \leq \zeta_{\mathcal{H}}^+(\neg\mathbf{q})(\mathbf{p}) + \zeta_{\mathcal{H}}^0(\neg\mathbf{q})(\mathbf{p}) + \zeta_{\mathcal{H}}^-(\neg\mathbf{q})(\mathbf{p}) \leq 1, \quad (5)$$

where

$$0 \leq \eta_{\mathcal{G}}^+(\mathbf{q})(\mathbf{p}) + \zeta_{\mathcal{H}}^+(\neg\mathbf{q})(\mathbf{p}) \leq 1, \quad (6)$$

$$0 \leq \eta_{\mathcal{G}}^0(\mathbf{q})(\mathbf{p}) + \zeta_{\mathcal{H}}^0(\neg\mathbf{q})(\mathbf{p}) \leq 1, \quad (7)$$

$$0 \leq \eta_{\mathcal{G}}^-(\mathbf{q})(\mathbf{p}) + \zeta_{\mathcal{H}}^-(\neg\mathbf{q})(\mathbf{p}) \leq 1, \quad (8)$$

and the degrees of positive memberships are $\eta_{\mathcal{G}}^+(\mathbf{q})(\mathbf{p})$ and $\zeta_{\mathcal{H}}^+(\neg\mathbf{q})(\mathbf{p})$, the degrees of neutral memberships are $\eta_{\mathcal{G}}^0(\mathbf{q})(\mathbf{p})$ and $\zeta_{\mathcal{H}}^0(\neg\mathbf{q})(\mathbf{p})$ and those of negative memberships are $\eta_{\mathcal{G}}^-(\mathbf{q})(\mathbf{p})$ and $\zeta_{\mathcal{H}}^-(\neg\mathbf{q})(\mathbf{p})$, respectively.

Notice that the set $\neg\mathcal{Q}$ contains opposite parameters as compared to the parameters in $\mathcal{Q}$.

The following example explains the Definition 5.

Example 1. Suppose a university ABC has some merit scholarships and to choose the most appropriate student from the five shortlisted candidates, a committee of senior professors is designed to perform this task. Let $\mathcal{R} = \{q_1 = \text{intelligent}, q_2 = \text{hard working}, q_3 = \text{regular}, q_4 = \text{cooperative}, q_5 = \text{well-mannered}\}$ be the collection of favorable attributes and let the respective not-set be $\neg\mathcal{R} = \{\neg q_1 = \text{dull}, \neg q_2 = \text{lazy}, \neg q_3 = \text{irregular}, \neg q_4 = \text{non-cooperative}, \neg q_5 = \text{bad-mannered}\}$. After a detailed discussion between committee members, they decide to take the favorable set of decision attributes $\mathcal{Q} = \{q_1, q_3, q_5\} \subset \mathcal{R}$ and its corresponding 'not-set' in the evaluation process. Consider the set $\mathcal{P} = \{p_1, p_2, \dots, p_5\}$ represents five students bearing different qualities. The evaluation reports for these students are provided by the committee in the form of a P_RFBSS $\Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg\mathcal{Q})$ as follows, describing the qualities and weaknesses of students:

$$\begin{aligned}\mathcal{G}(q_1) &= \{ (p_1, 0.13, 0.03, 0.43), (p_2, 0.23, 0.33, 0.19), (p_3, 0.53, 0.13, 0.23), \\ &\quad (p_4, 0.63, 0.06, 0.13), (p_5, 0.73, 0.13, 0.12) \}, \\ \mathcal{G}(q_3) &= \{ (p_1, 0.44, 0.13, 0.02), (p_2, 0.45, 0.13, 0.32), (p_3, 0.13, 0.12, 0.11), \\ &\quad (p_4, 0.46, 0.26, 0.13), (p_5, 0.11, 0.16, 0.39) \}, \\ \mathcal{G}(q_5) &= \{ (p_1, 0.68, 0.21, 0.01), (p_2, 0.11, 0.56, 0.23), (p_3, 0.44, 0.23, 0.22), \\ &\quad (p_4, 0.56, 0.12, 0.26), (p_5, 0.78, 0.01, 0.12) \},\end{aligned}$$

$$\begin{aligned}\mathcal{H}(\neg q_1) &= \{(p_1, 0.26, 0.36, 0.23), (p_2, 0.02, 0.43, 0.12), (p_3, 0.33, 0.23, 0.43), \\ &\quad (p_4, 0.13, 0.02, 0.63), (p_5, 0.13, 0.53, 0.16)\}, \\ \mathcal{H}(\neg q_3) &= \{(p_1, 0.23, 0.33, 0.01), (p_2, 0.25, 0.12, 0.57), (p_3, 0.43, 0.25, 0.30), \\ &\quad (p_4, 0.19, 0.46, 0.25), (p_5, 0.12, 0.36, 0.19)\}, \\ \mathcal{H}(\neg q_5) &= \{(p_1, 0.06, 0.18, 0.48), (p_2, 0.69, 0.13, 0.03), (p_3, 0.24, 0.06, 0.12), \\ &\quad (p_4, 0.27, 0.22, 0.36), (p_5, 0.08, 0.41, 0.39)\}.\end{aligned}$$

For $p_i \in \mathcal{P}$, $q_j \in \mathcal{Q}$ and $\neg q_j \in \neg \mathcal{Q}$, Table 2 represents the above $P_{\text{RFBSS}} \Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$ with entries

$$a_{ij} = \langle \mathcal{G}(p_i)(q_j), \mathcal{H}(p_i)(\neg q_j) \rangle.$$

Table 2. Tabular form of the $P_{\text{RFBSS}} \Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$.

Ω	$q_1, \neg q_1$	$q_3, \neg q_3$	$q_5, \neg q_5$
p_1	$\langle (0.13, 0.03, 0.43), (0.26, 0.36, 0.23) \rangle$	$\langle (0.44, 0.13, 0.02), (0.23, 0.33, 0.11) \rangle$	$\langle (0.68, 0.21, 0.01), (0.06, 0.18, 0.48) \rangle$
p_2	$\langle (0.23, 0.33, 0.19), (0.02, 0.43, 0.12) \rangle$	$\langle (0.45, 0.13, 0.32), (0.25, 0.12, 0.57) \rangle$	$\langle (0.11, 0.56, 0.23), (0.69, 0.13, 0.03) \rangle$
p_3	$\langle (0.53, 0.13, 0.23), (0.33, 0.23, 0.43) \rangle$	$\langle (0.13, 0.12, 0.11), (0.43, 0.25, 0.30) \rangle$	$\langle (0.44, 0.23, 0.22), (0.24, 0.06, 0.12) \rangle$
p_4	$\langle (0.63, 0.06, 0.13), (0.13, 0.02, 0.63) \rangle$	$\langle (0.46, 0.26, 0.13), (0.19, 0.46, 0.25) \rangle$	$\langle (0.56, 0.12, 0.26), (0.27, 0.22, 0.36) \rangle$
p_5	$\langle (0.73, 0.13, 0.12), (0.13, 0.53, 0.16) \rangle$	$\langle (0.11, 0.16, 0.39), (0.12, 0.36, 0.19) \rangle$	$\langle (0.78, 0.01, 0.12), (0.08, 0.41, 0.39) \rangle$

According to the committee's report, the candidate p_5 scores as the most intelligent (q_1) relatively; however, the candidate p_2 is the least dull ($\neg q_1$) among all. This represents the condition that a nearly below-average student is not necessarily a dull student. Similarly, p_5 obtains the highest agreement score for being a well-mannered student (q_5); however, the relatively bigger neutral membership and smaller negative membership degree for p_1 (considering the same attribute) make the later candidate able to stand in a fair competition with the former one.

From Definition 5, the $P_{\text{RFBSS}} (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$ is a union of two $P_{\text{RFSSs}} (\mathcal{G}, \mathcal{Q})$ and $(\mathcal{H}, \neg \mathcal{Q})$, shown in Tables 3 and 4 with entries $\mathcal{G}(p_i)(q_j)$ and $\mathcal{H}(p_i)(\neg q_j)$, respectively.

Table 3. $P_{\text{RFSS}} (\mathcal{G}, \mathcal{Q})$.

$(\mathcal{G}, \mathcal{Q})$	q_1	q_3	q_5
p_1	(0.13, 0.03, 0.43)	(0.44, 0.13, 0.02)	(0.68, 0.21, 0.01)
p_2	(0.23, 0.33, 0.19)	(0.45, 0.13, 0.32)	(0.11, 0.56, 0.23)
p_3	(0.53, 0.13, 0.23)	(0.13, 0.12, 0.11)	(0.44, 0.23, 0.22)
p_4	(0.63, 0.06, 0.13)	(0.46, 0.26, 0.13)	(0.56, 0.12, 0.26)
p_5	(0.73, 0.13, 0.12)	(0.11, 0.16, 0.39)	(0.78, 0.01, 0.12)

Table 4. $P_{\text{RFSS}} (\mathcal{H}, \neg \mathcal{Q})$.

$(\mathcal{H}, \neg \mathcal{Q})$	$\neg q_1$	$\neg q_3$	$\neg q_5$
p_1	(0.26, 0.36, 0.23)	(0.23, 0.33, 0.11)	(0.06, 0.18, 0.48)
p_2	(0.02, 0.43, 0.12)	(0.25, 0.12, 0.57)	(0.69, 0.13, 0.03)
p_3	(0.33, 0.23, 0.43)	(0.43, 0.25, 0.30)	(0.24, 0.06, 0.12)
p_4	(0.13, 0.02, 0.63)	(0.19, 0.46, 0.25)	(0.27, 0.22, 0.36)
p_5	(0.13, 0.53, 0.16)	(0.12, 0.36, 0.19)	(0.08, 0.41, 0.39)

The following discusses subset relations among P_{RFBSSs} and illustrates this concept via a numerical example.

Definition 6. For a universal set $\mathcal{P}$ and two P_{RFBSSs} , $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over $\mathcal{P}$, the set Ω_2 is called a picture fuzzy bipolar soft subset of Ω_1 , denoted as $\Omega_2 \subseteq \Omega_1$, if:

1. $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$;

2. $\mathcal{G}_2(q) \subseteq \mathcal{G}_1(q)$, that is, $\eta_{\mathcal{G}_2}^+(q)(p) \leq \eta_{\mathcal{G}_1}^+(q)(p)$, $\eta_{\mathcal{G}_2}^0(q)(p) \leq \eta_{\mathcal{G}_1}^0(q)(p)$ and $\eta_{\mathcal{G}_2}^-(q)(p) \geq \eta_{\mathcal{G}_1}^-(q)(p)$, $\forall q \in \mathcal{Q}$;
3. $\mathcal{H}_1(\neg q) \subseteq \mathcal{H}_2(\neg q)$, that is, $\zeta_{\mathcal{H}_1}^+(\neg q)(p) \leq \zeta_{\mathcal{H}_2}^+(\neg q)(p)$, $\zeta_{\mathcal{H}_1}^0(\neg q)(p) \leq \zeta_{\mathcal{H}_2}^0(\neg q)(p)$ and $\zeta_{\mathcal{H}_1}^-(\neg q)(p) \geq \zeta_{\mathcal{H}_2}^-(\neg q)(p)$, $\forall \neg q \in \neg \mathcal{Q}$ and $p \in \mathcal{P}$.

Example 2. Consider the $P_{\text{RFBSS}} \Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$ as taken in Example 1. Suppose, in the next academic session, another set of five new candidates needs to be evaluated for the merit scholarship award. This time, the committee decides to drop the manners criterion from the decision attributes. Hence, for $\mathcal{Q}_1 = \{q_1 = \text{intelligent}, q_3 = \text{regular}\} \subseteq \mathcal{R}$ and $\neg \mathcal{Q}_1 = \{\neg q_1 = \text{dull}, \neg q_3 = \text{not regular}\} \subseteq \neg \mathcal{R}$, a new $P_{\text{RFBSS}} \Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ is provided in Table 5.

Table 5. Tabular form of the $P_{\text{RFBSS}} \Omega_1$.

Ω_1	$q_1, \neg q_1$	$q_3, \neg q_3$
p_1	$\langle (0.09, 0.01, 0.45), (0.27, 0.41, 0.11) \rangle$	$\langle (0.21, 0.12, 0.04), (0.34, 0.36, 0.09) \rangle$
p_2	$\langle (0.11, 0.09, 0.22), (0.07, 0.44, 0.02) \rangle$	$\langle (0.12, 0.11, 0.33), (0.36, 0.14, 0.20) \rangle$
p_3	$\langle (0.23, 0.02, 0.24), (0.35, 0.33, 0.12) \rangle$	$\langle (0.09, 0.02, 0.13), (0.44, 0.26, 0.29) \rangle$
p_4	$\langle (0.23, 0.03, 0.17), (0.15, 0.07, 0.02) \rangle$	$\langle (0.32, 0.23, 0.15), (0.32, 0.47, 0.02) \rangle$
p_5	$\langle (0.19, 0.11, 0.26), (0.14, 0.55, 0.06) \rangle$	$\langle (0.09, 0.14, 0.41), (0.22, 0.37, 0.13) \rangle$

It is clear from Definition 6 that $(\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \tilde{\subset} (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$. This subset relation indicates that the new students are not as intelligent and regular as their senior fellows.

The concept of the equality relation between P_{RFBSS} s is studied in the following definition:

Definition 7. Two P_{RFBSS} s $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over a universal set $\mathcal{P}$ are said to be equal if $\Omega_1 \tilde{\subset} \Omega_2$ and $\Omega_2 \tilde{\subset} \Omega_1$. The equality relation between them is denoted by $\Omega_1 \doteq \Omega_2$.

The following definition discusses the notion of a complement to any P_{RFBSS} and elaborates it with an example.

