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Concepts of Picture Fuzzy Line Graphs and Their Applications in Data Analysis

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Abstract: The process of bundling and clustering has no clear boundaries; hence, their analysis contains uncertainties. Thus, it is more suitable to deal with bundling and clustering by using fuzzy graphs. Since picture fuzzy sets (PFSs) are more accurate, compatible, and flexible compared to the other generalizations of fuzzy sets (FSs), hence, it would be more effective to present edge bundling and clustering using picture fuzzy line graphs (PFLGs). The aim of our study is to introduce the notions of picture fuzzy intersection graphs (PFIGs) and picture fuzzy line graphs (PFLGs). These concepts are the generalizations of fuzzy intersection graphs (FIGs) and fuzzy line graphs (FLGs), respectively. We begin our discussion by introducing some fresh and useful terminologies in the theory of fuzzy graphs such as fuzzy intersection number, picture fuzzy intersection number, etc., and we explore few interesting results related to them. Based on these concepts, first we introduce the notion of picture fuzzy intersection graphs (PFIGs) and discuss many important characteristics of these graphs. Afterwards, we introduce the notion of picture fuzzy line graphs (PFLGs) and discuss their various properties. We also investigate some structural properties of our newly established fuzzy graphs using weak isomorphism and isomorphism. Finally, we provide an outline of the applications of picture fuzzy line graphs (PFLGs) in terms of cluster-based picture fuzzy edge bundling (CBPFEB) and the picture fuzzy c-mean algorithm. Since asymmetrical clusters ensure that the databases remain identical across the clusters, our study is deeply related to symmetry.

Keywords: PFIGs; PFLGs; picture fuzzy intersection number; clustering and edge bundling based on PFLGs



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1. Introduction

Fuzzy sets (FSs) were introduced by Zadeh in [1], and it became a useful tool by which we can model problems involving uncertainties. Many applications of FSs have been explored in the literature. The concept of FSs is more flexible compared to those of crisp sets; hence, it has been applied in different fields of science. Recently, new fuzzy fractional integral operators have been described in [2]. Similarly, based on FSs, many new concepts of fuzzy control were explored in [3,4]. Many generalizations of FSs were also explored in the literature. The very first generalization of FSs, termed interval-valued fuzzy sets (IVFSs), was also initiated by Zadeh [5]. Thereafter, the term intuitionistic fuzzy set (IFS) was introduced by Atanassov in [6]. He introduced the term degree of non-membership in the classical fuzzy set theory. Basically, FS is concerned only with the membership function of any element of a set X . Additionally, in classical FSs, the non-membership degree is equal to the complement of the membership degree. However, IFS has the degrees of both membership and non-membership, which are not dependent on each other, and the sum of these degrees cannot be greater than 1. IFS has been extensively used in many fields including computer science, engineering, medicine, etc. [7]. IFS is comparatively more

efficient than the typical fuzzy sets for handling uncertainty because it has an extra margin, i.e., the “hesitation margin”. However, it has been observed that in IFs, the concept of the “degree of neutrality” cannot be considered. However, the neutrality degree has much importance in many real-life circumstances such as democratic election stations and so forth. Human beings normally tend to provide their judgments in the following types: yes, no, abstain, and refusal. In these types of situations, if we apply intuitionistic fuzzy sets theory, then the details of voting for non-candidates (refusal) may be not considered. For handling such scenarios, Cuong [8] initiated the concept of picture fuzzy sets (PFSs), which are the generalization of both the terms FSs and IFs. In PFSs, we have an extra degree of neutrality for any object under consideration. Recently, Khan et al. [9] introduced several relations on bipolar picture fuzzy sets (BPPFSs) and presented their applications.

Graph theory is a branch of combinatorics that is widely studied. Graph theory has become a useful tool by which we can solve many problems in different fields of mathematics including geometry, algebra, topology, operations research, etc. On the other hand, the notion of fuzzy graphs (FGs), based on fuzzy relation, was proposed by Rosenfeld [10]. Shannon et al. [11] initiated the concepts of intuitionistic fuzzy graphs (IFGs). Some applications of IFGs were explored in [12]. The notion of strong intuitionistic fuzzy graphs (SIFGs) was proposed in [13]. Recently, the generalization of IFGs, termed picture fuzzy graphs (PFGs), was introduced by Zua et al. [14]. They introduced various types of PFGs such as strong PFGs, complete PFGs, etc. They also initiated and applied the terms weak and co-weak isomorphisms to the PFGs. Several applications of PFGs in social networks were also explored by them. After this, picture fuzzy multigraphs (PFMGs) were introduced in [15]. Shoaib et al. [16] introduced some new operations on PFGs and explored their applications. Further extensions of PFGs including constant picture fuzzy graphs [17], balanced picture fuzzy graphs [18], etc., have also been explored. Recently, Khan et al. [19] introduced the concepts of bipolar picture fuzzy graphs along with some of its applications. Similarly, Khan et al. also introduced the notions of Cayley picture fuzzy graphs [20] and interval-valued picture fuzzy graphs [21]. Many articles on vague graph structures with applications toward different fields of the sciences have also been published [22–28].

Information visualization provides a simpler and more insightful understanding of the data. Edge bundling is one of the important techniques of visualizing data, and is also helpful for the analysis of data. However, by converting the data into a graph, the connections among the data can be distinguished more naturally. In such circumstances, edge bundling decreases the visual mess based on various approaches. Results show that the bundled edges are well-organized and arrange the data in a more sophisticated way. Edges can be shown as nodes in such a way that some node-clustering methods can be applied to edge clustering [29] by interpreting any system via a line graph. Bundling edges based on edge clustering is more effective, and such a clustering method was described in [30]. We have both asymmetrical and symmetrical types of clustering for analyzing data. A symmetrical cluster helps us detect whether the clusters are identical, which ensures that the databases remain identical.

On the other hand, the fuzzy line graph has its own importance among the other types of fuzzy graphs. Firstly, the concepts of fuzzy line graphs were initiated by Mordeson in [31]. The concept of intuitionistic fuzzy line graphs was initiated by Akram and Parvathi [32]. The notion of anti-fuzzy line graphs was introduced in [33].

In our study, we introduce the notions of picture fuzzy intersection graphs (PFIGs) and picture fuzzy line graphs (PFLGs). These are the generalized forms of fuzzy intersection graphs and fuzzy line graphs, respectively. In the beginning, we introduce the terms fuzzy intersection number, picture fuzzy intersection number, etc. We also discuss few interesting results based on the fuzzy intersection numbers. Afterwards, we introduce the terms picture fuzzy intersection graphs and picture fuzzy line graphs (PFLGs). We also discuss some interesting characteristics of the picture fuzzy intersection graphs and picture fuzzy line graphs (PFLGs). During our study, we discuss the weak isomorphism

and isomorphism of PFLGs as well. At the end, we describe the proposed methods of clustering presented in [29,30] in the setting of the picture fuzzy line graphs as applications of these graphs. Finally, we conclude whether the results of clustering or edge bundling via PFLGs are symmetrical.

