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Dynamical behavior of the fractional coupled Konopelchenko–Dubrovsky and (3 + 1)-dimensional modified Korteweg–de Vries–Zakharov–Kuznestsov equations

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Abstract
This work studies soliton solutions of time-fractional coupled Konopelchenko–Dubrovsky (CKDE) and (3 + 1)-dimensional modified Korteweg–de Vries–Zakharov–Kuznestsov (mKdVZKE) equations. These models are used to define the physical phenomena of ocean dynamics, plasma physics, and soliton theory. The unified method is used to solve these fractional models analytically. To deal with the time fractional part, conformable and local M derivatives are used. A fractional wave transformation is used to transform a fractional partial differential equation to an ordinary differential equation. Using the proposed scheme, soliton solutions are obtained in polynomial and rational forms. The behavior of a soliton solution is also analyzed at different fractional parameters. The results show that the proposed scheme is simple and easy to apply to all types of time-fractional nonlinear homogenous evolution equations encountered in various fields of science.

Keywords Konopelchenko–Dubrovsky equation · mKdVZKE · Fractional order conformable derivative (FCD) · Local M derivative (LMD) · Analytical solutions

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1 Introduction

In this modern era, the significance of nonlinear evolution equations is increasing day by day in different fields of science like optical fiber, physics, chemistry, plasma, fluid dynamics, hydrodynamics, and engineering (Alam and Xin 2019). Most NLEEs are solved using the traveling wave transformation, which converts the partial differential equation into an ordinary differential equation. Following their conversion, various methods are imposed to obtain their solutions. Habibul Bashar used the exp-method and the tanh function to solve the Heisenberg ferromagnetic spin chain (HFSC) equation and extract the traveling wave solutions (Bashar et al. 2021). Kang-jia Wang utilized three different techniques on the coupled Konno–Oono equation to determine the abundant analytical solutions (Wang 2021). He has also extracted the solutions for traveling waves by imposing the variational direct method and his frequency formula method (HFM) on the mBBM (Wang 2022a) and the local fractional wave method on the local fractional Camassa Holm Kadmostsev Petviashvili (LFCHKPE) (Wang 2023), and further, this method was also imposed on the Riemann wave model (Wang 2022b). A variational approach was used to solve the (2 + 1)-dimensional Maccari system, and explicit solutions were obtained (Wang and Si 2022). Taqi et al. (2019) solved the velocities using the cosine and sine hyperbolic methods and discovered their traveling wave exact solutions to Benjamin–Bona–Mahony type equations. Gepreel et al. (2011) solved the Kaup–Kupershmit and Kersten–Krasil–Shchik equations in mathematical physics and finds the analytical traveling wave solution. In biology, Hassan used the generalized G'/G as well as extended G'/G method for the biological population model, and as a result of traveling waves, exact solutions have been obtained (Khan et al. 2021). Ali and Mehanna (2021) has also derived the traveling wave and an approximate solution of the Gilson–Pickering equation (GPE). Wang et al. (2022a) derived the traveling wave solution of the (3 + 1)-dimensional Boiti–Leon equation with the Mittag–Leffler function-based method.

In this paper, the CKDE and mKdVZKE are solved with a unified method, and their polynomial and rational solutions are obtained. Some authors have tried to solve these equations with other methods, like Elsayed, who obtained the traveling wave solution of CKDE with the help of the modified simple equation method and also imposed some parameters to get their solitary wave solutions (Zayed and Gepreel 2010). Taghizadeh also solved CKDE with the first integral method (Taghizadeh and Mirzazadeh 2011). Alfalqi extracts both exact and numerical CKDE solutions (Alfalqi et al. 2019).

$$\begin{aligned} &U_t - U_{xxx} - 6mUU_x + \frac{3}{2}n^2U^2U_x - 3V_y + 3nU_xV = 0, \\ &V_x = U_y. \end{aligned} \tag{1.1}$$

This equation arises in mathematical physics in the context of nonlinear waves with weak dispersion. U (x, y, t) and V (x, y, t) are analytic dependent variables with spatial (x, y) and temporal (t) variables, respectively, and m and n are real constants. The CKDE is used in different fields like ocean dynamics, plasma physics, and fluid mechanics. The Gardner equation, Kadomtsev–Petviashvili (KPE), and modified KPE are special cases that can be obtained by varying the m and n parameters. When $m = 0$, the governing couple equation is reduced to the Kadomtsev–Petviashvili KP equation; when $n = 0$, it can be converted to the modified KP model; and when $U_y = 0$, it behaves like the Gardner equation (Kumar et al. 2014).