Definition 8. For an arbitrary $P_{\text{RFBSS}} \Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$ over the universal set $\mathcal{P}$, its complement $\Omega^c = (\mathcal{G}^c, \mathcal{H}^c, \mathcal{Q}^c, \neg \mathcal{Q}^c)$ is again a P_{RFBSS} over $\mathcal{P}$, where:

$$\mathcal{G}^c(q) = \{ \langle p, \eta_{\mathcal{G}}^-(q)(p), \eta_{\mathcal{G}}^0(q)(p), \eta_{\mathcal{G}}^+(q)(p) \rangle \mid p \in \mathcal{P} \},$$

$$\mathcal{H}^c(\neg q) = \{ \langle p, \zeta_{\mathcal{H}}^-(\neg q)(p), \zeta_{\mathcal{H}}^0(\neg q)(p), \zeta_{\mathcal{H}}^+(\neg q)(p) \rangle \mid p \in \mathcal{P} \},$$

for all $q \in \mathcal{Q}$ and $\neg q \in \neg \mathcal{Q}$.

Example 3. Consider again the $P_{\text{RFBSS}} \Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$, as taken in Example 1. Then, using Definition 8, its complement $\Omega^c = (\mathcal{G}^c, \mathcal{H}^c, \mathcal{Q}^c, \neg \mathcal{Q}^c)$ is calculated and displayed in Table 6.

Table 6. The complement of $P_{\text{RFBSS}} (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$.

Ω^c	$q_1, \neg q_1$	$q_3, \neg q_3$	$q_5, \neg q_5$
p_1	$\langle (0.43, 0.03, 0.13), (0.23, 0.36, 0.26) \rangle$	$\langle (0.02, 0.13, 0.44), (0.01, 0.33, 0.23) \rangle$	$\langle (0.01, 0.21, 0.68), (0.48, 0.18, 0.06) \rangle$
p_2	$\langle (0.19, 0.33, 0.23), (0.12, 0.43, 0.02) \rangle$	$\langle (0.32, 0.13, 0.45), (0.57, 0.12, 0.25) \rangle$	$\langle (0.23, 0.56, 0.11), (0.03, 0.13, 0.69) \rangle$
p_3	$\langle (0.23, 0.13, 0.53), (0.43, 0.23, 0.33) \rangle$	$\langle (0.11, 0.12, 0.13), (0.30, 0.25, 0.43) \rangle$	$\langle (0.22, 0.23, 0.44), (0.12, 0.06, 0.24) \rangle$
p_4	$\langle (0.13, 0.06, 0.63), (0.63, 0.02, 0.13) \rangle$	$\langle (0.13, 0.26, 0.46), (0.25, 0.46, 0.19) \rangle$	$\langle (0.26, 0.12, 0.56), (0.36, 0.22, 0.27) \rangle$
p_5	$\langle (0.12, 0.13, 0.73), (0.16, 0.53, 0.13) \rangle$	$\langle (0.39, 0.16, 0.11), (0.19, 0.36, 0.12) \rangle$	$\langle (0.12, 0.01, 0.78), (0.39, 0.41, 0.08) \rangle$

Two utmost cases of P_{RFBSS} s are investigated in the following definitions.

Definition 9. A P_{RFBSS} on a universal set $\mathcal{P}$ is called a relative null P_{RFBSS} , symbolized by $(\Phi, \mathfrak{V}, \mathcal{Q}, \neg \mathcal{Q})$, if $\Phi(q)(p) = (0, 0, 1)$ and $\mathfrak{V}(\neg q)(p) = (1, 0, 0) \forall q \in \mathcal{Q}, \neg q \in \neg \mathcal{Q}, p \in \mathcal{P}$.

Similarly, a P_{RFBSS} on a universal set $\mathcal{P}$ is referred to as a relative absolute P_{RFBSS} , represented by $(\mathfrak{V}, \Phi, \mathcal{Q}, \neg \mathcal{Q})$, if $\mathfrak{V}(l)(p) = (1, 0, 0)$ and $\Phi(\neg l)(p) = (0, 0, 1) \forall q \in \mathcal{Q}, \neg q \in \neg \mathcal{Q}, p \in \mathcal{P}$.

The following two definitions provide the concepts of fundamental operations (AND and OR) for P_{RFBSS} s, which are further explained with the support of a numerical example:

Definition 10. For any two P_{RFBSS} s $(\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over a universal set $\mathcal{P}$, the AND operation between them, denoted by $(\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \bar{\wedge} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$, is defined as $(\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \bar{\wedge} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) = (\Lambda, \Gamma, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2)$, where, for all $(q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2, (\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$ and $p \in \mathcal{P}$, the mappings $\Lambda : \mathcal{Q}_1 \times \mathcal{Q}_2 \rightarrow \mathcal{F}^{\mathcal{P}}$ and $\Gamma : \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2 \rightarrow \mathcal{F}^{\mathcal{P}}$ are given as:

$$\begin{aligned}\Lambda(q_i, q_j)(p) &= (\min(\eta_{\mathcal{G}_1}^+(q_i)(p), \eta_{\mathcal{G}_2}^+(q_j)(p)), \min(\eta_{\mathcal{G}_1}^0(q_i)(p), \eta_{\mathcal{G}_2}^0(q_j)(p)), \max(\eta_{\mathcal{G}_1}^-(q_i)(p), \eta_{\mathcal{G}_2}^-(q_j)(p))), \\ \Gamma(\neg q_i, \neg q_j)(p) &= (\max(\zeta_{\mathcal{H}_1}^+(\neg q_i)(p), \zeta_{\mathcal{H}_2}^+(\neg q_j)(p)), \min(\zeta_{\mathcal{H}_1}^0(\neg q_i)(p), \zeta_{\mathcal{H}_2}^0(\neg q_j)(p)), \\ &\quad \min(\zeta_{\mathcal{H}_1}^-(\neg q_i)(p), \zeta_{\mathcal{H}_2}^-(\neg q_j)(p))).\end{aligned}$$

Definition 11. For any two P_{RFBSS} s $(\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over a universal set $\mathcal{P}$, the OR operation among them, represented by $(\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \vee (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$, is defined as $(\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \vee (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) = (\Psi, \Upsilon, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2)$, where, for all $(q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2, (\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$ and $p \in \mathcal{P}$, the mappings $\Psi : \mathcal{Q}_1 \times \mathcal{Q}_2 \rightarrow \mathcal{F}^{\mathcal{P}}$ and $\Upsilon : \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2 \rightarrow \mathcal{F}^{\mathcal{P}}$ are provided by:

$$\begin{aligned}\Psi(q_i, q_j)(p) &= (\max(\eta_{\mathcal{G}_1}^+(q_i)(p), \eta_{\mathcal{G}_2}^+(q_j)(p)), \min(\eta_{\mathcal{G}_1}^0(q_i)(p), \eta_{\mathcal{G}_2}^0(q_j)(p)), \min(\eta_{\mathcal{G}_1}^-(q_i)(p), \eta_{\mathcal{G}_2}^-(q_j)(p))), \\ \Upsilon(\neg q_i, \neg q_j)(p) &= (\min(\zeta_{\mathcal{H}_1}^+(\neg q_i)(p), \zeta_{\mathcal{H}_2}^+(\neg q_j)(p)), \min(\zeta_{\mathcal{H}_1}^0(\neg q_i)(p), \zeta_{\mathcal{H}_2}^0(\neg q_j)(p)), \\ &\quad \max(\zeta_{\mathcal{H}_1}^-(\neg q_i)(p), \zeta_{\mathcal{H}_2}^-(\neg q_j)(p))).\end{aligned}$$

Example 4. Consider a company that manufactures four new laptops using different hardware and software combinations. Two tech experts are hired to check the quality of these new models according to certain attributes. Consider $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$ as representing the new laptops and $\mathcal{R} = \{q_1 = \text{high speed}, q_2 = \text{good battery life}, q_3 = \text{good connectivity}\}$ as representing the set of favorable parameters. The corresponding not set of attributes is $\neg \mathcal{R} = \{\neg q_1 = \text{lagging}, \neg q_2 = \text{low battery life}, \neg q_3 = \text{distorted connectivity}\}$. Assume that the first expert considers $\mathcal{Q}_1 = \{q_1, q_2\} \subseteq \mathcal{R}$ and the second expert takes $\mathcal{Q}_2 = \{q_1, q_3\} \subseteq \mathcal{R}$ as their favorable set of parameters to compare the qualities of new laptops. Then, the estimations of experts are provided as P_{RFBSS} s Ω_1 and Ω_2 , which are displayed in Tables 7 and 8, respectively.

Table 7. Tabular form of the $P_{\text{RFBSS}} \Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$.

Ω_1	$q_1, \neg q_1$	$q_2, \neg q_2$
p_1	$\langle (0.02, 0.01, 0.19), (0.04, 0.26, 0.31) \rangle$	$\langle (0.11, 0.31, 0.02), (0.12, 0.16, 0.21) \rangle$
p_2	$\langle (0.03, 0.06, 0.39), (0.13, 0.01, 0.26) \rangle$	$\langle (0.12, 0.26, 0.31), (0.11, 0.12, 0.59) \rangle$
p_3	$\langle (0.26, 0.31, 0.16), (0.13, 0.30, 0.01) \rangle$	$\langle (0.13, 0.21, 0.61), (0.11, 0.22, 0.01) \rangle$
p_4	$\langle (0.81, 0.01, 0.02), (0.13, 0.16, 0.36) \rangle$	$\langle (0.06, 0.02, 0.11), (0.09, 0.16, 0.52) \rangle$

Table 8. Tabular form of the $P_{\text{RFBSS}} \Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$.

Ω_2	$q_1, \neg q_1$	$q_3, \neg q_3$
p_1	$\langle (0.01, 0.16, 0.23), (0.03, 0.11, 0.45) \rangle$	$\langle (0.16, 0.19, 0.23), (0.19, 0.29, 0.32) \rangle$
p_2	$\langle (0.11, 0.12, 0.13), (0.21, 0.32, 0.09) \rangle$	$\langle (0.13, 0.14, 0.03), (0.26, 0.16, 0.39) \rangle$
p_3	$\langle (0.23, 0.61, 0.14), (0.13, 0.24, 0.09) \rangle$	$\langle (0.34, 0.02, 0.09), (0.26, 0.02, 0.01) \rangle$
p_4	$\langle (0.11, 0.23, 0.19), (0.22, 0.12, 0.16) \rangle$	$\langle (0.04, 0.01, 0.03), (0.11, 0.12, 0.13) \rangle$

Then, the AND and OR operations between the $P_{\text{RFBSSs}} \Omega_1$ and Ω_2 are shown in Tables 9 and 10, respectively.

Table 9. AND operation between Ω_1 and Ω_2 .

$\Omega_1 \bar{\wedge} \Omega_2$	$(q_1, q_1), (\neg q_1, \neg q_1)$	$(q_1, q_3), (\neg q_1, \neg q_3)$
p_1	$\langle (0.01, 0.01, 0.23), (0.04, 0.11, 0.31) \rangle$	$\langle (0.02, 0.01, 0.23), (0.19, 0.26, 0.31) \rangle$
p_2	$\langle (0.03, 0.06, 0.39), (0.21, 0.01, 0.09) \rangle$	$\langle (0.03, 0.06, 0.39), (0.26, 0.01, 0.26) \rangle$
p_3	$\langle (0.23, 0.31, 0.16), (0.13, 0.24, 0.01) \rangle$	$\langle (0.26, 0.02, 0.16), (0.26, 0.02, 0.01) \rangle$
p_4	$\langle (0.11, 0.01, 0.19), (0.22, 0.12, 0.16) \rangle$	$\langle (0.04, 0.01, 0.03), (0.13, 0.12, 0.13) \rangle$
$\Omega_1 \bar{\wedge} \Omega_2$	$(q_2, q_1), (\neg q_2, \neg q_1)$	$(q_2, q_3), (\neg q_2, \neg q_3)$
p_1	$\langle (0.01, 0.16, 0.23), (0.12, 0.11, 0.21) \rangle$	$\langle (0.11, 0.19, 0.23), (0.19, 0.16, 0.21) \rangle$
p_2	$\langle (0.11, 0.12, 0.31), (0.21, 0.12, 0.09) \rangle$	$\langle (0.12, 0.14, 0.31), (0.26, 0.12, 0.39) \rangle$
p_3	$\langle (0.13, 0.21, 0.61), (0.13, 0.22, 0.01) \rangle$	$\langle (0.13, 0.02, 0.61), (0.26, 0.02, 0.01) \rangle$
p_4	$\langle (0.06, 0.02, 0.19), (0.22, 0.12, 0.16) \rangle$	$\langle (0.04, 0.01, 0.11), (0.11, 0.12, 0.13) \rangle$

Table 10. OR operation between Ω_1 and Ω_2 .