2. Preliminaries

A graph is a pair $G = (V, E)$, where V and E are the set of vertices and edges, respectively. Until now, numerous types of graphs have been introduced and many of their applications explored in the literature. Among the other types of graphs, an intersection graph is very useful. An intersection graph of the given $G = (V, E)$ is the pair $P(X) = (X, Y)$, where $X = \{X_1, X_2, \dots, X_n\}$ consists of a distinct nonempty family of the subsets of V and $E = \{X_i X_j : X_i, X_j \in X, X_i \cap X_j \neq \emptyset, i \neq j\}$. Any graph can be written in the form of an intersection graph. The minimum number of elements in X such that G becomes an intersection graph on X is termed as an intersection number of a given graph G , and it is abbreviated as $\omega(G)$. A line graph is the graph associated with any graph $G = (V, E)$, and it is denoted by $L(G) = (Z, T)$, where $Z = \{\{z\} \cup \{u_z, v_z\} : z \in E, u_z, v_z \in V, z = u_z v_z\}$ while $T = \{X_z X_{z'} : X_z \cap X_{z'} \neq \emptyset; z, z' \in E, z \neq z'\}$ and $X_z = \{z\} \cup \{u_z, v_z\}, z \in E$. Basically, the line graph $L(G)$ of a simple graph G consists of the adjacencies between the edges of graph G . Furthermore, the vertices of $L(G)$ of any graph G are the edges of graph G . In $L(G)$, two of its vertices are adjacent if and only if a vertex is shared by their corresponding edges in graph G . We refer to [34] for further useful terminologies and discussions about intersection graphs and for line graphs one may consult [35].

Definition 1 ([1]). A fuzzy set η on S is the map from set S to the interval $[0, 1]$.

Definition 2 ([7]). Map $A = (\gamma_A, \zeta_A) : S \rightarrow [0, 1] \times [0, 1]$ with $\gamma_A(s) + \zeta_A(s) \leq 1$, for each $s \in S$, is said to be an IFSs on S , where $\gamma_A : S \rightarrow [0, 1]$ and $\zeta_A : S \rightarrow [0, 1]$ stand for the membership and non-membership values of $s \in S$, respectively.

Definition 3 ([8]). Map $A = (\mu_A, \xi_A, \nu_A) : S \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ with $\mu_A(s) + \xi_A + \nu_A(s) \leq 1$, for all $s \in S$, is said to be a PFS defined on S , where $\mu_A : S \rightarrow [0, 1]$, $\xi_A : S \rightarrow [0, 1]$, and $\nu_A : S \rightarrow [0, 1]$ represent the degrees of membership, neutral, and non-membership of each element of $s \in S$.

Definition 4 ([10]). Let $G = (V, E)$ be any graph. Then, the ordered pair (μ, σ) is said to be a fuzzy graph with underlying set V , where μ and σ are the fuzzy subsets of V and $V \times V$, respectively, with

$$\sigma(uv) \leq \min\{\mu(u), \mu(v)\}$$

for all $u, v \in V$.

Definition 5 ([7]). A pair $G = (A, B)$, where $A = (\eta_A, \theta_A)$ is an IFS on V and $B = (\eta_B, \theta_B)$ is an IFS on $V \times V$ with

$$\eta_B(xy) \leq \eta_A(x) \wedge \eta_A(y) \text{ and } \theta_B(xy) \geq \theta_A(x) \vee \theta_A(y)$$

for all $xy \in V \times V$, is said to be an IFG on $G = (V, E)$.

Definition 6 ([32]). A bijective homomorphism $\varphi : G_1 \rightarrow G_2$ of IFGs is said to be a weak vertex-isomorphism, if $\mu_{A_1}(x_1) = \mu_{A_2}(\varphi(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(\varphi(x_1))$, for all $x_1 \in V_1$.

Definition 7 ([32]). A bijective homomorphism $\varphi : G_1 \rightarrow G_2$ of IFGs is a weak line-isomorphism, if $\mu_{A_1}(x_1 y_1) = \mu_{A_2}(\varphi(x_1) \varphi(y_1)), \nu_{A_1}(x_1 y_1) = \nu_{A_2}(\varphi(x_1) \varphi(y_1))$, for all $x_1 y_1 \in E_1$.

Definition 8 ([32]). A bijective homomorphism $\varphi : G_1 \rightarrow G_2$ satisfying the conditions of Definitions 6 and 7 is said to be a weak isomorphism between IFGs G_1 and G_2 .

Definition 9 ([32]). Let $P(X) = (X, Y)$ be an intersection graph described on $G = (V, E)$. Let $A_1 = (\mu_{A_1}, \nu_{A_1})$ and $B_1 = (\mu_{B_1}, \nu_{B_1})$ be the IFSs on V and E , respectively, and $A_2 = (\mu_{A_2}, \nu_{A_2})$ and $B_2 = (\mu_{B_2}, \nu_{B_2})$ be IFSs on X and Y , respectively. Then, the intuitionistic fuzzy intersection graph (IFIG) of IFG $G = (A_1, B_1)$ is an IFG $P(G) = (A_2, B_2)$ such that for all $X_i, X_j \in X$, and $X_i X_j \in Y$, we have

1. $\mu_{A_2}(X_i) = \mu_{A_1}(v_i), \nu_{A_2}(X_i) = \nu_{A_1}(v_i);$
2. $\mu_{B_2}(X_i X_j) = \mu_{B_1}(v_i v_j), \nu_{B_2}(X_i X_j) = \nu_{B_1}(v_i v_j).$

Definition 10 ([32]). Let $G = (V, E)$ be the simple graph and $L(G) = (Z, T)$ be its line graph. Let $A_1 = (\eta_{A_1}, \theta_{A_1})$ and $B_1 = (\eta_{B_1}, \theta_{B_1})$ be the intuitionistic fuzzy subsets of V and E , respectively, and $A_2 = (\eta_{A_2}, \theta_{A_2})$ and $B_2 = (\eta_{B_2}, \theta_{B_2})$ are the IFSs on Z and T , respectively. Then, we say that $L(G) = (A_2, B_2)$ is the intuitionistic fuzzy line graph $L(G) = (A_2, B_2)$ of IFG $G = (A_1, B_1)$, if it satisfies

1. $\eta_{A_2}(X_x) = \eta_{B_1}(x) = \eta_{B_1}(u_x v_x);$
2. $\theta_{A_2}(X_x) = \theta_{B_1}(x) = \theta_{B_1}(u_x v_x);$
3. $\eta_{B_2}(X_x X_y) = \min(\eta_{B_1}(x), \mu_{B_1}(y));$
4. $\theta_{B_2}(X_x X_y) = \max(\theta_{B_1}(x), \theta_{B_1}(y));$

for all $X_x, X_y \in Z, X_x X_y \in T$.

Definition 11 ([14]). A pair $G = (A, B)$, where $A = (\mu_A, \xi_A, \nu_A)$ is a PFS on V and $B = (\mu_B, \xi_B, \nu_B)$ is a PFS on $V \times V$ with

$$\mu_B(xy) \leq \min\{\mu_A(x), \mu_A(y)\}, \xi_B(xy) \leq \min\{\xi_A(x), \xi_A(y)\}, \nu_B(xy) \geq \max\{\nu_A(x), \nu_A(y)\}$$

for all $xy \in V \times V$, is a PFG on $G = (V, E)$.

Definition 12 ([14]). A triplet $S(G) = (S_\mu(G), S_\xi(G), S_\nu(G))$, where

$$S_\mu(G) = \sum_{u \in V, u \neq v} \mu(u, v), S_\xi(G) = \sum_{u \in V, u \neq v} \xi(u, v), S_\nu(G) = \sum_{u \in V, u \neq v} \nu(u, v)$$

is the size of a PFG.