The (3 + 1)-dimensional fractional mKdVZKE is defined as Islam et al. (2014)

$$U_t + bU^2U_x + U_{xxx} + U_{xyy} + U_{xzz} = 0. \tag{1.2}$$

Here, U is a dependent variable like U(x, t), and the coefficient of U^2U_x is nonzero. Cenesiz et al. (2017) used the functional variation method to solve Eq. (1.2). Baleanu et al. (2015) extracts the multiple-type solutions of Eq. (1.2) using the first integral method. Al-Gafri Yaslan and Girgin (2019) determines the solutions as well as several different types of solutions to the mKdVZKE. Tasbozan et al. (2017) also constructed their solution by using the direct algebraic method. In this work, Eq. (1.2) is solved using a unified method, and their exact multiple solutions are extracted.

The layout of this paper is described as follows: Sect. 2 deal with the varieties of fractional derivatives; Sect. 3 concern an overview of the proposed scheme; and Sect.4 relate to the implementation of a unified method on FPDEs and their graphical representation. Section 5 describes the physical interpretation of governing models, and Sect. 6 concludes them.

2 Varities of fractional order derivaties

2.1 Conformable derivative (CD)

Let us assume the fractional order conformable derivative of function $u : [0, \infty) \to \mathbb{R}$ at $t > 0$ with $\beta \in (0, 1)$ is expressed as Kajouni et al. (2021):

$$D_t^{\beta} h(t) = \lim_{\epsilon \to 0} \frac{u\left(t + \epsilon t^{1-\beta}\right) - u(t)}{\epsilon}, \tag{2.1}$$

The fractional parameter β is substitute equal to zero then it is reduced in ordinary derivative. Some results related to fractional order (CD) defined in this way:

Theorem 1 *If u_1 and u_2 are differentiable function at $t > 0$ with $\beta \in (0, 1)$ then (Sousa and de Oliveira 2017):*

1. $D_t^{\beta}(C) = 0$, (*for constant number C its results is zero*).
2. $D_t^{\beta}(t^p) = pt^{p-\beta} \forall\, p \in R$.
3. $D_t^{\beta}(u_1 + u_2) = D_t^{\beta}(u_1) + D_t^{\beta}(u_2)$.
4. $D_t^{\beta}(u_1 - u_2) = D_t^{\beta}(u_1) - D_t^{\beta}(u_2)$.
5. $D_t^{\beta}(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 D_t^{\beta}(u_1) + \lambda_2 D_t^{\beta}(h_2)$, *where* λ_1, λ_2 *any real numbers*.
6. $D_t^{\beta}(\frac{u_1}{u_2}) = \frac{u_1 D_t^{\beta}(u_2) - u_2 D_t^{\beta}(u_1)}{h_2^2}$.
7. $D_t^{\beta}(u_1 * u_2) = u_2 D_t^{\beta}(u_1) + u_1 D_t^{\beta}(u_2)$.
8. $D_t^{\beta}(u_1\ o\ u_2) = u_1'(u_2(t)) D_t^{\beta}(u_2)(t)$.

2.2 Local M derivative (FLMD)

Let us assume the fractional order local M derivative of function $u : [0, \infty) \rightarrow \mathbb{R}$ at $t > 0$ with $\kappa \in (0, 1)$ is expressed as (Rafiq et al. 2021):

$$D_{M,t}^{\kappa:\lambda} u(t) = \lim_{\epsilon \to 0} \frac{u\left(tE_{\lambda}(\epsilon t^{-\kappa}) \right) - u(t)}{\epsilon}, \tag{2.2}$$

κ is fractional parameter and M shows about derived function that comprise the Mittage–Lefflar function $E_{\lambda}(.)$, $\forall\, \kappa > 0$ of one parameter.

put t = 0 in Eq. (2.2) then it behaves like,
$D_{M,t}^{\kappa:\lambda} u(0) = \lim_{t \to 0^+} D_{M,t}^{\kappa:\lambda} u(t)$.