$\Omega_1 \vee \Omega_2$	$(q_1, q_1), (\neg q_1, \neg q_1)$	$(q_1, q_3), (\neg q_1, \neg q_3)$
p_1	$\langle (0.02, 0.01, 0.19), (0.03, 0.11, 0.45) \rangle$	$\langle (0.16, 0.01, 0.19), (0.04, 0.26, 0.32) \rangle$
p_2	$\langle (0.11, 0.06, 0.13), (0.13, 0.01, 0.26) \rangle$	$\langle (0.13, 0.06, 0.03), (0.13, 0.01, 0.39) \rangle$
p_3	$\langle (0.26, 0.31, 0.14), (0.13, 0.24, 0.09) \rangle$	$\langle (0.34, 0.02, 0.09), (0.13, 0.02, 0.01) \rangle$
p_4	$\langle (0.81, 0.01, 0.02), (0.13, 0.12, 0.36) \rangle$	$\langle (0.81, 0.01, 0.02), (0.11, 0.12, 0.36) \rangle$
$\Omega_1 \vee \Omega_2$	$(q_2, q_1), (\neg q_2, \neg q_1)$	$(q_2, q_3), (\neg q_2, \neg q_3)$
p_1	$\langle (0.11, 0.16, 0.02), (0.03, 0.11, 0.45) \rangle$	$\langle (0.16, 0.19, 0.02), (0.12, 0.16, 0.32) \rangle$
p_2	$\langle (0.12, 0.12, 0.13), (0.21, 0.12, 0.09) \rangle$	$\langle (0.13, 0.14, 0.03), (0.11, 0.12, 0.59) \rangle$
p_3	$\langle (0.23, 0.21, 0.14), (0.11, 0.22, 0.09) \rangle$	$\langle (0.34, 0.02, 0.09), (0.11, 0.02, 0.01) \rangle$
p_4	$\langle (0.11, 0.02, 0.11), (0.09, 0.12, 0.52) \rangle$	$\langle (0.06, 0.01, 0.03), (0.09, 0.12, 0.52) \rangle$

The following verifies the commutative, associative and distributive laws with respect to P_{RFBSS} AND and OR operations.

Proposition 1. Let $\mathcal{P}$ be a universal set, and let $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$, $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ and $\Omega_3 = (\mathcal{G}_3, \mathcal{H}_3, \mathcal{Q}_3, \neg \mathcal{Q}_3)$ be three P_{RFBSSs} on $\mathcal{P}$. Then:

1. $\Omega_1 \bar{\wedge} \Omega_2 = \Omega_2 \bar{\wedge} \Omega_1$;
2. $\Omega_1 \vee \Omega_2 = \Omega_2 \vee \Omega_1$;
3. $\Omega_1 \bar{\wedge} (\Omega_2 \bar{\wedge} \Omega_3) = (\Omega_1 \bar{\wedge} \Omega_2) \bar{\wedge} \Omega_3$;
4. $\Omega_1 \vee (\Omega_2 \vee \Omega_3) = (\Omega_1 \vee \Omega_2) \vee \Omega_3$;
5. $\Omega_1 \bar{\wedge} (\Omega_2 \vee \Omega_3) = (\Omega_1 \bar{\wedge} \Omega_2) \vee (\Omega_1 \bar{\wedge} \Omega_3)$;
6. $\Omega_1 \vee (\Omega_2 \bar{\wedge} \Omega_3) = (\Omega_1 \vee \Omega_2) \bar{\wedge} (\Omega_1 \vee \Omega_3)$.

Proof.

1. Suppose that

$$\Omega_1 \bar{\wedge} \Omega_2 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \bar{\wedge} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) = (\Lambda, \Gamma, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2).$$

Then, by Definition 10, $\forall (q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, we have:

$$\begin{aligned} \Lambda(q_i, q_j) &= (\min(\eta_{\mathcal{G}_1}^+(q_i), \eta_{\mathcal{G}_2}^+(q_j)), \min(\eta_{\mathcal{G}_1}^0(q_i), \eta_{\mathcal{G}_2}^0(q_j)), \max(\eta_{\mathcal{G}_1}^-(q_i), \eta_{\mathcal{G}_2}^-(q_j))) \\ &= (\min(\eta_{\mathcal{G}_2}^+(q_j), \eta_{\mathcal{G}_1}^+(q_i)), \min(\eta_{\mathcal{G}_2}^0(q_j), \eta_{\mathcal{G}_1}^0(q_i)), \max(\eta_{\mathcal{G}_2}^-(q_j), \eta_{\mathcal{G}_1}^-(q_i))) \\ &= \tilde{\Lambda}(q_j, q_i). \end{aligned}$$

Similarly, $\forall (\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$, we have:

$$\begin{aligned}\Gamma(\neg q_i, \neg q_j) &= (\max(\eta_{\mathcal{G}_1}^+(\neg q_i), \eta_{\mathcal{G}_2}^+(\neg q_j)), \min(\eta_{\mathcal{G}_1}^0(\neg q_i), \eta_{\mathcal{G}_2}^0(\neg q_j)), \min(\eta_{\mathcal{G}_1}^-(\neg q_i), \eta_{\mathcal{G}_2}^-(\neg q_j))) \\ &= (\max(\eta_{\mathcal{G}_2}^+(\neg q_j), \eta_{\mathcal{G}_1}^+(\neg q_i)), \min(\eta_{\mathcal{G}_2}^0(\neg q_j), \eta_{\mathcal{G}_1}^0(\neg q_i)), \min(\eta_{\mathcal{G}_2}^-(\neg q_j), \eta_{\mathcal{G}_1}^-(\neg q_i))) \\ &= \tilde{\Gamma}(\neg q_j, \neg q_i).\end{aligned}$$

Such that, $\forall (q_j, q_i) \in \mathcal{Q}_2 \times \mathcal{Q}_1$ and $(\neg q_j, \neg q_i) \in \neg \mathcal{Q}_2 \times \neg \mathcal{Q}_1$,

$$(\tilde{\Lambda}, \tilde{\Gamma}, \mathcal{Q}_2 \times \mathcal{Q}_1, \neg \mathcal{Q}_2 \times \neg \mathcal{Q}_1) = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) \bar{\wedge} (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) = \Omega_2 \bar{\wedge} \Omega_1.$$

Hence, $\Omega_1 \bar{\wedge} \Omega_2 = \Omega_2 \bar{\wedge} \Omega_1$.

2. Suppose that

$$\Omega_1 \vee \Omega_2 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \vee (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) = (\Psi, \Upsilon, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2).$$

Then, by Definition 11, $\forall (q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, we have:

$$\begin{aligned}\Psi(q_i, q_j) &= (\max(\eta_{\mathcal{G}_1}^+(q_i), \eta_{\mathcal{G}_2}^+(q_j)), \min(\eta_{\mathcal{G}_1}^0(q_i), \eta_{\mathcal{G}_2}^0(q_j)), \min(\eta_{\mathcal{G}_1}^-(q_i), \eta_{\mathcal{G}_2}^-(q_j))) \\ &= (\max(\eta_{\mathcal{G}_2}^+(q_j), \eta_{\mathcal{G}_1}^+(q_i)), \min(\eta_{\mathcal{G}_2}^0(q_j), \eta_{\mathcal{G}_1}^0(q_i)), \min(\eta_{\mathcal{G}_2}^-(q_j), \eta_{\mathcal{G}_1}^-(q_i))) \\ &= \tilde{\Psi}(q_j, q_i).\end{aligned}$$

Similarly, $\forall (\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$, we have:

$$\begin{aligned}\Upsilon(\neg q_i, \neg q_j) &= (\min(\eta_{\mathcal{G}_1}^+(\neg q_i), \eta_{\mathcal{G}_2}^+(\neg q_j)), \min(\eta_{\mathcal{G}_1}^0(\neg q_i), \eta_{\mathcal{G}_2}^0(\neg q_j)), \max(\eta_{\mathcal{G}_1}^-(\neg q_i), \eta_{\mathcal{G}_2}^-(\neg q_j))) \\ &= (\min(\eta_{\mathcal{G}_2}^+(\neg q_j), \eta_{\mathcal{G}_1}^+(\neg q_i)), \min(\eta_{\mathcal{G}_2}^0(\neg q_j), \eta_{\mathcal{G}_1}^0(\neg q_i)), \max(\eta_{\mathcal{G}_2}^-(\neg q_j), \eta_{\mathcal{G}_1}^-(\neg q_i))) \\ &= \tilde{\Upsilon}(\neg q_j, \neg q_i).\end{aligned}$$

Such that, $\forall (q_j, q_i) \in \mathcal{Q}_2 \times \mathcal{Q}_1$ and $(\neg q_j, \neg q_i) \in \neg \mathcal{Q}_2 \times \neg \mathcal{Q}_1$,

$$(\tilde{\Psi}, \tilde{\Upsilon}, \mathcal{Q}_2 \times \mathcal{Q}_1, \neg \mathcal{Q}_2 \times \neg \mathcal{Q}_1) = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) \vee (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) = \Omega_2 \vee \Omega_1.$$

Hence, $\Omega_1 \vee \Omega_2 = \Omega_2 \vee \Omega_1$.

The remaining parts (3–6) can be verified with similar arguments. $\square$

In the following, we verify the famous De Morgan's laws for P_{RFBSS} AND and OR operations.

Proposition 2. Let $\mathcal{P}$ be a universal set, and let $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ be two P_{RFBSS} s on $\mathcal{P}$. Then:

1. $(\Omega_1 \bar{\wedge} \Omega_2)^c = (\Omega_1)^c \vee (\Omega_2)^c$;
2. $(\Omega_1 \vee \Omega_2)^c = (\Omega_1)^c \bar{\wedge} (\Omega_2)^c$.

Proof.

1. From Definitions 8 and 10,

$$\begin{aligned}(\Omega_1 \bar{\wedge} \Omega_2)^c &= ((\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \bar{\wedge} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2))^c \\ &= (\Lambda, \Gamma, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2)^c \\ &= (\Lambda^c, \Gamma^c, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2).\end{aligned}$$

where $\forall (q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, we have

$$\begin{aligned}\Lambda^c(q_i, q_j) &= (\min(\eta_{\mathcal{G}_1}^+(q_i), \eta_{\mathcal{G}_2}^+(q_j)), \min(\eta_{\mathcal{G}_1}^0(q_i), \eta_{\mathcal{G}_2}^0(q_j)), \max(\eta_{\mathcal{G}_1}^-(q_i), \eta_{\mathcal{G}_2}^-(q_j)))^c \\ &= (\max(\eta_{\mathcal{G}_1}^-(q_i), \eta_{\mathcal{G}_2}^-(q_j)), \min(\eta_{\mathcal{G}_1}^0(q_i), \eta_{\mathcal{G}_2}^0(q_j)), \min(\eta_{\mathcal{G}_1}^+(q_i), \eta_{\mathcal{G}_2}^+(q_j))) \\ &= (\max(\eta_{\mathcal{G}_1^c}^+(q_i), \eta_{\mathcal{G}_2^c}^+(q_j)), \min(\eta_{\mathcal{G}_1^c}^0(q_i), \eta_{\mathcal{G}_2^c}^0(q_j)), \min(\eta_{\mathcal{G}_1^c}^-(q_i), \eta_{\mathcal{G}_2^c}^-(q_j))) \\ &= \bar{\Lambda}(q_i, q_j).\end{aligned}$$

Similarly, $\forall (\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$, we have:

$$\begin{aligned}\Gamma^c(\neg q_i, \neg q_j) &= (\max(\eta_{\mathcal{G}_1}^+(\neg q_i), \eta_{\mathcal{G}_2}^+(\neg q_j)), \min(\eta_{\mathcal{G}_1}^0(\neg q_i), \eta_{\mathcal{G}_2}^0(\neg q_j)), \min(\eta_{\mathcal{G}_1}^-(\neg q_i), \eta_{\mathcal{G}_2}^-(\neg q_j)))^c \\ &= (\min(\eta_{\mathcal{G}_1}^-(\neg q_i), \eta_{\mathcal{G}_2}^-(\neg q_j)), \min(\eta_{\mathcal{G}_1}^0(\neg q_i), \eta_{\mathcal{G}_2}^0(\neg q_j)), \max(\eta_{\mathcal{G}_1}^+(\neg q_i), \eta_{\mathcal{G}_2}^+(\neg q_j))) \\ &= (\min(\eta_{\mathcal{G}_1^c}^+(\neg q_i), \eta_{\mathcal{G}_2^c}^+(\neg q_j)), \min(\eta_{\mathcal{G}_1^c}^0(\neg q_i), \eta_{\mathcal{G}_2^c}^0(\neg q_j)), \max(\eta_{\mathcal{G}_1^c}^-(\neg q_i), \eta_{\mathcal{G}_2^c}^-(\neg q_j))) \\ &= \bar{\Gamma}(\neg q_i, \neg q_j).\end{aligned}$$

Such that, $\forall (q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ and $(\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$,

$$\begin{aligned}(\bar{\Lambda}, \bar{\Gamma}, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2) &= (\mathcal{G}_1^c, \mathcal{H}_1^c, \mathcal{Q}_1, \neg \mathcal{Q}_1) \vee (\mathcal{G}_2^c, \mathcal{H}_2^c, \mathcal{Q}_2, \neg \mathcal{Q}_2) \\ &= (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)^c \vee (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)^c \\ &= (\Omega_1)^c \vee (\Omega_2)^c.\end{aligned}$$

Hence, $(\Omega_1 \bar{\wedge} \Omega_2)^c = (\Omega_1)^c \vee (\Omega_2)^c$.