Definition 13 ([14]). A homomorphism from a PFG $G_1 = (A_1, B_1)$ defined on $G_1 = (V_1, E_1)$ to a PFG $G_2 = (A_2, B_2)$ defined on $G_2 = (V_2, E_2)$ is the map $\varphi: V_1 \rightarrow V_2$ satisfying (a) $\mu_{A_1}(u_1) \leq \mu_{A_2}(\varphi(u_1)), \xi_{A_1}(u_1) \leq \xi_{A_2}(\varphi(u_1)), \nu_{A_1}(u_1) \geq \nu_{A_2}(\varphi(u_1))$ (b) $\mu_{A_1}(u_1 v_1) \leq \mu_{A_2}(\varphi(u_1)\varphi(v_1)); \xi_{A_1}(u_1 v_1) \leq \xi_{A_2}(\varphi(u_1)\varphi(v_1)); \nu_{A_1}(u_1 v_1) \geq \nu_{A_2}(\varphi(u_1)\varphi(v_1)).$

3. Picture Fuzzy Intersection Graphs and Picture Fuzzy Line Graphs

This section consists of two subsections. In the first subsection, we initiate the concepts of picture fuzzy intersection graphs (PFIGs), which provide the basis for the discussion of picture fuzzy line graphs (PFLGs). Then, in the next subsection, we introduce the term PFLGs and add important results related to PFLGs.

3.1. Picture Fuzzy Intersection Graphs (PFIGs)

In this section, we initiate the notion of PFIGs, which lay the foundation for the discussion of PFLGs. Here, first we introduce the new term fuzzy intersection number of the fuzzy graphs and then extend it towards PFGs.

Definition 14. Let $G^* = (V, E)$ be a crisp graph and $P(X) = (X, Y)$ be an intersection graph defined on it. Let $G = (A_1, B_1)$ be a PFG defined on $G^* = (V, E)$, where $A_1 = (\mu_{A_1}, \xi_{A_1}, \nu_{A_1})$ and $B_1 = (\mu_{B_1}, \xi_{B_1}, \nu_{B_1})$ are the PFSs on V and E , respectively. Then, the picture fuzzy intersection graph (PFIG) $\wp(G)$ of the PFG $G = (A_1, B_1)$ is a picture fuzzy graph $\wp(G) = (A_2, B_2)$ such that for all $X_i, X_j \in X, X_i X_j \in Y$ it satisfies the following:

1. $\mu_{A_2}(X_i) = \mu_{A_1}(v_i), \xi_{A_2}(X_i) = \xi_{A_1}(v_i), \nu_{A_2}(X_i) = \nu_{A_1}(v_i);$
2. $\mu_{B_2}(X_i X_j) = \mu_{B_1}(v_i v_j), \xi_{B_2}(X_i X_j) = \xi_{B_1}(v_i v_j), \nu_{B_2}(X_i X_j) = \nu_{B_1}(v_i v_j).$

Proposition 1. Let $P(X) = (X, Y)$ be an intersection graph of a crisp graph $G^* = (V, E)$. Let $G = (A_1, B_1)$ be a PFG on G^* . Then, we have the following:

1. $\wp(G) = (A_2, B_2)$ is a picture fuzzy subgraph of $P(X)$;
2. $(A_1, B_1) \cong (A_2, B_2).$

Proof. (1) The following is obtained.

$$\mu_{B_2}(X_i X_j) = \mu_{B_1}(v_i v_j) \leq \min\{\mu_{A_1}(v_i), \mu_{A_1}(v_j)\} = \min\{\mu_{A_2}(X_i), \mu_{A_2}(X_j)\}$$

$$\mu_{B_2}(X_i X_j) = \mu_{B_1}(v_i v_j) \leq \min\{\mu_{A_1}(v_i), \mu_{A_1}(v_j)\} = \min\{\mu_{A_2}(X_i), \mu_{A_2}(X_j)\}$$

$$\nu_{B_2}(X_i X_j) = \nu_{B_1}(v_i v_j) \geq \max\{\nu_{A_1}(v_i), \nu_{A_1}(v_j)\} = \min\{\nu_{A_2}(X_i), \nu_{A_2}(X_j)\}.$$

Hence, the result follows.

(2) Let $f : V \rightarrow S$ be the map defined by $f(v_i) = S_i, i = 1, \dots, n$. Obviously, function f is a one-to-one function of V onto X . Now, $v_i v_j \in E$ iff $X_i X_j \in Y$ and so

$$Y = \{f(v_i)f(v_j) : v_i v_j \in E\}$$

Since, $\mu_{A_2}(f(v_i)) = \mu_{A_2}(X_i) = \mu_{A_1}(v_i), \xi_{A_2}(f(v_i)) = \xi_{A_2}(X_i) = \xi_{A_1}(v_i), \nu_{A_2}(f(v_i)) = \nu_{A_2}(X_i) = \nu_{A_1}(v_i)$ and $\mu_{B_2}(f(v_i)f(v_j)) = \mu_{B_2}(X_i X_j) = \mu_{B_1}(v_i v_j), \xi_{B_2}(f(v_i)f(v_j)) = \xi_{B_2}(X_i X_j) = \xi_{B_1}(v_i v_j),$ and $\nu_{B_2}(f(v_i)f(v_j)) = \nu_{B_2}(X_i X_j) = \nu_{B_1}(v_i v_j).$ Thus, f is an isomorphism between (A_1, B_1) onto (A_2, B_2) , i.e., $(A_1, B_1) \cong (A_2, B_2).$ $\square$

Now first we define the fuzzy intersection number of the fuzzy graph and then we give the definition of the picture fuzzy intersection number of the PFG. For the fuzzy intersection number, we assume that $F(S) = (S, T)$ is the fuzzy intersection graph of any graph $G = (V, E)$. Let (σ, μ) be a fuzzy subgraph of $G = (V, E)$. We suppose that (δ, ξ) is the fuzzy subgraph of $F(S) = (S, T)$; then, clearly (δ, ξ) is the fuzzy intersection graph of $(\sigma, \mu).$

Definition 15. The fuzzy intersection number $\omega(\sigma, \mu)$ of a given fuzzy graph (σ, μ) can be defined as

$$\omega(\sigma, \mu) = \left(\sum_{S_i \neq S_j} \xi(S_i S_j) \right)$$

for all $S_i, S_j \in S, S_i S_j \in T$.

By considering Definition 14, we may define the picture fuzzy intersection number as follows.

Definition 16. Let $\wp(G)$ be a PFIG of PFG G . Then, the picture fuzzy intersection number $\omega(\wp(G))$ of the PFG G is described as $\omega(G) = (\omega_\mu(G), \omega_\xi(G), \omega_\nu(G)),$ where

$$\omega_\mu(G) = \sum_{X_i \neq X_j} \mu_{B_2}(X_i X_j), \omega_\xi(G) = \sum_{X_i \neq X_j} \xi_{B_2}(X_i X_j), \omega_\nu(G) = \sum_{X_i \neq X_j} \nu_{B_2}(X_i X_j)$$

for all $X_i, X_j \in X; X_i X_j \in Y$.

Corollary 1. If $\wp(G)$ is connected and $V \geq 3$, then $\omega(\wp(G)) \leq S(G)$, where $S(G)$ is the size of a PFG.

Corollary 2. If $\wp(G)$ has isolated point p_0 , then $\omega(\wp(G)) \leq S(G) + p_0$.

Example 1. Let $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_2v_3, v_3v_1\}$ be the set of vertices and edges of the graph $G^* = (V, E)$, respectively. Let $G = (A_1, B_1)$ be a PFG described by Table 1, and it is shown in Figure 1.