Theorem 2 *If u_1 and u_2 are differentiable functions at $t > 0$ with $\kappa \in (0, 1)$ then (Baleanu et al. 2015):*

1. $D_{M,t}^{\kappa:\lambda}(C) = 0$, (*for constant number C its results is zero*).
2. $D_{M,t}^{\kappa:\lambda}(u_1 + u_2) = D_{M,t}^{\kappa:\lambda}(u_1) + D_{M}^{\kappa:\lambda}(u_2)$.
3. $D_{M,t}^{\kappa:\lambda}(u_1 - u_2) = D_{M,t}^{\eta:\beta}(u_1) - D_{M}^{\kappa:\lambda}(u_2)$.
4. $D_{M,t}^{\kappa:\lambda}(\beta_1 u_1 + \beta_2 u_2) = \beta_1 D_{M,t}^{\eta:\beta}(u_1) + \beta_2 D_{M,t}^{\kappa:\lambda}(u_2)$, *where β_1, β_2 any real constant.*
5. $D_{M,t}^{\kappa:\lambda}(\frac{u_1}{u_2}) = \frac{u_1 D_{M,t}^{\kappa:\lambda}(u_2) - s_2 D_{M,t}^{\kappa:\lambda}(u_1)}{u_2^2}$.
6. $D_{M,t}^{\kappa:\lambda}(u_1 * u_2) = u_2 D_{M,t}^{\kappa:\lambda}(u_1) + u_1 D_{M,t}^{\kappa:\lambda}(u_2)$.
7. $D_{M,t}^{\kappa:\lambda}(u_1 \, o \, u_2) = u_1'(u_2(t)) D_{M,t}^{\kappa:\lambda}(u_2)(t)$.

3 Analysis of unified method

Fractional order differential equation in term of dependent variable S and independent variable as spatial (x, y, z) and temporal (t) coordinates are illustrated as,

$$G\left(x, y, z, t, \frac{\partial^{\beta} S}{\partial t^{\beta}}, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}, \frac{\partial^{2\beta} S}{\partial t^{2\beta}}, \dots \right) = 0, \quad t \geq 0, \beta \in (0, 1]. \tag{3.1}$$

Here G is polynomial function that comprises the variables together with multiple derivatives. Some steps to be performs for solving the differential equation with unified method.

Step 1 Changed the governing model Eq. (3.1) using fractional wave transformation into simple (ODE).

$$D\left(s, \frac{ds}{d\zeta}, \frac{d^2 s}{d\zeta^2}, \frac{d^3 s}{d\zeta^3} \dots \right) = 0. \tag{3.2}$$

D is the function that contains integers order derivatives. $\frac{ds}{d\zeta} = s'$

Step 2 Balanced the highest order (dispersive term) as well as nonlinear term in reduced differential equation through homogenous balance principal.

Step 3 Solve Eq. (3.2) and obtain the solutions in two ways.

(a) Polynomial form solution

This kind of solution are described with the help of series relation such as (Raza et al. 2021):

$$U(\zeta) = \sum_{i=0}^{n} E_i \psi^i(\zeta), \tag{3.3}$$

with

$$(\psi'(\zeta))^\rho = \sum_{i=0}^{\rho h} P_i \psi^i(\zeta), \quad \rho = 1, 2. \tag{3.4}$$

Here the E_i and P_i $(i = 0, 1, 2, \dots, n)$ are unknown values that determine with the help of consistency condition in such a way that the calculated coefficient from Eq. (3.3) are inserted in Eq. (3.2) and it must be satisfy. Further the constant values of parameter n and h are calculated by inserting a relationship using homogenous balance principal. The suggested method demonstrate their solution in elementary or elliptic form by putting the parameter $\rho = 1$ or $\rho = 2$.