2. From Definitions 8 and 11,

$$\begin{aligned}(\Omega_1 \vee \Omega_2)^c &= ((\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \vee (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2))^c \\ &= (\Psi, Y, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2)^c \\ &= (\Psi^c, Y^c, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2).\end{aligned}$$

where $\forall (q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, we have:

$$\begin{aligned}\Psi^c(q_i, q_j) &= (\max(\eta_{\mathcal{G}_1}^+(q_i), \eta_{\mathcal{G}_2}^+(q_j)), \min(\eta_{\mathcal{G}_1}^0(q_i), \eta_{\mathcal{G}_2}^0(q_j)), \min(\eta_{\mathcal{G}_1}^-(q_i), \eta_{\mathcal{G}_2}^-(q_j)))^c \\ &= (\min(\eta_{\mathcal{G}_1}^-(q_i), \eta_{\mathcal{G}_2}^-(q_j)), \min(\eta_{\mathcal{G}_1}^0(q_i), \eta_{\mathcal{G}_2}^0(q_j)), \max(\eta_{\mathcal{G}_1}^+(q_i), \eta_{\mathcal{G}_2}^+(q_j))) \\ &= (\min(\eta_{\mathcal{G}_1^c}^+(q_i), \eta_{\mathcal{G}_2^c}^+(q_j)), \min(\eta_{\mathcal{G}_1^c}^0(q_i), \eta_{\mathcal{G}_2^c}^0(q_j)), \max(\eta_{\mathcal{G}_1^c}^-(q_i), \eta_{\mathcal{G}_2^c}^-(q_j))) \\ &= \bar{\Psi}(q_i, q_j).\end{aligned}$$

Similarly, $\forall (\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$, we have:

$$\begin{aligned}Y^c(\neg q_i, \neg q_j) &= (\min(\eta_{\mathcal{G}_1}^+(\neg q_i), \eta_{\mathcal{G}_2}^+(\neg q_j)), \min(\eta_{\mathcal{G}_1}^0(\neg q_i), \eta_{\mathcal{G}_2}^0(\neg q_j)), \max(\eta_{\mathcal{G}_1}^-(\neg q_i), \eta_{\mathcal{G}_2}^-(\neg q_j)))^c \\ &= (\max(\eta_{\mathcal{G}_1}^-(\neg q_i), \eta_{\mathcal{G}_2}^-(\neg q_j)), \min(\eta_{\mathcal{G}_1}^0(\neg q_i), \eta_{\mathcal{G}_2}^0(\neg q_j)), \min(\eta_{\mathcal{G}_1}^+(\neg q_i), \eta_{\mathcal{G}_2}^+(\neg q_j))) \\ &= (\max(\eta_{\mathcal{G}_1^c}^+(\neg q_i), \eta_{\mathcal{G}_2^c}^+(\neg q_j)), \min(\eta_{\mathcal{G}_1^c}^0(\neg q_i), \eta_{\mathcal{G}_2^c}^0(\neg q_j)), \min(\eta_{\mathcal{G}_1^c}^-(\neg q_i), \eta_{\mathcal{G}_2^c}^-(\neg q_j))) \\ &= \bar{Y}(\neg q_i, \neg q_j).\end{aligned}$$

Such that, $\forall (q_i, q_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ and $(\neg q_i, \neg q_j) \in \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2$,

$$\begin{aligned}(\Psi, \bar{Y}, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg \mathcal{Q}_1 \times \neg \mathcal{Q}_2) &= (\mathcal{G}_1^c, \mathcal{H}_1^c, \mathcal{Q}_1, \neg \mathcal{Q}_1) \bar{\wedge} (\mathcal{G}_2^c, \mathcal{H}_2^c, \mathcal{Q}_2, \neg \mathcal{Q}_2) \\ &= (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)^c \bar{\wedge} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)^c \\ &= (\Omega_1)^c \bar{\wedge} (\Omega_2)^c.\end{aligned}$$

Hence, $(\Omega_1 \vee \Omega_2)^c = (\Omega_1)^c \bar{\wedge} (\Omega_2)^c$.

□

The next four definitions give the notions of extended (restricted) union and intersection, respectively, along with illustrative numerical examples.

Definition 12. For any two PRFBSSs $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over a universal set $\mathcal{P}$, their extended union, represented by $\Omega_1 \uplus_{\mathcal{E}} \Omega_2$, is again a PRFBSS $((\mathcal{G}_1 \uplus \mathcal{G}_2), (\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2), \mathcal{Q}_1 \cup \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2)$ on $\mathcal{P}$, which is defined as follows:

$$(\mathcal{G}_1 \uplus \mathcal{G}_2)(q) = \begin{cases} \mathcal{G}_1(q), & \text{if } q \in \mathcal{Q}_1 - \mathcal{Q}_2, \\ \mathcal{G}_2(q), & \text{if } q \in \mathcal{Q}_2 - \mathcal{Q}_1, \\ \mathcal{G}_1(q) \cup \mathcal{G}_2(q), & \text{if } q \in \mathcal{Q}_1 \cap \mathcal{Q}_2. \end{cases}$$

$$(\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2)(\neg q) = \begin{cases} \mathcal{H}_1(\neg q), & \text{if } \neg q \in (\neg \mathcal{Q}_1) - (\neg \mathcal{Q}_2), \\ \mathcal{H}_2(\neg q), & \text{if } \neg q \in (\neg \mathcal{Q}_2) - (\neg \mathcal{Q}_1), \\ \mathcal{H}_1(\neg q) \cap \mathcal{H}_2(\neg q), & \text{if } \neg q \in (\neg \mathcal{Q}_1) \cap (\neg \mathcal{Q}_2). \end{cases}$$

where

$$\mathcal{G}_1(q) \cup \mathcal{G}_2(q) = \{ \langle p, \max(\eta_{\mathcal{G}_1}^+, \eta_{\mathcal{G}_2}^+)(q)(p), \min(\eta_{\mathcal{G}_1}^0, \eta_{\mathcal{G}_2}^0)(q)(p), \min(\eta_{\mathcal{G}_1}^-, \eta_{\mathcal{G}_2}^-)(q)(p) \rangle \mid p \in \mathcal{P}, q \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \},$$

$$\mathcal{H}_1(\neg q) \cap \mathcal{H}_2(\neg q) = \{ \langle p, \min(\zeta_{\mathcal{H}_1}^+, \zeta_{\mathcal{H}_2}^+)(\neg q)(p), \min(\zeta_{\mathcal{H}_1}^0, \zeta_{\mathcal{H}_2}^0)(\neg q)(p), \max(\zeta_{\mathcal{H}_1}^-, \zeta_{\mathcal{H}_2}^-)(\neg q)(p) \rangle \mid p \in \mathcal{P}, \neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2 \}.$$

Definition 13. For any two PRFBSSs $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over a universal set $\mathcal{P}$, their restricted union denoted by $\Omega_1 \uplus_{\mathcal{R}} \Omega_2$ is a PRFBSS $((\mathcal{G}_1 \uplus \mathcal{G}_2), (\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2), \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2)$ on $\mathcal{P}$, such that $\forall q \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \neq \emptyset$ and $\neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2 \neq \emptyset$,

$$(\mathcal{G}_1 \uplus \mathcal{G}_2)(q) = \mathcal{G}_1(q) \cup \mathcal{G}_2(q) \quad \text{and} \quad (\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2)(\neg q) = \mathcal{H}_1(\neg q) \cap \mathcal{H}_2(\neg q).$$

Proposition 3. Let $\mathcal{P}$ be a universal set and let $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$, $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ and $\Omega_3 = (\mathcal{G}_3, \mathcal{H}_3, \mathcal{Q}_3, \neg \mathcal{Q}_3)$ be three PRFBSSs on $\mathcal{P}$. Then:

1. $\Omega_1 \uplus_{\mathcal{E}} \Omega_2 = \Omega_2 \uplus_{\mathcal{E}} \Omega_1$;
2. $\Omega_1 \uplus_{\mathcal{R}} \Omega_2 = \Omega_1 \uplus_{\mathcal{R}} \Omega_1$;
3. $\Omega_1 \uplus_{\mathcal{E}} (\Omega_2 \uplus_{\mathcal{E}} \Omega_3) = (\Omega_1 \uplus_{\mathcal{E}} \Omega_2) \uplus_{\mathcal{E}} \Omega_3$;
4. $\Omega_1 \uplus_{\mathcal{R}} (\Omega_2 \uplus_{\mathcal{R}} \Omega_3) = (\Omega_1 \uplus_{\mathcal{R}} \Omega_2) \uplus_{\mathcal{R}} \Omega_3$.

Proof.

1. From Definition 12:

$$\begin{aligned} \Omega_1 \uplus_{\mathcal{E}} \Omega_2 &= (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \uplus_{\mathcal{E}} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) \\ &= (\mathcal{G}_1 \uplus \mathcal{G}_2, \mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2, \mathcal{Q}_1 \cup \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2). \end{aligned}$$

Such that $\forall q \in \mathcal{Q}_1 \cup \mathcal{Q}_2$:

$$\mathcal{G}_1 \uplus \mathcal{G}_2 = \begin{cases} \mathcal{G}_1 & \text{if } q \in \mathcal{Q}_1 - \mathcal{Q}_2, \\ \mathcal{G}_2 & \text{if } q \in \mathcal{Q}_2 - \mathcal{Q}_1, \\ \mathcal{G}_1 \cup \mathcal{G}_2 & \text{if } q \in \mathcal{Q}_1 \cap \mathcal{Q}_2. \end{cases} = \begin{cases} \mathcal{G}_2 & \text{if } q \in \mathcal{Q}_2 - \mathcal{Q}_1, \\ \mathcal{G}_1 & \text{if } q \in \mathcal{Q}_1 - \mathcal{Q}_2, \\ \mathcal{G}_2 \cup \mathcal{G}_1 & \text{if } q \in \mathcal{Q}_2 \cap \mathcal{Q}_1. \end{cases} = \mathcal{G}_2 \uplus \mathcal{G}_1.$$

Similarly, $\forall \neg q \in \neg \mathcal{Q}_1 \cup \neg \mathcal{Q}_2$,

$$\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2 = \begin{cases} \mathcal{H}_1 & \text{if } \neg q \in \neg \mathcal{Q}_1 - \neg \mathcal{Q}_2, \\ \mathcal{H}_2 & \text{if } \neg q \in \neg \mathcal{Q}_2 - \neg \mathcal{Q}_1, \\ \mathcal{H}_1 \cap \mathcal{H}_2 & \text{if } \neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2. \end{cases} = \begin{cases} \mathcal{H}_2 & \text{if } \neg q \in \neg \mathcal{Q}_2 - \neg \mathcal{Q}_1, \\ \mathcal{H}_1 & \text{if } \neg q \in \neg \mathcal{Q}_1 - \neg \mathcal{Q}_2, \\ \mathcal{H}_2 \cap \mathcal{H}_1 & \text{if } \neg q \in \neg \mathcal{Q}_2 \cap \neg \mathcal{Q}_1. \end{cases} = \mathcal{H}_2 \mathbin{\mathbb{M}} \mathcal{H}_1.$$

This implies that

$$(\mathcal{G}_1 \uplus \mathcal{G}_2, \mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2, \mathcal{Q}_1 \cup \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2) = (\mathcal{G}_2 \uplus \mathcal{G}_1, \mathcal{H}_2 \mathbin{\mathbb{M}} \mathcal{H}_1, \mathcal{Q}_2 \cup \mathcal{Q}_1, \neg \mathcal{Q}_2 \cap \neg \mathcal{Q}_1).$$

Hence, $\Omega_1 \uplus_{\mathcal{E}} \Omega_2 = \Omega_2 \uplus_{\mathcal{E}} \Omega_1$.

2. From Definition 13:

$$\begin{aligned}\Omega_1 \uplus_{\mathcal{R}} \Omega_2 &= (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \uplus_{\mathcal{R}} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) \\ &= (\mathcal{G}_1 \uplus \mathcal{G}_2, \mathcal{H}_1 \uplus \mathcal{H}_2, \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2).\end{aligned}$$

Such that $\forall q \in \mathcal{Q}_1 \cap \mathcal{Q}_2$,

$$\mathcal{G}_1 \uplus \mathcal{G}_2 = \mathcal{G}_1 \sqcup \mathcal{G}_2 = \mathcal{G}_2 \sqcup \mathcal{G}_1 = \mathcal{G}_2 \uplus \mathcal{G}_1,$$

and $\forall \neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2$,

$$\mathcal{H}_1 \uplus \mathcal{H}_2 = \mathcal{H}_1 \sqcup \mathcal{H}_2 = \mathcal{H}_2 \sqcup \mathcal{H}_1 = \mathcal{H}_2 \uplus \mathcal{H}_1.$$

This implies that

$$(\mathcal{G}_1 \uplus \mathcal{G}_2, \mathcal{H}_1 \uplus \mathcal{H}_2, \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2) = (\mathcal{G}_2 \uplus \mathcal{G}_1, \mathcal{H}_2 \uplus \mathcal{H}_1, \mathcal{Q}_2 \cap \mathcal{Q}_1, \neg \mathcal{Q}_2 \cap \neg \mathcal{Q}_1).$$

Hence, $\Omega_1 \uplus_{\mathcal{R}} \Omega_2 = \Omega_2 \uplus_{\mathcal{R}} \Omega_1$.