Table 1. Picture fuzzy graph in Example 1.

	v_1	v_2	v_3
μ_{A_1}	0.2	0.2	0.1
ξ_{A_1}	0.2	0.3	0.3
ν_{A_1}	0.3	0.1	0.2
	v_1v_2	v_2v_3	v_3v_1
μ_{B_1}	0.2	0.1	0.1
ξ_{B_1}	0.2	0.2	0.2
ν_{B_1}	0.3	0.2	0.3

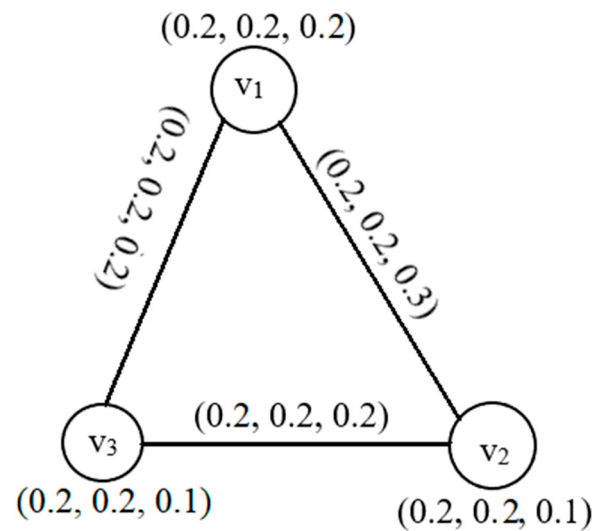


Figure 1. Picture fuzzy graph.

The size of a G is $S(G) = (0.6, 0.6, 0.7)$. Now, we consider that $P(X) = (X, Y)$ is an intersection graph such that $X = \{X_1 = \{v_1, v_2\}, X_2 = \{v_2, v_3\}, X_3 = \{v_1, v_3\}\}$ and $Y = \{X_1X_2, X_2X_3, X_3X_1\}$. Let $A_2 = (\mu_{A_2}, \xi_{A_2}, \nu_{A_2})$ and $B_2 = (\mu_{B_2}, \xi_{B_2}, \nu_{B_2})$ be the PFSs on X and Y , respectively, which are presented in Figure 2. Then, by simple computations, we have the following.

$$\mu_{A_2}(X_1) = \mu_{A_1}(v_1) = 0.2, \mu_{A_2}(X_2) = \mu_{A_1}(v_2) = 0.2, \mu_{A_2}(X_3) = \mu_{A_1}(v_3) = 0.2,$$

$$\xi_{A_2}(X_1) = \xi_{A_1}(v_1) = 0.2, \xi_{A_2}(X_2) = \xi_{A_1}(v_2) = 0.2, \xi_{A_2}(X_3) = \xi_{A_1}(v_3) = 0.2,$$

$$\nu_{A_2}(X_1) = \nu_{A_1}(v_1) = 0.3, \nu_{A_2}(X_2) = \nu_{A_1}(v_2) = 0.2, \nu_{A_2}(X_3) = \nu_{A_1}(v_3) = 0.2,$$

$$\mu_{B_2}(X_1X_2) = \mu_{B_1}(v_1v_2) = 0.2, \mu_{B_2}(X_2X_3) = \mu_{B_1}(v_2v_3) = 0.2, \mu_{B_2}(X_3X_1) = \mu_{B_1}(v_3v_1) = 0.2,$$

$$\xi_{B_2}(X_1X_2) = \xi_{B_1}(v_1v_2) = 0.2, \xi_{B_2}(X_2X_3) = \xi_{B_1}(v_2v_3) = 0.2, \xi_{B_2}(X_3X_1) = \xi_{B_1}(v_3v_1) = 0.2,$$

$$\nu_{B_2}(X_1X_2) = \nu_{B_1}(v_1v_2) = 0.3, \nu_{B_2}(X_2X_3) = \nu_{B_1}(v_2v_3) = 0.2, \nu_{B_2}(X_3X_1) = \nu_{B_1}(v_3v_1) = 0.2.$$

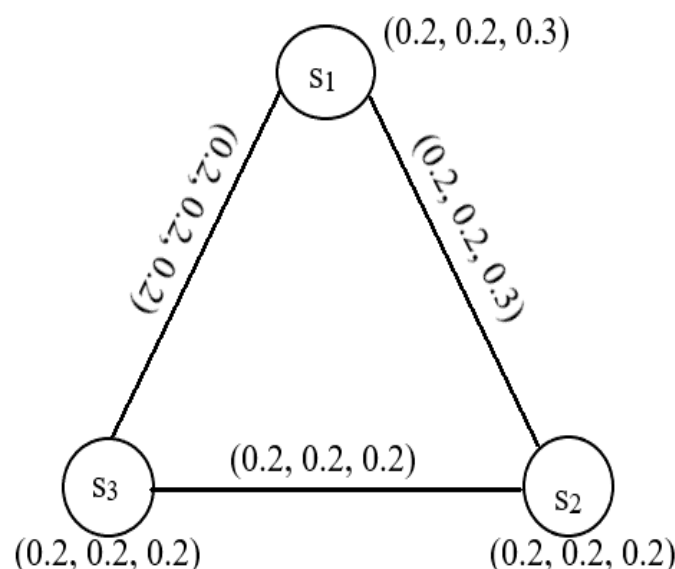


Figure 2. Picture fuzzy intersection graph.

Finally, it is easy to verify that $\wp(G) = (A_2, B_2)$ is a PFIG for which its intersection number is $\omega(P(G)) = (0.6, 0.6, 0.7)$.

Proposition 2. PFIG of a PFG is a PFG.

Proof. Let G^* be any graph. Additionally, let $G = (A_1, B_1)$ and $\wp(G) = (A_2, B_2)$ be a PFG and PFIG, respectively. We will prove that $\wp(G)$ is also PFG. Following the definition of PFG, we have the following.

$$\mu_{B_2}(X_i X_j) = \mu_{B_1}(v_i v_j) \leq \min(\mu_{A_1}(v_i), \mu_{A_1}(v_j))$$

$$\xi_{B_2}(X_i X_j) = \xi_{B_1}(v_i v_j) \leq \min(\xi_{A_1}(v_i), \xi_{A_1}(v_j))$$

$$\nu_{B_2}(X_i X_j) = \nu_{B_1}(v_i v_j) \geq \max(\nu_{A_1}(v_i), \nu_{A_1}(v_j))$$

Hence, PFIG is a PFG. $\square$

3.2. Picture Fuzzy Line Graphs (PFLGs)

In this section, we initiate the notion of PFLGs and provide few of its important characterizations.

Definition 17. Let $G^* = (V, E)$ be the simple graph and $L(G^*) = (Z, T)$ be its line graph. Let $A_1 = (\mu_{A_1}, \xi_{A_1}, \nu_{A_1})$ and $B_1 = (\mu_{B_1}, \xi_{B_1}, \nu_{B_1})$ be picture fuzzy subsets of V and E , respectively. Let $A_2 = (\mu_{A_2}, \xi_{A_2}, \nu_{A_2})$ and $B_2 = (\mu_{B_2}, \xi_{B_2}, \nu_{B_2})$ be PFSSs on Z and T , respectively. Then, the picture fuzzy line graph (PFLG) $L(G) = (A_2, B_2)$ of the PFG $G = (A_1, B_1)$ can be described as follows:

1. $\mu_{A_2}(X_x) = \mu_{B_1}(x) = \mu_{B_1}(u_x v_x);$
2. $\xi_{A_2}(X_x) = \xi_{B_1}(x) = \xi_{B_1}(u_x v_x);$
3. $\nu_{A_2}(X_x) = \nu_{B_1}(x) = \nu_{B_1}(u_x v_x);$
4. $\mu_{B_2}(X_x X_y) = \min(\mu_{B_1}(x), \mu_{B_1}(y));$
5. $\xi_{B_2}(X_x X_y) = \min(\xi_{B_1}(x), \xi_{B_1}(y));$
6. $\nu_{B_2}(X_x X_y) = \max(\nu_{B_1}(x), \nu_{B_1}(y));$

for all $X_x, X_y \in Z, X_x X_y \in T$.