(b) Rational form solution

This kind of solution are described with the help of series relation such as (Wang et al. 2022b):

$$U(\zeta) = \frac{\sum_{i=0}^{n} E_i \psi^i(\zeta)}{\sum_{i=0}^{r} F_i \psi^i(\zeta)}, \quad n \geq r, \tag{3.5}$$

$$(\psi'(\zeta))^\rho = \sum_{i=0}^{\rho h} P_i \psi^i(\zeta), \quad \rho = 1, 2. \tag{3.6}$$

Here the E_i F_i and P_i $(i = 0, 1, 2, \dots, n)$ are unknown values that determine with the help of consistency condition in such a way that the calculated coefficient from Eq. (3.6) are inserted in equation Eq. (3.2) and it must be satisfy. Further the constant values of parameter n and h are calculated by inserting a relationship using homogenous balance principal. The suggested method demonstrate their solution in elementary or elliptic form by putting the parameter $\rho = 1$ or $\rho = 2$.

Step 3 For getting the values of coefficient solved the system of equation through Maple Software.

Step 4 Finally got the exact solution of NLEEs.

4 Application

This section deals with the application of differential equation that is fractional CKDE and (3 + 1)-dimensional mKdVZKE solved with utilizing the unified method and get out the solutions in two form that is polynomial and rational. Further their subcases also describes according to substituting the values of parameters. The CKDE is written as,

$$\begin{aligned} & U_t - U_{xxx} - 6mUU_x + \frac{3}{2}n^2U^2U_x - 3V_y + 3nU_xV = 0, \\ & V_x = U_y. \end{aligned} \tag{4.1}$$

The Eq. (4.1) is written in the form of temporal fractional order conformable derivative (TFCD) as,

$$\begin{aligned} & D_t^{\kappa}U - U_{xxx} - 6mUU_x + \frac{3}{2}n^2U^2U_x - 3V_y + 3nU_xV = 0, \\ & V_x = U_y. \end{aligned} \tag{4.2}$$

$D_t^{\kappa}U$ describe about conformable derivative of function U in term of independent variable t. By considering the value of κ =1 then fractional form of Eq. (4.2) transformed into its original CKDE. But our concentration to solve Eq. (4.2) via unified method so for this purpose we used fractional wave transformation of following form,

$$U(x,y,t) = U(\zeta), \quad V(x,y,t) = V(\zeta), \quad \text{and} \quad \zeta = x + y - v\frac{t^{\kappa}}{\kappa},$$

and reduced the CKDE into simple differential equation like that,

$$-vU' - U''' - 6mUU' + \frac{3}{2}n^2U^2U' - 3V' + 3nU'V = 0. \tag{4.3}$$

$$U' = V'. \tag{4.4}$$

integrate Eq. (4.4) and put $U = V$ and integrating constant ($K = 0$) in Eq. (4.3) as a conclusion,

$$(v+3)U + \frac{3}{2}(2m-n)U^2 - \frac{1}{2}n^2U^3 + U'' = 0. \tag{4.5}$$

Equation (4.5) exit in its simple differential form in which dispersive and nonlinear terms are presents, Now we solve this equation with unified method and extract the analytical solutions in polynomial as well as rational forms.

4.1 Polynomial solutions

For the discovery of polynomial like solutions of Eq. (4.5) we assumed the relation as follow,

$$U(\zeta) = \sum_{i=0}^{n} E_i\psi^i(\zeta), \tag{4.6}$$

$$(\psi'(\zeta))^{\rho} = \sum_{i=0}^{\rho h} P_i \psi^i(\zeta), \quad \rho = 1, 2. \tag{4.7}$$

In Eq. (4.6) E_i and P_i in Eq. (4.7) represent the unknowns values. Before calculating these coefficient firstly, determine the balancing number n by homogenous balancing principal through equating the highest order derivative U'' and nonlinear term U^3 and obtain the number $n = 1$ and setup a relation like this, $n = h - 1, \forall\, h \geqslant 2$.