The remaining parts (3 and 4) can be proven similarly. $\square$

Definition 14. For any two PRFBSSs $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over a universal set $\mathcal{P}$, their extended intersection, represented by $\Omega_1 \cap_{\mathcal{E}} \Omega_2$, is again a PRFBSS $((\mathcal{G}_1 \cap \mathcal{G}_2), (\mathcal{H}_1 \cap \mathcal{H}_2), \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2)$ on $\mathcal{P}$, which is defined as:

$$\begin{aligned}(\mathcal{G}_1 \cap \mathcal{G}_2)(q) &= \begin{cases} \mathcal{G}_1(q), & \text{if } q \in \mathcal{Q}_1 - \mathcal{Q}_2, \\ \mathcal{G}_2(q), & \text{if } q \in \mathcal{Q}_2 - \mathcal{Q}_1, \\ \mathcal{G}_1(q) \cap \mathcal{G}_2(q), & \text{if } q \in \mathcal{Q}_1 \cap \mathcal{Q}_2. \end{cases} \\ (\mathcal{H}_1 \cap \mathcal{H}_2)(\neg q) &= \begin{cases} \mathcal{H}_1(\neg q), & \text{if } \neg q \in (\neg \mathcal{Q}_1) - (\neg \mathcal{Q}_2), \\ \mathcal{H}_2(\neg q), & \text{if } \neg q \in (\neg \mathcal{Q}_2) - (\neg \mathcal{Q}_1), \\ \mathcal{H}_1(\neg q) \cap \mathcal{H}_2(\neg q), & \text{if } \neg q \in (\neg \mathcal{Q}_1) \cap (\neg \mathcal{Q}_2). \end{cases}\end{aligned}$$

where

$$\begin{aligned}\mathcal{G}_1(q) \cap \mathcal{G}_2(q) &= \{ \langle p, \min(\eta_{\mathcal{G}_1}^+, \eta_{\mathcal{G}_2}^+)(q)(p), \min(\eta_{\mathcal{G}_1}^0, \eta_{\mathcal{G}_2}^0)(q)(p), \\ &\quad \max(\eta_{\mathcal{G}_1}^-, \eta_{\mathcal{G}_2}^-)(q)(p) \rangle \mid p \in \mathcal{P}, q \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \}, \\ \mathcal{H}_1(\neg q) \cap \mathcal{H}_2(\neg q) &= \{ \langle p, \max(\zeta_{\mathcal{H}_1}^+, \zeta_{\mathcal{H}_2}^+)(\neg q)(p), \min(\zeta_{\mathcal{H}_1}^0, \zeta_{\mathcal{H}_2}^0)(\neg q)(p), \\ &\quad \min(\zeta_{\mathcal{H}_1}^-, \zeta_{\mathcal{H}_2}^-)(\neg q)(p) \rangle \mid p \in \mathcal{P}, \neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2 \}.\end{aligned}$$

Definition 15. For any two PRFBSSs $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ over a universal set $\mathcal{P}$, their restricted intersection, denoted by $\Omega_1 \cap_{\mathcal{R}} \Omega_2$, is a PRFBSS $((\mathcal{G}_1 \cap \mathcal{G}_2), (\mathcal{H}_1 \cap \mathcal{H}_2), \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2)$ on $\mathcal{P}$, such that $\forall q \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \neq \emptyset$ and $\neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2 \neq \emptyset$,

$$(\mathcal{G}_1 \cap \mathcal{G}_2)(q) = \mathcal{G}_1(q) \cap \mathcal{G}_2(q) \quad \text{and} \quad (\mathcal{H}_1 \cap \mathcal{H}_2)(\neg q) = \mathcal{H}_1(\neg q) \cap \mathcal{H}_2(\neg q).$$

Example 5. Reconsider the PRFBSSs Ω_1 and Ω_2 in Example 4. Suppose the company wants to analyze the laptops again, both by taking the maximum scores from the two reports to shortlist the best laptops and by taking the minimum scores from the two reports to shortlist the worst ones. Consequently, the extended union $\Omega_1 \uplus_{\mathcal{E}} \Omega_2$ and the extended intersection $\Omega_1 \cap_{\mathcal{E}} \Omega_2$ are represented by Tables 11 and 12, respectively.

Table 11. Extended union between Ω_1 and Ω_2 .

$\Omega_1 \uplus_{\mathcal{E}} \Omega_2$	q_1	q_2	q_3
p_1	$\langle (0.02, 0.01, 0.19), (0.03, 0.11, 0.45) \rangle$	$\langle (0.11, 0.31, 0.02), (0.12, 0.16, 0.21) \rangle$	$\langle (0.16, 0.19, 0.23), (0.19, 0.29, 0.32) \rangle$
p_2	$\langle (0.11, 0.06, 0.13), (0.13, 0.01, 0.26) \rangle$	$\langle (0.12, 0.26, 0.31), (0.11, 0.12, 0.59) \rangle$	$\langle (0.13, 0.14, 0.03), (0.26, 0.16, 0.39) \rangle$
p_3	$\langle (0.26, 0.31, 0.14), (0.13, 0.24, 0.09) \rangle$	$\langle (0.13, 0.21, 0.61), (0.11, 0.22, 0.01) \rangle$	$\langle (0.34, 0.02, 0.09), (0.26, 0.02, 0.01) \rangle$
p_4	$\langle (0.81, 0.01, 0.02), (0.13, 0.12, 0.36) \rangle$	$\langle (0.06, 0.02, 0.11), (0.09, 0.16, 0.52) \rangle$	$\langle (0.04, 0.01, 0.03), (0.11, 0.12, 0.13) \rangle$

Table 12. Extended intersection between Ω_1 and Ω_2 .

$\Omega_1 \cap_{\mathcal{E}} \Omega_2$	q_1	q_2	q_3
p_1	$\langle (0.01, 0.01, 0.23), (0.04, 0.11, 0.31) \rangle$	$\langle (0.11, 0.31, 0.02), (0.12, 0.16, 0.21) \rangle$	$\langle (0.16, 0.19, 0.23), (0.19, 0.29, 0.32) \rangle$
p_2	$\langle (0.03, 0.06, 0.39), (0.21, 0.01, 0.09) \rangle$	$\langle (0.12, 0.26, 0.31), (0.11, 0.12, 0.59) \rangle$	$\langle (0.13, 0.14, 0.03), (0.26, 0.16, 0.39) \rangle$
p_3	$\langle (0.23, 0.31, 0.16), (0.13, 0.24, 0.01) \rangle$	$\langle (0.13, 0.21, 0.61), (0.11, 0.22, 0.01) \rangle$	$\langle (0.34, 0.02, 0.09), (0.26, 0.02, 0.01) \rangle$
p_4	$\langle (0.11, 0.01, 0.19), (0.22, 0.12, 0.16) \rangle$	$\langle (0.06, 0.02, 0.11), (0.09, 0.16, 0.52) \rangle$	$\langle (0.04, 0.01, 0.03), (0.11, 0.12, 0.13) \rangle$

These analyses indicate the properties of laptops by considering the maximum and minimum grades using the reports of both experts. However, the combined analyses come out to yield reports with same scores for the attributes not considered mutually. This further asks for more filtered analyses considering only the attributes considered by both of the experts. To deal with this, Tables 13 and 14 provide the restricted union $\Omega_1 \uplus_{\mathcal{R}} \Omega_2$ and restricted intersection $\Omega_1 \cap_{\mathcal{R}} \Omega_2$, respectively.

Table 13. Restricted union between Ω_1 and Ω_2 .

$\Omega_1 \uplus_{\mathcal{R}} \Omega_2$	q_1
p_1	$\langle (0.02, 0.01, 0.19), (0.03, 0.11, 0.45) \rangle$
p_2	$\langle (0.11, 0.06, 0.13), (0.13, 0.01, 0.26) \rangle$
p_3	$\langle (0.26, 0.31, 0.14), (0.13, 0.24, 0.09) \rangle$
p_4	$\langle (0.81, 0.01, 0.02), (0.13, 0.12, 0.36) \rangle$

Table 14. Restricted intersection between Ω_1 and Ω_2 .

$\Omega_1 \cap_{\mathcal{R}} \Omega_2$	q_1
p_1	$\langle (0.01, 0.01, 0.23), (0.04, 0.11, 0.31) \rangle$
p_2	$\langle (0.03, 0.06, 0.39), (0.21, 0.01, 0.09) \rangle$
p_3	$\langle (0.23, 0.31, 0.16), (0.13, 0.24, 0.01) \rangle$
p_4	$\langle (0.11, 0.01, 0.19), (0.22, 0.12, 0.16) \rangle$

The following shows the commutative and associative properties of extended and restricted $P_{\mathcal{R}}\text{FBSS}$ intersection.

Proposition 4. Let $\mathcal{P}$ be a universal set and let $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$, $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ and $\Omega_3 = (\mathcal{G}_3, \mathcal{H}_3, \mathcal{Q}_3, \neg \mathcal{Q}_3)$ be three $P_{\mathcal{R}}\text{FBSS}$ s on $\mathcal{P}$. Then:

1. $\Omega_1 \cap_{\mathcal{E}} \Omega_2 = \Omega_2 \cap_{\mathcal{E}} \Omega_1$;
2. $\Omega_1 \cap_{\mathcal{R}} \Omega_2 = \Omega_2 \cap_{\mathcal{R}} \Omega_1$;
3. $\Omega_1 \cap_{\mathcal{E}} (\Omega_2 \cap_{\mathcal{E}} \Omega_3) = (\Omega_1 \cap_{\mathcal{E}} \Omega_2) \cap_{\mathcal{E}} \Omega_3$;
4. $\Omega_1 \cap_{\mathcal{R}} (\Omega_2 \cap_{\mathcal{R}} \Omega_3) = (\Omega_1 \cap_{\mathcal{R}} \Omega_2) \cap_{\mathcal{R}} \Omega_3$.

Proof.

1. From Definition 14:

$$\begin{aligned}\Omega_1 \cap_{\mathcal{E}} \Omega_2 &= (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \cap_{\mathcal{E}} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) \\ &= (\mathcal{G}_1 \cap \mathcal{G}_2, \mathcal{H}_1 \cap \mathcal{H}_2, \mathcal{Q}_1 \cup \mathcal{Q}_2, \neg \mathcal{Q}_1 \cup \neg \mathcal{Q}_2).\end{aligned}$$

Such that $\forall q \in \mathcal{Q}_1 \cup \mathcal{Q}_2$,

$$\mathcal{G}_1 \mathbin{\mathbb{M}} \mathcal{G}_2 = \begin{cases} \mathcal{G}_1 & \text{if } q \in \mathcal{Q}_1 - \mathcal{Q}_2, \\ \mathcal{G}_2 & \text{if } q \in \mathcal{Q}_2 - \mathcal{Q}_1, \\ \mathcal{G}_1 \bar{\cap} \mathcal{G}_2 & \text{if } q \in \mathcal{Q}_1 \cap \mathcal{Q}_2. \end{cases} = \begin{cases} \mathcal{G}_2 & \text{if } q \in \mathcal{Q}_2 - \mathcal{Q}_1, \\ \mathcal{G}_1 & \text{if } q \in \mathcal{Q}_1 - \mathcal{Q}_2, \\ \mathcal{G}_2 \bar{\cap} \mathcal{G}_1 & \text{if } q \in \mathcal{Q}_2 \cap \mathcal{Q}_1. \end{cases} = \mathcal{G}_2 \mathbin{\mathbb{M}} \mathcal{G}_1.$$

Similarly, $\forall \neg q \in \neg \mathcal{Q}_1 \cup \neg \mathcal{Q}_2$,

$$\mathcal{H}_1 \mathbin{\mathbb{W}} \mathcal{H}_2 = \begin{cases} \mathcal{H}_1 & \text{if } \neg q \in \neg \mathcal{Q}_1 - \neg \mathcal{Q}_2, \\ \mathcal{H}_2 & \text{if } \neg q \in \neg \mathcal{Q}_2 - \neg \mathcal{Q}_1, \\ \mathcal{H}_1 \cup \mathcal{H}_2 & \text{if } \neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2. \end{cases} = \begin{cases} \mathcal{H}_2 & \text{if } \neg q \in \neg \mathcal{Q}_2 - \neg \mathcal{Q}_1, \\ \mathcal{H}_1 & \text{if } \neg q \in \neg \mathcal{Q}_1 - \neg \mathcal{Q}_2, \\ \mathcal{H}_2 \cup \mathcal{H}_1 & \text{if } \neg q \in \neg \mathcal{Q}_2 \cap \neg \mathcal{Q}_1. \end{cases} = \mathcal{H}_2 \mathbin{\mathbb{W}} \mathcal{H}_1.$$

This implies that

$$(\mathcal{G}_1 \mathbin{\mathbb{M}} \mathcal{G}_2, \mathcal{H}_1 \mathbin{\mathbb{W}} \mathcal{H}_2, \mathcal{Q}_1 \cup \mathcal{Q}_2, \neg \mathcal{Q}_1 \cup \neg \mathcal{Q}_2) = (\mathcal{G}_2 \mathbin{\mathbb{M}} \mathcal{G}_1, \mathcal{H}_2 \mathbin{\mathbb{W}} \mathcal{H}_1, \mathcal{Q}_2 \cup \mathcal{Q}_1, \neg \mathcal{Q}_2 \cup \neg \mathcal{Q}_1).$$

Hence, $\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{E}} \Omega_2 = \Omega_2 \mathbin{\mathbb{M}}_{\mathcal{E}} \Omega_1$.

2. From Definition 15:

$$\begin{aligned} \Omega_1 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_2 &= (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1) \mathbin{\mathbb{M}}_{\mathcal{R}} (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2) \\ &= (\mathcal{G}_1 \mathbin{\mathbb{M}} \mathcal{G}_2, \mathcal{H}_1 \mathbin{\mathbb{W}} \mathcal{H}_2, \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2). \end{aligned}$$

Such that $\forall q \in \mathcal{Q}_1 \cap \mathcal{Q}_2$,

$$\mathcal{G}_1 \mathbin{\mathbb{M}} \mathcal{G}_2 = \mathcal{G}_1 \bar{\cap} \mathcal{G}_2 = \mathcal{G}_2 \bar{\cap} \mathcal{G}_1 = \mathcal{G}_2 \mathbin{\mathbb{M}} \mathcal{G}_1,$$

and $\forall \neg q \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2$,

$$\mathcal{H}_1 \mathbin{\mathbb{W}} \mathcal{H}_2 = \mathcal{H}_1 \cup \mathcal{H}_2 = \mathcal{H}_2 \cup \mathcal{H}_1 = \mathcal{H}_2 \mathbin{\mathbb{W}} \mathcal{H}_1.$$

This implies that

$$(\mathcal{G}_1 \mathbin{\mathbb{M}} \mathcal{G}_2, \mathcal{H}_1 \mathbin{\mathbb{W}} \mathcal{H}_2, \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2) = (\mathcal{G}_2 \mathbin{\mathbb{M}} \mathcal{G}_1, \mathcal{H}_2 \mathbin{\mathbb{W}} \mathcal{H}_1, \mathcal{Q}_2 \cap \mathcal{Q}_1, \neg \mathcal{Q}_2 \cap \neg \mathcal{Q}_1).$$

Hence, $\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_2 = \Omega_2 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_1$.