Example 2. Let $G^* = (V, E)$ be any graph, where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{x_1 = v_1v_2, x_2 = v_2v_3, x_3 = v_3v_4, x_4 = v_4v_1\}$. Let A_1 be a picture fuzzy subset of V and B_1 be a picture fuzzy subset of E , which are described in Table 2 and Figure 3.

Table 2. Picture fuzzy in Example 2.

	v_1	v_2	v_3	v_4
μ_{A_1}	0.2	0.3	0.3	0.3
ξ_{A_1}	0.1	0.2	0.1	0.2
ν_{A_1}	0.3	0.1	0.2	0.2
	x_1	x_2	x_3	x_4
μ_{B_1}	0.2	0.3	0.3	0.2
ξ_{B_1}	0.1	0.1	0.1	0.1
ν_{B_1}	0.4	0.2	0.3	0.3

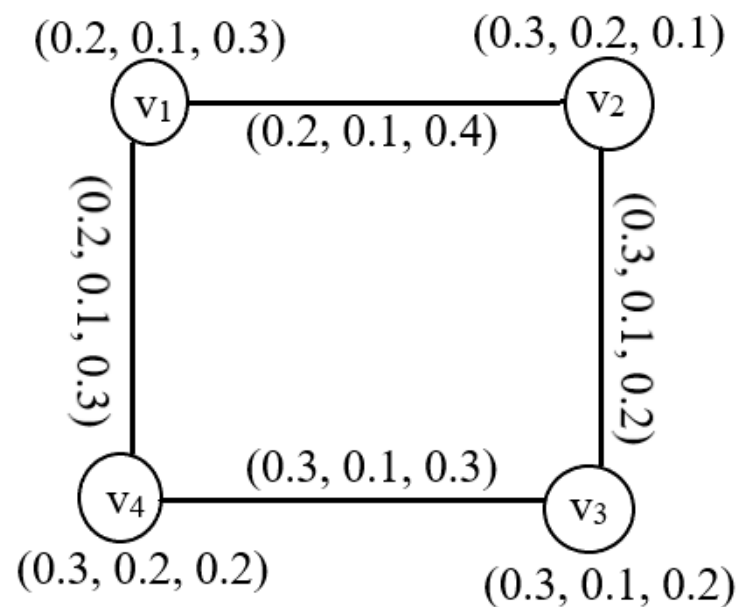


Figure 3. Picture fuzzy graph.

It is easy to verify that G is a PFG. Next, consider a line graph $L(G^*) = (Z, T)$, where $Z = \{X_{x_1}, X_{x_2}, X_{x_3}, X_{x_4}\}$ and $T = \{X_{x_1}X_{x_2}, X_{x_2}X_{x_3}, X_{x_3}X_{x_4}, X_{x_4}X_{x_1}\}$. Let $A_2 = (\mu_{A_2}, \xi_{A_2}, \nu_{A_2})$ and $B_2 = (\mu_{B_2}, \xi_{B_2}, \nu_{B_2})$ be PFSs of Z and T , respectively, and they are shown in Figure 4. Then, we have the following.

$$\mu_{A_2}(X_{x_1}) = 0.2, \mu_{A_2}(X_{x_2}) = 0.3, \mu_{A_2}(X_{x_3}) = 0.3, \mu_{A_2}(X_{x_4}) = 0.2$$

$$\xi_{A_2}(X_{x_1}) = 0.1, \xi_{A_2}(X_{x_2}) = 0.1, \xi_{A_2}(X_{x_3}) = 0.1, \xi_{A_2}(X_{x_4}) = 0.1$$

$$\nu_{A_2}(X_{x_1}) = 0.4, \nu_{A_2}(X_{x_2}) = 0.2, \nu_{A_2}(X_{x_3}) = 0.3, \nu_{A_2}(X_{x_4}) = 0.3$$

$$\mu_{B_2}(X_{x_1}X_{x_2}) = 0.2, \mu_{B_2}(X_{x_2}X_{x_3}) = 0.3, \mu_{B_2}(X_{x_3}X_{x_4}) = 0.2, \mu_{B_2}(X_{x_4}X_{x_1}) = 0.2$$

$$\xi_{B_2}(X_{x_1}X_{x_2}) = 0.1, \xi_{B_2}(X_{x_2}X_{x_3}) = 0.1, \xi_{B_2}(X_{x_3}X_{x_4}) = 0.1, \xi_{B_2}(X_{x_4}X_{x_1}) = 0.1$$

$$\nu_{B_2}(X_{x_1}X_{x_2}) = 0.4, \nu_{B_2}(X_{x_2}X_{x_3}) = 0.3, \nu_{B_2}(X_{x_3}X_{x_4}) = 0.3, \nu_{B_2}(X_{x_4}X_{x_1}) = 0.4$$

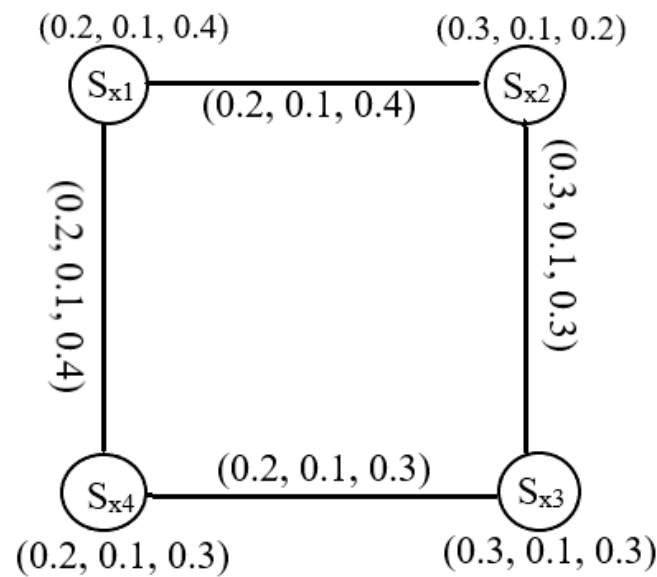


Figure 4. Picture fuzzy line graph.

Evidently, $L(G)$ is a PFLG.

Proposition 3. Every PFLG is a strong PFG.

Proof. The proof is obvious. $\square$

Proposition 4. $L(G) = (A_2, B_2)$ is a PFLG of the PFG $G = (A_1, B_1)$ if and only if for each $X_x X_y \in T$,

$$\mu_{B_2}(X_x X_y) = \min(\mu_{A_2}(X_x), \mu_{A_2}(X_y))$$

$$\xi_{B_2}(X_x X_y) = \min(\xi_{A_2}(X_x), \xi_{A_2}(X_y))$$

$$\nu_{B_2}(X_x X_y) = \max(\nu_{A_2}(X_x), \nu_{A_2}(X_y)).$$

Proof. Let $\mu_{B_2}(X_x X_y) = \min(\mu_{A_2}(X_x), \mu_{A_2}(X_y))$, for all $X_x X_y \in T$. Then,

$$\mu_{B_2}(X_x X_y) = (\mu_{A_2}(X_x) \wedge \mu_{A_2}(X_y)) = (\mu_{A_2}(x) \wedge \mu_{A_2}(y)),$$

$$\xi_{B_2}(X_x X_y) = (\xi_{A_2}(X_x) \wedge \xi_{A_2}(X_y)) = (\xi_{A_2}(x) \wedge \xi_{A_2}(y)),$$

$$\nu_{B_2}(X_x X_y) = (\nu_{A_2}(X_x) \vee \nu_{A_2}(X_y)) = (\nu_{A_2}(x) \vee \nu_{A_2}(y)).$$

A PFS of $A_1 = (\mu_{A_1}, \xi_{A_1}, \nu_{A_1})$ satisfies the following sufficient conditions.

$$\mu_{B_1}(xy) \leq (\mu_{A_1}(x) \wedge \mu_{A_1}(y))$$

$$\xi_{B_1}(xy) \leq (\xi_{A_1}(x) \wedge \xi_{A_1}(y))$$

$$\nu_{B_1}(xy) \geq (\nu_{A_1}(x) \vee \nu_{A_1}(y)).$$

The converse is obvious. $\square$

Proposition 5. Let $L(G) = (Z, T)$ be a PFLG of the PFG G . Then, $L(G^*)$ is a line graph of crisp graph G^* .