Case (1) Introduce the parameters, $h = 2$ *and* $\rho = 1$ and gain the outcomes,

$$U(\zeta) = E_0 + E_1\, \psi(\zeta). \tag{4.8}$$

$$\psi'(\zeta) = P_0 + P_1\, \psi(\zeta) + P_2\, \psi^2(\zeta), \tag{4.9}$$

Now, put up the Eq. (4.8) in Eq. (4.5) and performed some operation after this a system of equations are generated, that solved with Maple software for coefficients like $(E_o, E_1, P_o, P_1, P_2, m, n, v)$.

$$E_0 = -\frac{P_1}{n}, E_1 = E_1, P_0 = P_0, P_1 = P_1, P_2 = -\frac{1}{2} E_1\, n, m = \frac{n}{2}, n = n, v = \frac{1}{2} {P_1}^2 + E_1\, nP_0 - 3, \tag{4.10}$$

The derived parameters from Eq. (4.10) are plugged in Eq. (4.8) with Eq. (4.9) after performing some mathematical steps gain a remarkable result,

$$U(x, y, t) = -\frac{\tan\left(\frac{1}{2}\,\zeta\, K\right) K}{n}, \tag{4.11}$$

Here $K = \sqrt{-2\, E_1\, nP_0 - {P_1}^2}$ and ζ is wave transformation that related to (CD).

The 2D and 3D illustration of polynomial form solution of coupled (KD) equation case (1) using fractional order (CD), and the unknown values for their plotting are assumed as,($n = 0.1, E_1 = 1, P_0 = 5, P_1 = -0.06, y = 0.8, \kappa = 0.5$) and observe that amplitude of soliton wave increase and increases the value of fractional parameter $k = 0.5, 0.7, 0.9$.

Case (2) In this present case introduce the parameters, $\rho = 2$ *and* $h = 2$ and obtain the outcomes,

$$U(\zeta) = E_0 + E_1\, \psi(\zeta), \tag{4.12}$$

$$\psi'(\zeta) = \psi(\zeta)\sqrt{P_0 + P_1\, \psi(\zeta) + P_2\, \psi^2(\zeta)}, \tag{4.13}$$

Now, put up the Eq. (4.12) in Eq. (4.5) and performed some operation after this a system of equations are generated, that solved with Maple software for coefficients like $(E_o, E_1, P_o, P_1, P_2, m, n, v)$.

$$P_0 = -v - 3,\ P_1 = -2\, mE_1 + nE_1,\ P_2 = \frac{1}{4} n^2 {E_1}^2,\ E_0 = 0,\ v = v,\ n = n,\ m = m,\ E_1 = E_1, \tag{4.14}$$

The derived parameters from Eq. (4.14) are plugged in Eq. (4.12) with Eq. (4.13) after performing some mathematical steps gain a remarkable result,

$$U(\zeta) = -\frac{4(v+3)E_1 \mathrm{e}^{\zeta J}}{1 + \left((v+4)n^2 - 4\,nm + 4\,m^2\right){E_1}^2\left(\mathrm{e}^{\zeta J}\right)^2 - 2\,E_1(n-2\,m)\mathrm{e}^{\zeta J}}, \tag{4.15}$$

$J = \sqrt{-\beta - 3}$ and ζ is wave transformation that related to (CD).

The 2D and 3D illustration of polynomial form solution of coupled (KD) equation case (2) using fractional order (CD), and the unknown values for their plotting are assumed as, $(\kappa = 0.5, m = 5, n = 0.1, v = -4, y = 0.8, E_1 = 1)$ and observe that amplitude of soliton wave increase and increases the value of fractional parameter $k = 0.5, 0.7, 0.9$.

4.2 Rational solutions

To extract out the rational solutions of Eq. (4.5) we considered the relation as follow,

$$\begin{aligned} U(\zeta) &= \frac{\sum_{i=0}^{n} E_i \psi^i(\zeta)}{\sum_{i=0}^{r} F_i \psi^i(\zeta)}, \\ (\psi'(\zeta))^\rho &= \sum_{i=0}^{\rho h} P_i \psi^i(\zeta), \quad \rho = 1, 2. \end{aligned} \tag{4.16}$$

In Eq. (4.16) E_i, F_i and P_i represent the unknowns values. Before calculating these coefficient firstly, determine the balancing number n by homogenous balancing principal through equating the highest order derivative U'' and nonlinear term U^3 and obtain the number $n = 1$ and setup a relation like this, $n - r = h - 1$, for h=1 it provides $n = r$. and observe that amplitude of soliton wave increase and increases the value of fractional parameter $k = 0.5, 0.7, 0.9$.