The remaining parts (3 and 4) can be proven in the same way as the above parts. $\square$

Proposition 5. Let $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$, $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ and $\Omega_3 = (\mathcal{G}_3, \mathcal{H}_3, \mathcal{Q}_3, \neg \mathcal{Q}_3)$ be three $P_{\mathcal{R}}\text{FBSSs}$ on the universe $\mathcal{P}$. Then:

1. $\Omega_1 \mathbin{\mathbb{W}}_{\mathcal{E}} (\Omega_2 \mathbin{\mathbb{M}}_{\mathcal{E}} \Omega_3) = (\Omega_1 \mathbin{\mathbb{W}}_{\mathcal{E}} \Omega_2) \mathbin{\mathbb{M}}_{\mathcal{E}} (\Omega_1 \mathbin{\mathbb{W}}_{\mathcal{E}} \Omega_3);$
2. $\Omega_1 \mathbin{\mathbb{W}}_{\mathcal{R}} (\Omega_2 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_3) = (\Omega_1 \mathbin{\mathbb{W}}_{\mathcal{R}} \Omega_2) \mathbin{\mathbb{M}}_{\mathcal{R}} (\Omega_1 \mathbin{\mathbb{W}}_{\mathcal{R}} \Omega_3);$
3. $\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{E}} (\Omega_2 \mathbin{\mathbb{W}}_{\mathcal{E}} \Omega_3) = (\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{E}} \Omega_2) \mathbin{\mathbb{W}}_{\mathcal{E}} (\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{E}} \Omega_3);$
4. $\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{R}} (\Omega_2 \mathbin{\mathbb{W}}_{\mathcal{R}} \Omega_3) = (\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_2) \mathbin{\mathbb{W}}_{\mathcal{R}} (\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_3).$

Proof. The proof can be easily deduced from the arguments used in proofs of Propositions 3 and 4. $\square$

The following lemma discusses the concepts of minimal and maximal $P_{\mathcal{R}}\text{FBSSs}$.

Lemma 1. Let $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ be two $P_{\mathcal{R}}\text{FBSSs}$ on a universe $\mathcal{P}$. Then:

1. $\Omega_1 \mathbin{\mathbb{W}}_{\mathcal{E}} \Omega_2$ is the minimal $P_{\mathcal{R}}\text{FBSS}$ over $\mathcal{P}$ containing both Ω_1 and Ω_2 .
2. $\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_2$ is the maximal $P_{\mathcal{R}}\text{FBSS}$ over $\mathcal{P}$ that is contained in both Ω_1 and Ω_2 .

Proof. It is directly followed by Definitions 12 and 15. $\square$

Theorem 1. Let $\Omega_1 = (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg \mathcal{Q}_1)$ and $\Omega_2 = (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg \mathcal{Q}_2)$ be two P_R FBSSs on the universe $\mathcal{P}$. Then:

1. $(\Omega_1 \uplus_{\mathcal{E}} \Omega_2)^c = (\Omega_1)^c \mathbin{\mathbb{M}}_{\mathcal{E}} (\Omega_2)^c$;
2. $(\Omega_1 \uplus_{\mathcal{R}} \Omega_2)^c = (\Omega_1)^c \mathbin{\mathbb{M}}_{\mathcal{R}} (\Omega_2)^c$;
3. $(\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{E}} \Omega_2)^c = (\Omega_1)^c \uplus_{\mathcal{E}} (\Omega_2)^c$;
4. $(\Omega_1 \mathbin{\mathbb{M}}_{\mathcal{R}} \Omega_2)^c = (\Omega_1)^c \uplus_{\mathcal{R}} (\Omega_2)^c$.

Proof. 1. By Definitions 8 and 12, we obtain

$$(\Omega_1 \uplus_{\mathcal{E}} \Omega_2)^c = ((\mathcal{G}_1 \uplus \mathcal{G}_2)^c, (\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2)^c, \mathcal{Q}_1 \cup \mathcal{Q}_2, \neg \mathcal{Q}_1 \cup \neg \mathcal{Q}_2),$$

where:

$$(\mathcal{G}_1 \uplus \mathcal{G}_2)^c(\mathbf{q}) = \begin{cases} \mathcal{G}_1^c(\mathbf{q}), & \text{if } \mathbf{q} \in \mathcal{Q}_1 - \mathcal{Q}_2, \\ \mathcal{G}_2^c(\mathbf{q}), & \text{if } \mathbf{q} \in \mathcal{Q}_2 - \mathcal{Q}_1, \\ \mathcal{G}_1^c(\mathbf{q}) \cap \mathcal{G}_2^c(\mathbf{q}), & \text{if } \mathbf{q} \in \mathcal{Q}_1 \cap \mathcal{Q}_2. \end{cases}$$

$$= (\mathcal{G}_1^c \mathbin{\mathbb{M}} \mathcal{G}_2^c)(\mathbf{q}), \text{ by Definition 14, and}$$

$$(\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2)^c(\neg \mathbf{q}) = \begin{cases} \mathcal{H}_1^c(\neg \mathbf{q}), & \text{if } \neg \mathbf{q} \in (\neg \mathcal{Q}_1) - (\neg \mathcal{Q}_2), \\ \mathcal{H}_2^c(\neg \mathbf{q}), & \text{if } \neg \mathbf{q} \in (\neg \mathcal{Q}_2) - (\neg \mathcal{Q}_1), \\ \mathcal{H}_1^c(\neg \mathbf{q}) \cup \mathcal{H}_2^c(\neg \mathbf{q}), & \text{if } \neg \mathbf{q} \in (\neg \mathcal{Q}_1) \cap (\neg \mathcal{Q}_2). \end{cases}$$

$$= (\mathcal{H}_1^c \uplus \mathcal{H}_2^c)(\neg \mathbf{q}), \text{ by Definition 14.}$$

Hence, $(\Omega_1 \uplus_{\mathcal{E}} \Omega_2)^c = (\Omega_1)^c \mathbin{\mathbb{M}}_{\mathcal{E}} (\Omega_2)^c$.

2. By Definitions 8 and 13, we obtain:

$$(\Omega_1 \uplus_{\mathcal{R}} \Omega_2)^c = ((\mathcal{G}_1 \uplus \mathcal{G}_2)^c, (\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2)^c, \mathcal{Q}_1 \cap \mathcal{Q}_2, \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2),$$

where, for all $\mathbf{q} \in \mathcal{Q}_1 \cap \mathcal{Q}_2$,

$$(\mathcal{G}_1 \uplus \mathcal{G}_2)^c(\mathbf{q}) = \mathcal{G}_1^c(\mathbf{q}) \cap \mathcal{G}_2^c(\mathbf{q})$$

$$= (\mathcal{G}_1^c \mathbin{\mathbb{M}} \mathcal{G}_2^c)(\mathbf{q}), \text{ by Definition 15.}$$

Similarly, for all $\neg \mathbf{q} \in \neg \mathcal{Q}_1 \cap \neg \mathcal{Q}_2$,

$$(\mathcal{H}_1 \mathbin{\mathbb{M}} \mathcal{H}_2)^c(\neg \mathbf{q}) = \mathcal{H}_1^c(\neg \mathbf{q}) \cup \mathcal{H}_2^c(\neg \mathbf{q})$$

$$= (\mathcal{H}_1^c \uplus \mathcal{H}_2^c)(\neg \mathbf{q}), \text{ by Definition 15.}$$

Hence, $(\Omega_1 \uplus_{\mathcal{R}} \Omega_2)^c = (\Omega_1)^c \mathbin{\mathbb{M}}_{\mathcal{R}} (\Omega_2)^c$.

The remaining parts (3 and 4) can be easily followed using similar arguments. $\square$

4. Application to MCDM

This section first discusses an algorithm under the initiated P_R FBSSs, and then implements it on a daily-life MCDM problem to rank available alternatives subject to the governing criterion. The following Algorithm 1 is used to choose the best alternative or to find the rankings of alternatives when the estimations are given in the form of a P_R FBSS.

Algorithm 1: Selection of best alternative under P_RFBSS environment.

- (1) **Input:**
 - (i) a universe $\mathcal{P}$ having n alternatives,
 - (ii) a set $\mathcal{R}$ of decision-attributes,
 - (iii) a not set $\neg\mathcal{R}$ of decision-attributes opposite to those in $\mathcal{R}$,
 - (iv) a P_RFBSS $\Omega = (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg\mathcal{Q})$ in tabular form based on the decision-maker's opinions where $\mathcal{Q} \subseteq \mathcal{R}$, and $\neg\mathcal{Q} \subseteq \neg\mathcal{R}$.
- (2) Construct the score-value table using Definition 3 by finding the scores of all the PFNs ($\mathcal{F}_{ij}$) regarding favorable set of attributes $\mathcal{Q} \subseteq \mathcal{R}$, which are given in the associated PFSS $(\mathcal{G}, \mathcal{Q})$. Then compute the object-wise sum (i.e., $\sum_i s(\mathcal{F}_{ij})$) of score values in the last column of this table.
- (3) Construct the score-value-table using Definition 3 by finding the scores of all the PFNs ($\neg\mathcal{F}_{ij}$) regarding unfavorable set of attributes $\neg\mathcal{Q} \subseteq \neg\mathcal{R}$, which are given in the associated PFSS $(\mathcal{H}, \neg\mathcal{Q})$. Then compute the object-wise sum (i.e., $\sum_i s(\neg\mathcal{F}_{ij})$) of score values in the last column of this table.
- (4) Calculate the final scores for all the objects using formula $s_i = \sum_i s(\mathcal{F}_{ij}) - \sum_i s(\neg\mathcal{F}_{ij})$.
- (5) Find k for which $s_k = \max(s_i)$.

Output: The last step declares p_k as the most appropriate alternative. In case of multiple values of k , any of the alternative p_k with same score can be considered as an optimal alternative.

To better understand the processing of Algorithm 1, readers may refer to flowchart diagram (see Figure 2).

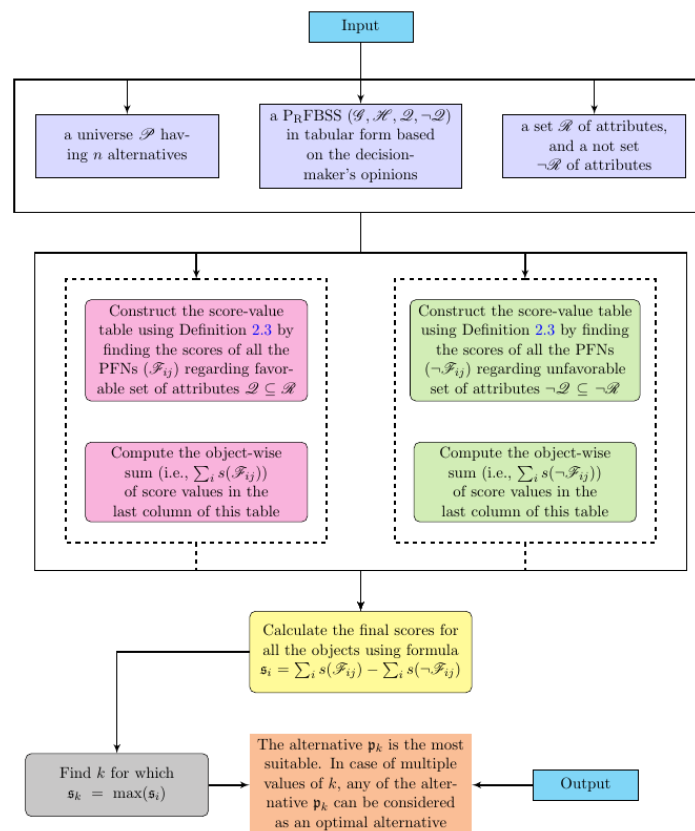


Figure 2. Graphical structure of Algorithm 1.

Application: Selection of Best Graphic Designer for a Studio

Graphic design is one of the largest industries due to its significance in marketing, advertisements, showcasing, modeling, movies, and so on. Nowadays, the graphic design industry needs to grow and glow in all disciplines ranging from arts to sciences to keep up with the ever-evolving digital tools and to stay competitive with the digital platforms in showcasing and implement marketing strategies, advertising, and designing. Graphic design is mainly based on the customer's needs and the graphic designer's particular tools to achieve the required goals. A graphic designer communicates ideas and information through visual concepts by turning their ideas into digital images and presentations. Using different graphic design software such as Adobe Photoshop, Adobe Illustrator, Corel Draw, Blender, Crello, etc., they create immersive visuals that inspire audiences. Graphic design being a huge field, which includes designing everything from posters to logos, packaging designs, brand identity designs, etc., graphic designers are employed all over the world in many places based on their experience, expertise and interests relevant to the specific areas they excel in. Today, the graphic designing market stands globally at a massive total of approximately USD 43.4 billion, with the industry growing at a rate of about 2.5% each year. The growth of the graphic design industry has been at 0.6% annually since 2017, but a massive growth of 3.7% is expected in 2022 (<https://www.ibisworld.com>) (accessed on 3 October 2022).