Proof. Let $G = (A_1, B_1)$ be a PFG and $L(G)$ be a PFLG. Then, $\mu_{A_2}(X_x) = \mu_{B_1}(x)$, $\xi_{A_2}(X_x) = \xi_{B_1}(x)$, $\nu_{A_2}(X_x) = \nu_{B_1}(x)$, for all $x \in E$, and so $X_x \in Z \Leftrightarrow xa \in E$. Additionally,

$$\mu_{B_2}(X_x X_y) = \min(\mu_{B_1}(x), \mu_{B_1}(y))$$

$$\xi_{B_2}(X_x X_y) = \min(\xi_{B_1}(x), \xi_{B_1}(y))$$

$$\nu_{B_2}(X_x X_y) = \max(\nu_{B_1}(x), \nu_{B_1}(y))$$

for all $X_x X_y \in T$, and so

$$T = \{X_x X_y : X_x \cap X_y \neq \emptyset, x, y \in E, x \neq y\}.$$

Hence, the result follows. $\square$

The following proposition reflects when a PFG is a PFLG.

Proposition 6. $L(G) = (A_2, B_2)$ is a PFLG if and only if $L(G^*) = (Z, T)$ is a line graph and for all $uv \in T$,

$$\mu_{B_2}(uv) = \min(\mu_{A_2}(u), \mu_{A_2}(v))$$

$$\xi_{B_2}(uv) = \min(\xi_{A_2}(u), \xi_{A_2}(v))$$

$$\nu_{B_2}(uv) = \max(\nu_{A_2}(u), \nu_{A_2}(v))$$

Proof. The result follows from Propositions 4 and 5. $\square$

Definition 18. An isomorphism $\varphi : G_1 \rightarrow G_2$ of the PFGs is said to be a weak vertex-isomorphism, if $\mu_{A_1}(x_1) = \mu_{A_2}(\varphi(x_1))$, $\xi_{A_1}(x_1) = \xi_{A_2}(\varphi(x_1))$, $\nu_{A_1}(x_1) = \nu_{A_2}(\varphi(x_1))$, for all $x_1 \in V_1$.

Definition 19. An isomorphism $\varphi : G_1 \rightarrow G_2$ of the PFGs is said to be a weak line-isomorphism, if $\mu_{A_1}(x_1 y_1) = \mu_{A_2}(\varphi(x_1) \varphi(y_1))$, $\xi_{A_1}(x_1 y_1) = \xi_{A_2}(\varphi(x_1) \varphi(y_1))$, $\nu_{A_1}(x_1 y_1) = \nu_{A_2}(\varphi(x_1) \varphi(y_1))$, for all $x_1 y_1 \in E_1$.

Remark 1. An isomorphism $\varphi : G_1 \rightarrow G_2$ satisfying Definitions 18 and 19 is said to be a weak isomorphism of the PFGs G_1 and G_2 . It is also important to note that the weak isomorphism may not preserve the weights of the edges but it preserves the weights of the vertices.

Theorem 1. Let $L(G) = (A_2, B_2)$ be the PFLG for a PFG $G = (A_1, B_1)$. Let the underlying crisp graph $G^* = (V, E)$ be connected. Then, the following is the case:

1. There exists a weak isomorphism of G onto $L(G)$ if and only if G^* is cyclic and for an $v \in V$, $x \in E$, $\mu_{A_1}(v) = \mu_{B_1}(x)$, $\xi_{A_1}(v) = \xi_{B_1}(x)$, $\nu_{A_1}(v) = \nu_{B_1}(x)$; i.e., $A_1 = (\mu_{A_1}, \xi_{A_1}, \nu_{A_1})$ and $B_1 = (\mu_{B_1}, \xi_{B_1}, \nu_{B_1})$ are the constant functions on V and E , respectively, taking on the same value.
2. φ is an isomorphism if it is a weak isomorphism of G onto $L(G)$.

Proof. Let φ is a weak isomorphism of G onto $L(G)$. Following [36] (Theorem 8.2, p. 72), $G^* = (V, E)$ is a cycle. Let $V = \{u_1, u_2, \dots, u_n\}$ and $E = \{x_1 = u_1 u_2, x_2 = u_2 u_3, \dots, x_n = u_n u_1\}$, where $u_1 u_2 u_3 \dots u_n$ is a cycle. We define PFSs as

$$\mu_{A_1}(u_i) = s_i, \xi_{A_1}(u_i) = s'_i, \nu_{A_1}(u_i) = s''_i$$

and

$$\mu_{B_1}(x_i) = \mu_{B_1}(u_i u_{i+1}) = r_i, \xi_{B_1}(x_i) = \xi_{B_1}(u_i u_{i+1}) = r'_i, \nu_{B_1}(x_i) = \nu_{B_1}(u_i u_{i+1}) = r''_i, \\ i = 1, 2, \dots, n, \nu_{n+1} = u_1.$$

Then, for $s_{n+1} = s_1, s_{n+1}' = s_1', s_{n+1}'' = s_1''$, we have

$$r_i \leq \min(s_i, s_{i+1}), r_i' \leq \min(s_i', s_{i+1}'), r_i'' \geq \max(s_i'', s_{i+1}''), i = 1, 2, \dots, n. \quad (1)$$

Now

$$Z = \{X_{x_1}, X_{x_2}, X_{x_3}, \dots, X_{x_n}\}, T = \{X_{x_1}X_{x_2}, X_{x_2}X_{x_3}, \dots, X_{x_n}X_{x_2}\}.$$

Hence, for $r_{n+1} = r_1$, we obtain

$$\mu_{A_2}(X_{x_i}) = \mu_{B_1}(x_i) = r_i, \xi_{A_2}(X_{x_i}) = \xi_{B_1}(x_i) = r_i', \nu_{A_2}(X_{x_i}) = \nu_{B_1}(x_i) = r_i'',$$

$$\mu_{B_2}(X_{x_i}X_{x_{i+1}}) = \min(\mu_{B_1}(x_i), \mu_{B_1}(x_{i+1})) = \min(r_i, r_{i+1}),$$

$$\mu_{B_2}(X_{x_i}X_{x_{i+1}}) = \min(\mu_{B_1}(x_i), \mu_{B_1}(x_{i+1})) = \min(r_i', r_{i+1}'),$$

$$\mu_{B_2}(X_{x_i}X_{x_{i+1}}) = \min(\mu_{B_1}(x_i), \mu_{B_1}(x_{i+1})) = \min(r_i'', r_{i+1}'')$$

for $i = 1, 2, \dots, n, v_{n+1} = v_1$. As φ is an isomorphism of G^* onto $L(G^*)$ so φ is a bijective map of V onto $\mathbb{Z}$. Since φ also preserves an adjacency, φ induces the a permutation of $\{1, 2, \dots, n\}$ with

$$\varphi(u_i) = X_{u_{\pi(i)}u_{\pi(i)+1}}.$$

Moreover,

$$u_i u_{i+1} \rightarrow \varphi(u_i) \varphi(u_{i+1}) = X_{u_{\pi(i)}u_{\pi(i)+1}} X_{u_{\pi(i+1)}u_{\pi(i+1)+1}}, i = 1, 2, \dots, n-1.$$