Case (1) In this present case we Introduce the parameters, $(h = 1, \rho = 2)$ and gain the outcomes,

$$U(\zeta) = \frac{E_0 + E_1\psi(\zeta)}{F_0 + F_1\psi(\zeta)}, \tag{4.17}$$

$$\psi'(\zeta) = \sqrt{P_0 + P_1\psi(\zeta) + P_2\psi^2(\zeta)}, \tag{4.18}$$

Now, put up the Eq. (4.17) in Eq. (4.5) and performed some operation after this a system of equations are generated, that solved with Maple software for coefficients like $(E_o, E_1, F_1, F_2, P_o, P_1, P_2, m, n, v)$.

$$\begin{aligned} &P_0 = \frac{1}{4}\,\frac{n^2{E_0}^2 - 8\,mE_0F_0 + 4\,nE_0F_0 - 4\,v{F_0}^2 - 12\,{F_0}^2}{{F_1}^2},\ P_1 = \frac{-2\,mE_0 + nE_0 - 6\,F_0 - 2\,vF_0}{F_1}, \\ &P_2 = -v - 3,\ E_1 = 0,\ F_0 = F_0,\ F_1 = F_1,\ E_0 = E_0,\ m = m\,v = v,\ n = n, \end{aligned} \tag{4.19}$$

The derived parameters from Eq. (4.19) are plugged in Eq. (4.17) with Eq. (4.18) after performing some mathematical steps gain a remarkable result,

$$U(\zeta) = -\frac{8\,E_0F_1\mathrm{e}^{\zeta G}(-v-3)^{3/2}}{4\,\mathrm{e}^{2\eta G}{F_1}^2v + 4\,G\mathrm{e}^{\eta G}nE_0F_1 - 8\,G\mathrm{e}^{\zeta G}mE_0F_1 - n^2v{E_0}^2 + 12\,\mathrm{e}^{2\eta G}{F_1}^2 - 4\,n^2{E_0}^2 + 4\,nm{E_0}^2 - 4\,v^2{E_0}^2}, \tag{4.20}$$

$G = \sqrt{-\beta - 3}$ and ζ is wave transformation that related to (CD).

The 2D and 3D illustration of Rational form solution of CKDE case (1) using fractional order (CD), and the unknown values for their plotting are assumed as ($\kappa = 0.9, m = 2, v = -5, y = 0.8, E_0 = 2, F_0 = 4, F_1 = 6$) and observe that amplitude of soliton wave increase and increases the value of fractional parameter $k = 0.5, 0.7, 0.9$.

4.3 (3 + 1)-Dimensional mKdVZKE

The mKdVZKE Eq. (3.2) expressed as,

$$U_t + bU^2U_x + U_{xxx} + U_{xyy} + U_{xzz} = 0. \tag{4.21}$$

The governing nonlinear evolution solved with unified method in two ways one is polynomial form and another is rational form. The Eq. (4.21) is expressed in the form of temporal local M derivative (LCD) as,

$$D^{\kappa:\lambda}_{M,t}U + bU^2U_x + U_{xxx} + U_{xyy} + U_{xzz} = 0. \tag{4.22}$$

$D^{\kappa:\lambda}_{M,t}U$ describe about local M derivative of function U in term of independent variable t. By considering the value of κ =1 then fractional form of Eq. (4.22) transformed into its original mKdVZKE. But our concentration to solve Eq. (4.22) via unified method so for this purpose we used fractional wave transformation of following form then,

$$U(x, y, t) = G(\zeta), \quad \zeta = x + y + z - \frac{v}{\kappa}\Gamma(\lambda + 1)t^{\kappa},$$

reduced the Eq. (4.22) into simple differential equation (ODE).

$$-vG' + bG^2G' + G''' + G''' + G''' = 0. \tag{4.23}$$

For own their simplicity integrate the Eq. (4.23) and put up integrating constant ($K = 0$) in Eq. (4.23) as a conclusion,

$$-vG + \frac{bG^3}{3} + 3G'' = 0. \tag{4.24}$$

Equation (4.24) exist in its simple differential form in which dispersive and nonlinear terms are presents. Further $'$ shows about differentiation and v about velocity of solition. Now we solve this equation with unified method and extract their analytical solutions in following forms.