Among the many different uses of graphic designing, digital marketing and advertisement are the most important. Companies and businesses are switching over to digital marketing strategies because they are much more cost-efficient than traditional methods of advertising. Graphic design plays a key role in digital marketing because it is such a visual industry. Having a strong and appealing design for a company's website, social media pages and other marketing channels can help the company to connect with its audience and increase its brand loyalty. Digital marketing is all about reaching potential customers on their terms. Customers are constantly online and are looking for businesses that cater to their needs easily and conveniently. Graphic design is an essential aspect of digital marketing because it is one of the most visual forms of marketing. It uses images, visuals and other images to promote products or services and reach customers. The visual appeal of the designs in digital marketing campaigns is extremely important because it can make or break a brand. A business loses out on potential customers if the representative website and social media pages are cluttered, messy or unclear. However, if the website and social media pages are extremely clean, organized and visually appealing, one will be able to reach a wider audience and convert more customers. However, choosing the most appropriate graphic designer for the business is not a piece of cake. Graphic designers must be chosen that will perform for the recruiter in the best way.

Consider a multinational online marketing company that needs to fill in the vacancy of a graphic designer to help their clients reach their customers effectively. Suppose $\mathcal{P} = \{p_1, p_2, \dots, p_{15}\}$ is the set of fifteen candidates who applied for the vacancy. Now, the head of the graphic design studio of the company collects personalized demo tests from each candidate by giving them a model task to promote a certain product through a catchy video advertisement in a way that keeps the audience engaged and interested within prescribed time limits, followed by critical comments and discussion with the candidates about their creation. Based on these tests, the head of the studio decides to pick out the best candidate considering multiple criteria. Suppose $\mathcal{R} = \{q_1, q_2, \dots, q_7\}$ is the set of favorable criteria considered by the selector, where, for $q = 1, 2, \dots, 7$, the criterion q_j can serve as active listening, good communication skills, time management, improvement based on critical comments, patience, co-operation and effective storytelling. Accordingly, the considered not-set of parameters is $\neg\mathcal{R} = \{\neg q_1 = \text{impassivity}, \neg q_2 = \text{bad communication skills}, \neg q_3 = \text{disorganization}, \neg q_4 = \text{stubbornness}, \neg q_5 = \text{impatience}, \neg q_6 = \text{noncooperation}, \neg q_7 = \text{impatience}\}$. After a brief discussion among the members of the selection committee appointed by the studio, the head of the studio decides to evaluate each candidate under a favorable subset of parameters $\mathcal{Q} = \{q_1, q_2, q_3, q_5, q_7\}$ of $\mathcal{R}$.

According to the estimations of the committee members about the applicants, a P_{RFBSS} $(\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$ is constructed which describes the qualities of the candidates (see Table 15).

Table 15. Tabular form of the $P_{\text{RFBSS}} (\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$.

$(\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$	q_1	q_2	q_3
p_1	$\langle (0.01, 0.09, 0.32), (0.06, 0.19, 0.36) \rangle$	$\langle (0.92, 0.02, 0.01), (0.01, 0.09, 0.02) \rangle$	$\langle (0.16, 0.17, 0.19), (0.17, 0.18, 0.19) \rangle$
p_2	$\langle (0.21, 0.01, 0.11), (0.13, 0.81, 0.02) \rangle$	$\langle (0.16, 0.01, 0.02), (0.11, 0.01, 0.02) \rangle$	$\langle (0.39, 0.11, 0.26), (0.11, 0.01, 0.02) \rangle$
p_3	$\langle (0.11, 0.19, 0.66), (0.72, 0.11, 0.01) \rangle$	$\langle (0.16, 0.26, 0.16), (0.36, 0.19, 0.02) \rangle$	$\langle (0.11, 0.12, 0.33), (0.11, 0.23, 0.39) \rangle$
p_4	$\langle (0.13, 0.11, 0.12), (0.23, 0.16, 0.02) \rangle$	$\langle (0.27, 0.01, 0.09), (0.69, 0.01, 0.13) \rangle$	$\langle (0.12, 0.23, 0.34), (0.13, 0.23, 0.16) \rangle$
p_5	$\langle (0.34, 0.23, 0.21), (0.16, 0.01, 0.02) \rangle$	$\langle (0.16, 0.26, 0.31), (0.21, 0.23, 0.33) \rangle$	$\langle (0.16, 0.21, 0.23), (0.39, 0.01, 0.11) \rangle$
p_6	$\langle (0.39, 0.19, 0.06), (0.21, 0.22, 0.12) \rangle$	$\langle (0.34, 0.11, 0.01), (0.12, 0.11, 0.19) \rangle$	$\langle (0.09, 0.02, 0.06), (0.09, 0.13, 0.26) \rangle$
p_7	$\langle (0.13, 0.19, 0.26), (0.19, 0.01, 0.23) \rangle$	$\langle (0.26, 0.13, 0.21), (0.12, 0.33, 0.15) \rangle$	$\langle (0.16, 0.13, 0.45), (0.17, 0.01, 0.16) \rangle$
p_8	$\langle (0.19, 0.29, 0.30), (0.12, 0.07, 0.08) \rangle$	$\langle (0.11, 0.12, 0.33), (0.23, 0.11, 0.09) \rangle$	$\langle (0.62, 0.13, 0.11), (0.02, 0.31, 0.33) \rangle$
p_9	$\langle (0.11, 0.22, 0.33), (0.33, 0.22, 0.11) \rangle$	$\langle (0.19, 0.23, 0.16), (0.11, 0.21, 0.13) \rangle$	$\langle (0.16, 0.23, 0.17), (0.11, 0.13, 0.33) \rangle$
p_{10}	$\langle (0.16, 0.23, 0.31), (0.16, 0.34, 0.21) \rangle$	$\langle (0.16, 0.23, 0.02), (0.20, 0.21, 0.31) \rangle$	$\langle (0.16, 0.20, 0.01), (0.01, 0.16, 0.02) \rangle$
p_{11}	$\langle (0.23, 0.31, 0.11), (0.23, 0.16, 0.19) \rangle$	$\langle (0.56, 0.16, 0.16), (0.16, 0.32, 0.41) \rangle$	$\langle (0.11, 0.01, 0.01), (0.01, 0.13, 0.39) \rangle$
p_{12}	$\langle (0.16, 0.02, 0.21), (0.11, 0.22, 0.31) \rangle$	$\langle (0.11, 0.16, 0.23), (0.16, 0.31, 0.22) \rangle$	$\langle (0.11, 0.21, 0.30), (0.11, 0.21, 0.31) \rangle$
p_{13}	$\langle (0.16, 0.11, 0.23), (0.13, 0.11, 0.12) \rangle$	$\langle (0.16, 0.16, 0.02), (0.13, 0.71, 0.02) \rangle$	$\langle (0.19, 0.03, 0.13), (0.36, 0.19, 0.11) \rangle$
p_{14}	$\langle (0.01, 0.06, 0.70), (0.16, 0.23, 0.21) \rangle$	$\langle (0.16, 0.01, 0.11), (0.12, 0.01, 0.30) \rangle$	$\langle (0.11, 0.20, 0.30), (0.12, 0.30, 0.24) \rangle$
p_{15}	$\langle (0.13, 0.23, 0.11), (0.21, 0.09, 0.06) \rangle$	$\langle (0.59, 0.23, 0.11), (0.02, 0.19, 0.59) \rangle$	$\langle (0.12, 0.13, 0.16), (0.13, 0.12, 0.01) \rangle$
$(\mathcal{G}, \mathcal{H}, \mathcal{Q}, \neg \mathcal{Q})$	q_5	q_7	
p_1	$\langle (0.01, 0.03, 0.11), (0.02, 0.13, 0.06) \rangle$	$\langle (0.01, 0.02, 0.03), (0.07, 0.69, 0.01) \rangle$	
p_2	$\langle (0.32, 0.21, 0.01), (0.02, 0.01, 0.02) \rangle$	$\langle (0.02, 0.03, 0.01), (0.02, 0.11, 0.16) \rangle$	
p_3	$\langle (0.56, 0.01, 0.19), (0.11, 0.13, 0.19) \rangle$	$\langle (0.16, 0.16, 0.13), (0.14, 0.19, 0.23) \rangle$	
p_4	$\langle (0.26, 0.31, 0.01), (0.21, 0.41, 0.01) \rangle$	$\langle (0.03, 0.16, 0.19), (0.17, 0.11, 0.21) \rangle$	
p_5	$\langle (0.13, 0.23, 0.23), (0.13, 0.01, 0.01) \rangle$	$\langle (0.21, 0.31, 0.02), (0.69, 0.01, 0.02) \rangle$	
p_6	$\langle (0.11, 0.39, 0.03), (0.01, 0.02, 0.96) \rangle$	$\langle (0.01, 0.79, 0.01), (0.01, 0.01, 0.02) \rangle$	
p_7	$\langle (0.86, 0.01, 0.01), (0.13, 0.73, 0.01) \rangle$	$\langle (0.06, 0.09, 0.03), (0.08, 0.07, 0.08) \rangle$	
p_8	$\langle (0.26, 0.02, 0.03), (0.19, 0.29, 0.32) \rangle$	$\langle (0.81, 0.01, 0.02), (0.73, 0.01, 0.01) \rangle$	
p_9	$\langle (0.69, 0.01, 0.02), (0.03, 0.01, 0.01) \rangle$	$\langle (0.01, 0.02, 0.04), (0.06, 0.09, 0.23) \rangle$	
p_{10}	$\langle (0.01, 0.01, 0.02), (0.39, 0.46, 0.02) \rangle$	$\langle (0.11, 0.13, 0.19), (0.23, 0.11, 0.16) \rangle$	
p_{11}	$\langle (0.11, 0.23, 0.26), (0.13, 0.12, 0.11) \rangle$	$\langle (0.26, 0.13, 0.16), (0.91, 0.02, 0.01) \rangle$	
p_{12}	$\langle (0.23, 0.11, 0.01), (0.01, 0.02, 0.07) \rangle$	$\langle (0.11, 0.12, 0.13), (0.13, 0.16, 0.23) \rangle$	
p_{13}	$\langle (0.11, 0.13, 0.23), (0.16, 0.19, 0.16) \rangle$	$\langle (0.26, 0.31, 0.32), (0.21, 0.22, 0.23) \rangle$	
p_{14}	$\langle (0.23, 0.45, 0.01), (0.01, 0.02, 0.06) \rangle$	$\langle (0.41, 0.23, 0.33), (0.56, 0.11, 0.23) \rangle$	
p_{15}	$\langle (0.11, 0.12, 0.31), (0.13, 0.23, 0.27) \rangle$	$\langle (0.11, 0.23, 0.11), (0.69, 0.01, 0.02) \rangle$	

The committee decides to use Algorithm 1 to find the best designer based on the P_{RFBSS} report. Using Definition 3, the score values of all P_{RFNs} in both P_{RFSSs} $(\mathcal{G}, \mathcal{Q})$ and $(\mathcal{H}, \neg \mathcal{Q})$ are computed and then shown in Tables 16 and 17, respectively.

Table 16. Score-values table for $P_{\text{RFSS}} (\mathcal{G}, \mathcal{Q})$.

$(\mathcal{G}, \mathcal{Q})$	q_1	q_2	q_3	q_5	q_7	$\sum s(\mathcal{F})$
p_1	0.3275	1.4100	0.5225	0.4475	0.4900	3.1975
p_2	0.6525	0.6475	0.7325	0.7625	0.5075	3.3025
p_3	0.2325	0.5150	0.4150	0.9625	0.5550	2.6800
p_4	0.5425	0.7225	0.3925	0.6775	0.3950	2.7300
p_5	0.6775	0.4400	0.4425	0.4575	0.6225	2.6400
p_6	0.8125	0.8075	0.5550	0.4975	0.3075	2.9800
p_7	0.4525	0.6225	0.4025	1.3525	0.5225	3.3525
p_8	0.4675	0.4150	1.0325	0.7400	1.2975	3.9525
p_9	0.3900	0.5525	0.5175	1.1775	0.4850	3.1225
p_{10}	0.4475	0.5925	0.6050	0.4975	0.4825	2.6250
p_{11}	0.5975	0.9400	0.6025	0.4225	0.6475	3.2100
p_{12}	0.5500	0.4550	0.4075	0.6975	0.5150	2.6250
p_{13}	0.5175	0.6100	0.6175	0.4625	0.5225	2.7300
p_{14}	0.1450	0.6025	0.4100	0.6125	0.6875	2.4575
p_{15}	0.5175	0.9775	0.5075	0.4250	0.4975	2.9250

Table 17. Score-values table for $P_{RFSS}(\mathcal{H}, \neg \mathcal{Q})$.

$(\mathcal{H}, \neg \mathcal{Q})$	$\neg q_1$	$\neg q_2$	$\neg q_3$	$\neg q_5$	$\neg q_7$	$\sum s(\neg \mathcal{F})$
p ₁	0.3325	0.4775	0.5300	0.4575	0.3925	2.1900
p ₂	0.4175	0.5975	0.5975	0.5075	0.4625	2.5825
p ₃	1.1875	0.8025	0.3575	0.4825	0.4775	3.3075
p ₄	0.6800	1.1225	0.4925	0.6025	0.5375	3.4350
p ₅	0.6475	0.4875	0.8325	0.6225	1.1775	3.7675
p ₆	0.5950	0.4975	0.4275	0.0250	0.4975	2.0425
p ₇	0.5725	0.4625	0.5875	0.4425	0.5225	2.5875
p ₈	0.5625	0.6575	0.2775	0.4575	1.2000	3.1550
p ₉	0.7200	0.4925	0.4125	0.5225	0.4225	2.5695
p ₁₀	0.4700	0.4925	0.4600	0.7650	0.6225	2.8100
p ₁₁	0.5950	0.3750	0.2825	0.4950	1.4000	3.1475
p ₁₂	0.4000	0.4725	0.4025	0.4700	0.4750	2.2200
p ₁₃	0.5425	0.4425	0.7575	0.5325	0.5400	2.8150
p ₁₄	0.4475	0.4675	0.4250	0.4750	0.9175	2.7325
p ₁₅	0.6575	0.1775	0.5950	0.4375	0.1775	2.0450

Finally, Table 18 provides the final scores for each candidate.