Thus,

$$s_i = \mu_{A_1}(u_i) \leq \mu_{A_2}(\varphi(u_i)) = \mu_{A_2}(S_{u_{\pi(i)}u_{\pi(i)+1}}) = \mu_{B_1}(u_{\pi(i)}u_{\pi(i)+1}) = r_{\pi(i)},$$

$$s_i = \xi_{A_1}(u_i) \leq \xi_{A_2}(\varphi(u_i)) = \xi_{A_2}(S_{u_{\pi(i)}u_{\pi(i)+1}}) = \xi_{B_1}(u_{\pi(i)}u_{\pi(i)+1}) = r_{\pi(i)}',$$

$$s_i = \nu_{A_1}(u_i) \geq \nu_{A_2}(\varphi(u_i)) = \nu_{A_2}(S_{u_{\pi(i)}u_{\pi(i)+1}}) = \nu_{B_1}(u_{\pi(i)}u_{\pi(i)+1}) = r_{\pi(i)}'',$$

and

$$\begin{aligned} r_i &= \mu_{B_1}(u_i u_{i+1}) \leq \mu_{B_2}(\varphi(u_i) \varphi(u_{i+1})) \\ &= \mu_{B_2}(S_{u_{\pi(i)}u_{\pi(i)+1}} S_{u_{\pi(i+1)}u_{\pi(i+1)+1}}) \\ &= \min(\mu_{B_1}(u_{\pi(i)}u_{\pi(i)+1}), \mu_{B_1}(u_{\pi(i+1)}u_{\pi(i+1)+1})) \\ &= \min(r_{\pi(i)}, r_{\pi(i+1)}). \end{aligned}$$

Similarly,

$$\begin{aligned} r_i' &= \xi_{B_1}(u_i u_{i+1}) \leq \xi_{B_2}(\varphi(u_i) \varphi(u_{i+1})) \\ &= \xi_{B_2}(X_{u_{\pi(i)}u_{\pi(i)+1}} X_{u_{\pi(i+1)}u_{\pi(i+1)+1}}) \\ &= \min(\xi_{B_1}(u_{\pi(i)}u_{\pi(i)+1}), \xi_{B_1}(u_{\pi(i+1)}u_{\pi(i+1)+1})) \\ &= \min(r_{\pi(i)}', r_{\pi(i+1)}'). \end{aligned}$$

Additionally,

$$\begin{aligned} r''_i &= v_{B_1}(u_i u_{i+1}) \geq v_{B_2}(\varphi(u_i) \varphi(u_{i+1})) \\ &= v_{B_2}(X_{u_{\pi(i)} u_{\pi(i)+1}} X_{u_{\pi(i+1)} u_{\pi(i+1)+1}}) \\ &= \max\left(v_{B_1}(u_{\pi(i)} u_{\pi(i)+1}), v_{B_1}(u_{\pi(i+1)} u_{\pi(i+1)+1})\right) \\ &= \max(r''_{\pi(i)}, r''_{\pi(i+1)}) \end{aligned}$$

for $i = 1, 2, \dots, n$. That is,

$$s_i \leq r_{\pi(i)}, s'_i \leq r'_{\pi(i)}, s''_i \geq r''_{\pi(i)} \quad (2)$$

Moreover,

$$r_i \leq \min(r_{\pi(i)}, r_{\pi(i+1)}), r'_i \leq \min(r'_{\pi(i)}, r'_{\pi(i+1)}), r''_i \geq \min(r''_{\pi(i)}, r''_{\pi(i+1)}) \quad (3)$$

Thus, $r_i \leq r_{\pi(i)}$, $r'_i \leq r'_{\pi(i)}$, and $r''_i \geq r''_{\pi(i)}$, and so $r_{\pi(i)} \leq r_{\pi(\pi(i))}$, $r'_{\pi(i)} \leq r'_{\pi(\pi(i))}$, and $r''_{\pi(i)} \geq r''_{\pi(\pi(i))}$ for all $i = 1, 2, \dots, n$.

Continuing in this way, we obtain the following:

$$\begin{aligned} r_i &\leq r_{\pi(i)} \leq \dots \leq r_{\pi^j(i)} \leq r_i, \\ r'_i &\leq r'_{\pi(i)} \leq \dots \leq r'_{\pi^j(i)} \leq r'_i, \\ r''_i &\geq r''_{\pi(i)} \geq \dots \geq r''_{\pi^j(i)} \geq r''_i, \end{aligned}$$

where π^{j+1} is the identity map. So, $r_i = r_{\pi(i)}$, $r'_i = r'_{\pi(i)}$, and $r''_i = r''_{\pi(i)}$, for all $i = 1, 2, \dots, n$. However, by Equation (3), we also have $r_i \leq r_{\pi(i+1)} = r_{i+1}$, $r'_i \leq r'_{\pi(i+1)} = r'_{i+1}$, and $r''_i \geq r''_{\pi(i+1)} = r''_{i+1}$, which together with $r_{n+1} = r_1$, $r'_{n+1} = r'_1$, and $r''_{n+1} = r''_1$ implies $r_i = r_1$, $r'_i = r'_1$ and $r''_i = r''_1$, for all $i = 1, 2, \dots, n$. Hence, by Equations (1) and (2), we obtain the following.

$$\begin{aligned} r_1 &= \dots = r_n = s_1 = \dots = s_n, \\ r'_1 &= \dots = r'_n = s'_1 = \dots = s'_n, \\ r''_1 &= \dots = r''_n = s''_1 = \dots = s''_n. \end{aligned}$$

Thus, (ii) holds. The converse part of (i) is straightforward. $\square$

4. Applications of PFLGs towards Edge Clustering and c-means Algorithm

Classical (or hard) clustering is fixed and specifies each data point relative to only one cluster; however, fuzzy clustering allocates a membership value to each data point towards each feasible cluster and allocates many points to the cluster that has the highest “membership value”. Fuzzy clustering can be thought of as a forerunner of the hard clustering, because it has also the purpose of partitioning the data points into different sets for control or categorization. While dealing with hard clustering, we have to fix the boundaries between groups, but this is not the case with many natural systems.

On the other hands, information visualization permits easy and insightful understanding about the given data. Edge bundling is a technique by which we can perform a visual analysis. By the transformation of data into a network diagram, the connections among the groups in a given datum can be distinguished automatically. In such circumstances, edge bundling decreases visual confusion by bundling the edges based on different categories. Results reflect that the bundles of edges are systematized in only some relationships. Alternatively, the bundles can be considered as clusters of edges. By transforming a network into a line graph, edges can be transformed as nodes, and several node-clustering methods

could be applied to edge clustering. We make the bundle of edges on the basis of the results of edge clustering. This technique is new in the concepts of fuzzy edge bundling and fuzzy edge clustering. Using this, many edges can be clearly bundled while a few edges belonging to different clusters are not included in these bundles.