4.3.1 Polynomial form solutions

For the discovery of polynomial like solutions of Eq. (4.24) we assumed the relation as follow,

$$G(\zeta) = \sum_{i=0}^{n} E_i \psi^i(\zeta), \tag{4.25}$$

$$(\psi'(\zeta))^{\rho} = \sum_{i=0}^{\rho h} P_i \psi^i(\zeta), \quad \rho = 1, 2. \tag{4.26}$$

In Eq. (4.25) E_i and P_i in Eq. (4.26) represent the unknowns values. Before calculating these coefficient firstly, determine the balancing number n by homogenous balancing principal through equating the highest order derivative U'' and nonlinear term U^3 and obtain the number $n = 1$ and setup a relation like this, $n = h - 1, \forall\, h \geqslant 2$.

Case (**1**) Introduce the parameters, h = 2 and $\rho = 1$ and gain the outcomes,

$$\begin{aligned} G(\zeta) &= E_0 + E_1 \psi(\zeta), \\ \phi'(\zeta) &= P_0 + P_1 \psi(\zeta) + P_2 \psi^2(\zeta), \end{aligned} \tag{4.27}$$

Now, put up the Eq. (4.27) in Eq. (4.24) and performed some operation after this a system of equations are generated, that solved with Maple software for coefficients like $(E_0, E_1, P_0, P_1, P_2, b, v)$.

$$E_0 = \frac{1}{2}\frac{E_1 P_1}{P_2},\ b = \frac{-18 {P_2}^2}{{E_1}^2},\ v = -\frac{3}{2}{P_1}^2 + 6 P_0 P_2, E_1 = E_1, P_0 = P_0, P_1 = P_1, P_2 = P_2, \tag{4.28}$$

The derived parameters from Eq. (4.28) are plugged in Eq. (4.27) and after performing some mathematical steps gain a remarkable result,

$$G(x, y, t) = \frac{E_1 \tan\left(\frac{1}{2}\zeta\, Y\right) Y}{2 S_2}, \tag{4.29}$$

$Y = \sqrt{4 P_0 P_2 - {P_1}^2}$ and ζ is wave transformation that related to (LMD).

The 2D and 3D illustrations of polynomial form solution of the mKdVZKE case (1) using fractional order (LMD), and the unknown values for their plotting are assumed as, $(E_1 = 0.6, P_0 = 2, P_1 = -3, P_2 = 1, \kappa = 0.6, \lambda = 0.6, y = 0.44, z = 0.3)$ and observe that amplitude of soliton wave increase and increases the value of fractional parameter $k = 0.6, 0.7, 0.8$.

Case (**2**) In this present case we Introduce the parameters, $(h = 2, \rho = 2)$ and gain the outcomes,

$$\begin{aligned} G(\zeta =) E_0 + E_1 \psi(\zeta), \\ \psi'(\zeta) = \psi(\zeta)\sqrt{P_0 + P_1 \psi(\zeta) + P_2 \psi^2(\zeta)}. \end{aligned} \tag{4.30}$$

Now, put up the Eq. (4.30) in Eq. (4.24) and performed some operation after this a system of equations are generated, that solved with Maple software for coefficients like $(E_0, E_1, P_0, P_1, P_2, b, v)$.

$$\begin{aligned} P_0 &= -\frac{2}{9} b {E_0}^2, \quad P_1 = -\frac{2}{9} b E_0 E_1, \\ P_2 &= -\frac{1}{18} b {E_1}^2, \quad v = \frac{1}{3} b {E_0}^2, E_0 = E_0, E_1 = E_1, b = b. \end{aligned} \tag{4.31}$$

The derived parameters from Eq. (4.31) are plugged in Eq. (4.30) and after performing some mathematical steps gain a remarkable result,

$$G(x,y,t) = -\frac{E_0\left(4\,bE_0E_1\mathrm{e}^{\frac{1}{3}\zeta R} - 9\right)}{4\,bE_0E_1\mathrm{e}^{\frac{1}{3}\zeta R} + 9}, \tag{4.32}$$

$R = \sqrt{-2\,bE_0{}^2}$, and ζ is wave transformation that related to (LMD).