Table 18. Final scores table.

$\mathcal{P}$	$\sum s(\mathcal{F})$	$\sum s(\neg \mathcal{F})$	$s = \sum s(\mathcal{F}) - \sum s(\neg \mathcal{F})$
p ₁	3.1975	2.1900	1.0075
p ₂	3.3025	2.5825	0.7200
p ₃	2.6800	3.3075	−0.6275
p ₄	2.7300	3.4350	−0.7050
p ₅	2.6400	3.7675	−1.1275
p ₆	2.9800	2.0425	0.9375
p ₇	3.3525	2.5875	0.7650
p ₈	3.9525	3.1550	0.7975
p ₉	3.1225	2.5695	0.5530
p ₁₀	2.6250	2.8100	−0.1850
p ₁₁	3.2100	3.1475	0.0625
p ₁₂	2.6250	2.2200	0.4050
p ₁₃	2.7300	2.8150	−0.0850
p ₁₄	2.4575	2.7325	−0.2750
p ₁₅	2.9250	2.0450	0.8800

Clearly, the candidate p₁ aced the selection procedure, scoring higher than all other candidates. Therefore, the head of the studio decides to hire candidate p₁ for the post of graphic designer.

5. Comparison

In this section, we discuss the comparative analysis of the initiated model with some previous models, qualitatively and quantitatively. Furthermore, this section highlights the overall advantages and limitations of the proposed model.

5.1. Advantages

The previously existing models, namely, P_{RFSS} s, B_{RSS} s and P_{RFSS} s, prove to be very powerful in their existing domains; however, they all have limitations. For instance, models such as P_{RFSS} s are unable to differentiate between the symmetrically opposing sets of decision attributes or criteria (bipolarity of decision attributes), whereas B_{RSS} s fails to deal with complicated uncertainties requiring the consideration of degrees of satisfaction, dissatisfaction and neutrality. The initiated P_{RFBSS} s as a generalization of these structures allows the consideration of positive, negative and neutral membership degrees while differentiating the decision attributes with respect to their bipolarities. In this

way, the initiated model not only overcomes the limitations in the previous tools but also offers modeling and handling of decision-making problems from a relatively larger domain as compared to its predecessors. In short, P_{RFBS} s are stronger, more efficient, more applicable and more generalizable than the previous models, including P_{RFS} s, B_{RFS} s, fuzzy B_{RSS} s and P_{RFS} s.

5.2. Comparison

The developed hybrid model (P_{RFBS} s) is the combination of two efficient models, namely, P_{RFS} s [20] and B_{RSS} s [51]. The reason for this development comes from the limitations of these models, i.e., P_{RFS} s cannot handle the bipolarity of decision attributes, whereas B_{RSS} s cannot handle uncertainties efficiently. Being a generalization of these models, the proposed P_{RFBS} model overcomes these limitations efficiently. To show the supremacy of P_{RFBS} s, the problem discussed in the previous section is again solved using fuzzy B_{RSS} s [52] and P_{RFS} s [28]. The comparison of the final scores and the final ranking given by these three models is given in Tables 19 and 20, respectively. From this comparison, it is clear that fuzzy B_{RSS} s failed to make a distinction between p_1 and p_7 . This is because of the information lost in the form of negative and neutral membership degrees. Similarly, P_{RFS} s failed to differentiate between the ranks of p_4 , p_{13} and p_{10} , p_{12} . Furthermore, the optimal decision offered by P_{RFS} s does not comply with the decisions offered by the other two models. The reason for this variation in results is the inability of P_{RFS} s to consider the bipolarity of decision attributes. On the other hand, the proposed P_{RFBS} s successfully provides a ranking of each alternative at a different rank from the other alternatives by considering the maximum conditions. In a similar way, models such as B_{RSS} s, P_{RFS} s, intuitionistic fuzzy soft sets, etc., will fail to provide a solution as efficient as P_{RFBS} s. This proves the efficiency of the proposed model over the previous models. For a pictorial representation of the discussed comparison, readers may refer to Figure 3. In addition, Table 21 presents a comparative qualitative analysis between the applicability and properties of the proposed model and previous models.

Table 19. Quantitative comparison of the developed model with different existing models.

Models	Fuzzy B_{RSS} s [52]	P_{RFS} s [28]	Proposed P_{RFBS} s
p_1	0.78	3.1975	1.0075
p_2	0.71	3.3025	0.7200
p_3	−0.34	2.6800	−0.6275
p_4	−0.65	2.7300	−0.7050
p_5	−0.58	2.6400	−1.1275
p_6	0.50	2.9800	0.9375
p_7	0.78	3.3525	0.7650
p_8	0.70	3.9525	0.7975
p_9	0.52	3.1225	0.5530
p_{10}	−0.39	2.6250	−0.1850
p_{11}	−0.17	3.2100	0.0625
p_{12}	0.20	2.6250	0.4050
p_{13}	−0.11	2.7300	−0.0850
p_{14}	−0.05	2.4575	−0.2750
p_{15}	−0.12	2.9250	0.8800

Table 20. Comparison among the ranking of the presented model with different existing models.

Hybrid Models	Rankings
Fuzzy B_{RSS} s [52]	$p_1 = p_7 > p_2 > p_8 > p_9 > p_6 > p_{12} > p_{14} > p_{13} > p_{15} > p_{11} > p_3 > p_{10} > p_4 > p_5$
P_{RFS} s [28]	$p_8 > p_7 > p_2 > p_{11} > p_1 > p_9 > p_6 > p_{15} > p_4 = p_{13} > p_3 > p_5 > p_{10} = p_{12} > p_{14}$
Proposed P_{RFBS} s	$p_1 > p_6 > p_{15} > p_8 > p_7 > p_2 > p_9 > p_{12} > p_{11} > p_{13} > p_{10} > p_{14} > p_3 > p_4 > p_5$

Table 21. Qualitative comparison between decision models.

Decision Model	Advantages	Limitations
Fuzzy set [1]	Handles uncertainties in data sets depicting partial truths by using fuzzy memberships ranging from 0 to 1.	Does not consider a membership degree for disagreement.
Intuitionistic fuzzy set [5]	Uses fuzzy membership and non-membership degrees with their mutual sum bounded by unity.	Does not consider the impartiality or neutral opinion.
Picture fuzzy set [18–20]	Uses positive, neutral and negative membership degrees with their mutual sum bounded by unity.	Fails to handle information affected by multiple decision attributes or parameters.
Soft set [11]	Allows decision making with multiple decision attributes by considering parameterized families of sets together.	Cannot handle uncertain problems dealing with partial truths in the data set.
Fuzzy soft set [14]	Allows decision making with multiple decision attributes by considering parameterized families of fuzzy sets.	Fails to depict a measure of disagreement in the parameterized data set.
Intuitionistic Fuzzy soft set [15]	Deals with intuitionistic fuzzy information in soft environments.	Does not consider the neutral membership degrees.
q -Rung orthopair fuzzy soft set [16]	Extends the membership and non-membership degrees by generalizing the rank of IFS memberships.	Does not consider the neutrality of the information.
Interval-valued q -rung orthopair fuzzy soft set [17]	Increases flexibility by using fuzzy intervals instead of discrete fuzzy memberships in q -rung orthopair fuzzy information.	Fails to consider any measures of neutral membership concerning the information.
Picture fuzzy soft set [28]	Handles picture fuzzy information under the effect of multiple decision attributes.	Cannot depict the bipolarity of decision attributes.
q -Rung picture fuzzy set, q -Rung picture linguistic numbers [46]	Generalizes the sum-restriction condition for picture fuzzy sets by combining with q -rung orthopair fuzzy sets, and uses q -rung picture linguistic Heronian mean operators for decision making.	Fails to differentiate between the bipolar sets of decision attributes.
Picture fuzzy N -soft set [45]	Handles multinary data in picture fuzzy soft environment.	Cannot depict the bipolarity of decision attributes.
Complex picture fuzzy N -soft set [47]	Uses complex picture fuzzy numbers to interpret multinary data in N -soft environment.	Fails to depict bipolarity of decision attributes.
Bipolar soft set [51]	Handles the bipolarity of decision attributes by considering two symmetrically opposite attribute sets.	Fails to handle uncertain information in the data set.
Fuzzy bipolar soft set [52]	Deals with bipolar soft information in fuzzy environment.	Fails to consider non-membership degrees in bipolar soft environment.
q -Rung orthopair fuzzy bipolar soft set [55]	Deals with q -rung orthopair fuzzy information in bipolar soft environment.	Cannot give a measure for neutrality in the decision makers opinions.
Proposed Picture fuzzy bipolar soft set	Allows the handling of bipolar soft information in picture fuzzy environment.	Free from all the limitations discussed above.

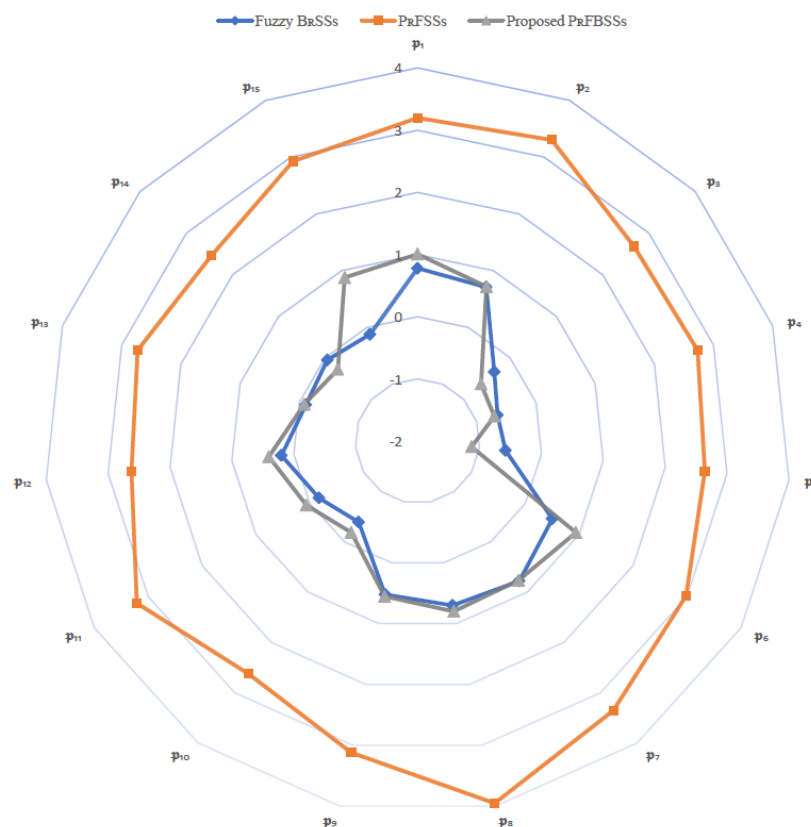


Figure 3. Comparison among the initiated P_R FBSS model and existing Fuzzy BSS [52] and P_R FSS [28] models.

5.3. Limitations

Like the gaps discussed in the previous literature, our proposed model also has some limitations which need to be addressed in the future. One of these limitations comes from the membership sum constraint of the P_R FSSs, which restricts the proposed model to work in situations where the sum of positive, negative and neutral membership degrees exceeds unity. Another problem is the limitation of the proposed model and algorithm to a single expert. This makes the model less efficient for group decision-making scenarios.

6. Conclusions and Future Directions

Many complicated decision problems in society are affected by several criteria. For instance, job recruitment for various designations need an effective investigation of multiple attributes of the candidate. Consequently, decision-making problems often require consideration of several criteria affecting the decisions in one place. As a helping hand, multi-criteria decision-making (MCDM) tools offer efficient solutions while considering multiple decision criteria affecting the decision-making process. Among these tools, picture fuzzy soft sets (P_R FSSs) allow solutions to complicated uncertain situations considering the degrees of agreement, disagreement and neutrality in a picture fuzzy soft environment. However, P_R FSSs cannot handle the bipolarity of criteria and are ineffective in problems concerning bipolar criteria. On the other hand, bipolar soft sets (BrSSs) can easily deal with bipolarity of criterion. However, unlike P_R FSSs, BrSSs fail to consider uncertainties in the information efficiently. To fill these gaps, we have presented a new hybrid MCDM model called picture fuzzy bipolar soft sets (P_R FBSSs) by the fusion of P_R FSSs and BrSSs. The proposed model is capable of handling picture fuzzy information under two symmetrically opposing sets of decision attributes. Moreover, we have investigated some novel properties of the proposed P_R FBSS model such as sub-set, super-set, equality, complement, relative null and

absolute P_R FBSSs, extended union and intersection and restricted union and intersection. Furthermore, two fundamental operations of P_R FBSSs, namely, AND and OR operations are studied. The commutative, associative and distributive properties along with De Morgan's laws about these initiated notions and operations are presented and verified. An application regarding the recruitment of a graphic designer for a studio is discussed under the novel P_R FBSSs. The best designer is selected using the proposed algorithm to show the effective applicability of the initiated model in different real-life complex situations involving uncertainties in P_R FBSS environments. Finally, concerning both qualitative and quantitative perspectives, we have studied a detailed comparative analysis of the proposed model with certain existing models.

One limitation of the model is its inability to discuss problems where the sum of positive, neutral and negative membership degrees exceeds unity. Another limitation is the restriction of decision making to a single expert's opinions. For future works, one can extend our initiated model to

- Rough picture fuzzy bipolar soft sets;
- Picture fuzzy bipolar soft expert sets;
- Picture fuzzy bipolar soft aggregation operators;
- q -Rung picture fuzzy bipolar soft sets;
- Interval-valued picture fuzzy bipolar soft sets;
- Rough picture fuzzy bipolar soft expert sets.

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