In addition to the above, Fuzzy clustering techniques are very useful when the dataset consists of many subgroups of points with imprecise boundaries. Conventional techniques have been widely studied and applied on real-world statistics, but users must acquaint with the results a priori, which conclude how many numbers of clusters to search for. Moreover, different iterative algorithms have been introduced for the best possible number of clusters. Usually, three algorithms (fuzzy c-means, Gustafson–Kessel, and an iterative version of Gustafson–Kessel) based on fuzzy sets have been described for clustering.

In a traditional dataset, a crisp cutoff valued was usually considered. However, it has been revealed that the α -cut of fuzzy sets provided better results in order to avoid the non-required points. Results specify that the α -cut procedure reduces more subgroups for selecting a suitable number of sub-clusters within the given dataset and, hence, is more efficient compared to that of the traditional methods of clustering.

4.1. Layout of Application of PFLG as Cluster-Based Picture Fuzzy Edge Bundling (CBPFEB)

Generally, the method of CBEB based on PFLGs is demonstrated in Figures 5 and 6. Initially, we transform an underlying PFG into a PFLG. This means that the edges of the graphs are converted into vertices and are connected if a pair of the original edges is connected with them. After transforming the graph to PFG, we utilize a community detection method to form a PFLG. The detected communities on the PFLG will be the clusters of edges in the underlying PFG. At the end, the edge bundling method will be applied to the PFG. After using the cluster information, the algorithm for the bundled-edges can be weighted. All those edges lying in the same cluster are firmly bundled (membership-value), and the edges belonging to the dissimilar clusters are loosely bundled (non-membership value). The edges lying in the intersection of two bundles will be considered as a partially bundled (neutral membership value).

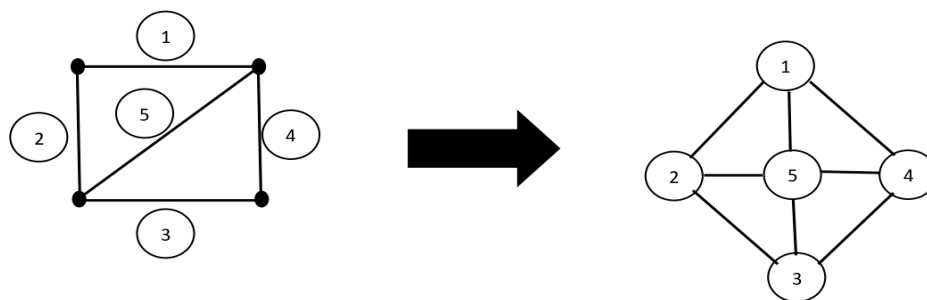


Figure 5. PFG to PFLG.

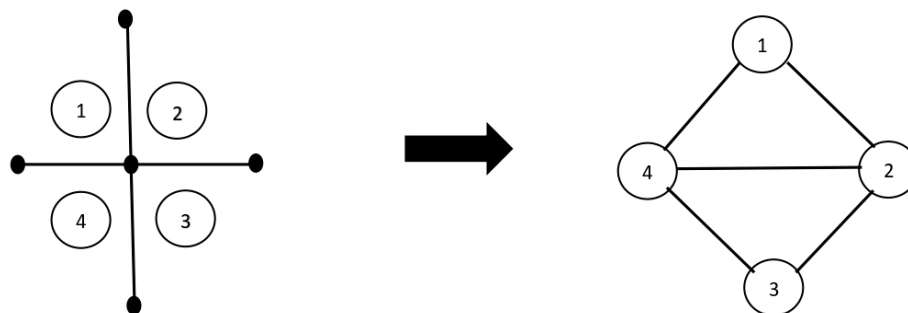


Figure 6. Edge clustering based on PFLG.

After converting a PFG to a PFLG, we follow the steps below for cluster-based edge bundling using PFLGs:

- i. The PFG can be the best for fuzzy partition visually. The $(\alpha, \beta, \Upsilon)$ -cut graph also corresponds to the $(\alpha, \beta, \Upsilon)$ -cut partition.
- ii. After obtaining the picture fuzzy cut-off values, we divide the data into three categories: membership, neutral membership, and non-membership values.
- iii. Now, the edges can be easily bundled based on their respective values.
- iv. We fixed the strength of the class (say A) based on the highest membership values for a specific cluster. This means that it will not be included in any other class.
- v. By eliminating the class A, we can continue bundling the other vertices termed neutral and non-membership classes. In this way, we make the clustering procedure easy for the other classes.
- vi. Finally, this technique decreases the data overall and easily bundles the edges, hence resulting in clustering.

At the end, we succeed to in obtaining clusters that are more symmetrical.

4.2. Proposed Picture Fuzzy c-mean Algorithm Based on PFLGs

Using PFLGs, we can categorize the data with three allocated values, solely lying in the X bundle with a membership value, solely not lying in X bundle with a non-membership value, and lying in the intersection of two bundles with a neutral membership value. After this, the bundles divide the data into different sets or categories more accurately, and it is easy to study the clustering of edges via PFLGs. In this regard, we extend the fuzzy c-means algorithm presented in [30] in terms of a picture fuzzy c-means algorithm while considering the dataset as edges and information (connections) as the vertices, which is basically a conversion of the PFG of the dataset into a PFLG. In this way, by using picture fuzzy α -cuts, we can make determinations with more accuracy when eliminating non-relevant points. As there are three membership values in PFSs, the produced results would indicate that the picture fuzzy α -cut method will eliminate more points compared to the crisp cutoff values and fuzzy cutoff values. Hence, the picture fuzzy clustering algorithm is more capable with respect to selecting the optimum picture fuzzy number of sub-clusters within a point set, leading to a proper indication of regions of interest for further expert analysis. Consequently, we obtain more symmetrical clusters compared to that of the traditional and fuzzy c-mean algorithm.

5. Conclusions

Picture fuzzy models are more precise, flexible, and compatible compared to that of other existing fuzzy models. In this manuscript, we have introduced the concepts of PFLGs and FIGs. These are the generalizations of the concepts of picture fuzzy line graphs (FLGs) and intuitionistic fuzzy line graphs (IFLGs). For fruitful discussions about picture fuzzy intersection graphs, first, we have shifted some concepts of classical graphs towards fuzzy graphs such as the fuzzy intersection number, picture fuzzy intersection number, etc., which hasnot beencarried out beforein the theory of fuzzy graphs. We have also produced some interesting results related to these newly established concepts. Afterwards, we have introduced the notions of PFIGs and PFLGs and discussed some impotant characteristics of these graphs. During this step, we have also established few connections among PFGs, PFIGs, and PFLGs, such as the following: every PFIG of a PFG is a PFG, every PFLG is a strong PFG, and so on. We have also found some connections between picture fuzzy line graphs and crisp line graphs. Consequently, the results presented in our study are the direct extensions of the concepts of intersection graphs and line graphs in a classical graph theory. We have also investigated some structural properties of PFIGs and PFLGs using different types of homomorphisms and isomorphisms. Throughout our research study, we have furnished our results by providing suitable examples and counterexamples. Overall, we have not only provided the theoretical aspects of PFIGs and PFLGs but also presented their applications. Moreover, with respect to the applications of PFLGs, we have proposed cluster-based picture fuzzy edge bundling and the picture fuzzy c-mean algorithm. Furthermore, we are planning to creat different algorithms for

more investigations on the picture fuzzy c-mean algorithm based on PFLGs, which can be helpful for finding the symmetrical aspects of the given data. In this regard, we are planning to prepare another article in which we provide different algorithms for the proposed applications in this work. One can extend this study towards bipolar picture fuzzy sets by introducing bipolar picture fuzzy intersection graphs and bipolar picture fuzzy line graphs. Without a doubt, the results presented in this article would be helpful for solving many problems in transportation models, networks, image recognition, etc.

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