The 2D and 3D illustrations of polynomial form solution of the mKdVZKE case (2) using fractional order (LMD), and the unknown values for their plotting are assumed as, $(a_1 = 0.6, a_0 = 0.2, \beta = 0.6, y = 0.44, z = 0.3, \gamma = 0.6, \alpha = -4)$ and observe that amplitude of soliton wave increase and increases the value of fractional parameter $k = 0.6, 0.7, 0.8$.

4.3.2 Rational solutions

To extract out the rational solutions of Eq. (4.24) we considered the relation as follow,

$$\begin{aligned} G(\zeta) &= \frac{\sum_{i=0}^{n} E_i\psi^i(\zeta)}{\sum_{i=0}^{r} F_i\psi^i(\zeta)}, \\ (\psi'(\zeta))^{\rho} &= \sum_{i=0}^{\rho h} P_i\psi^i(\zeta), \quad \rho = 1, 2. \end{aligned} \tag{4.33}$$

In Eq. (4.33) E_i, F_i and P_i represent the unknowns values. Before calculating these coefficient firstly, determine the balancing number n by homogenous balancing principal through equating the highest order derivative U'' and nonlinear term U^3 and obtain the number $n = 1$ and setup a relation like this, $n - r = h - 1$, for h = 1 it provides $n = r$.

Case (1) In this present case we Introduce the parameters, $(h = 1, \rho = 2)$ and gain the outcomes,

$$\begin{aligned} G(\zeta) &= \frac{E_0 + E_1\psi(\zeta)}{F_0 + F_1\Phi(\zeta)}, \\ \psi'(\zeta) &= \sqrt{S_0 + S_1\psi(\zeta) + P_2\psi^2(\zeta)}. \end{aligned} \tag{4.34}$$

Now, put up the Eq. (4.34) in Eq. (4.24) and performed some operation after this a system of equations are generated, that solved with Maple software for coefficients like $(E_o, E_1, F_0, F_1, P_o, P_1, P_2, m, n, v)$.

$$\begin{aligned} &P_0 = -\frac{1}{6}\,\frac{v\left(E_0{}^2F_1{}^2 + 2\,E_0E_1F_0F_1 + E_1{}^2F_0{}^2\right)}{E_1{}^2F_1{}^2}, P_1 = -\frac{2}{3}\,\frac{v\left(E_0F_1 + E_1F_0\right)}{E_1F_1}, \\ &P_2 = -\frac{2}{3}\,v, b = \frac{3vF_1{}^2}{E_1{}^2}, E_0 = E_0, E_1 = E_1, F_0 = F_0, F_1 = F_1, v = v. \end{aligned} \tag{4.35}$$

The derived parameters from Eq. (4.35) are plugged in Eq. (4.34) and after performing some mathematical steps gain a remarkable result,

$$G(x,y,t)=\frac{\left(\left(\left(\mathrm{e}^{\frac{1}{3}\eta L}F_1-\frac{1}{3}LF_0\right)E_1-\frac{1}{3}LE_0F_1\right)v-3E_0F_1\left(-\frac{2}{3}v\right)^{\frac{3}{2}}\right)E_1}{\left(\left(\left(\mathrm{e}^{\frac{1}{3}\eta L}F_1-\frac{1}{3}LF_0\right)E_1-\frac{1}{3}LE_0F_1\right)v-3E_1F_0\left(-\frac{2}{3}v\right)^{\frac{3}{2}}\right)F_1}. \tag{4.36}$$

$\sqrt{-6v}=L$ and ζ is wave transformation related to (LMD).

The 3D and 2D illustration of Rational form solution of the mKdVZKE case (1) using fractional order (LMD), and the unknown values for their plotting are assumed a s , $(\kappa=0.6,\lambda=0.6,v=-0.9,y=0.44,z=0.8,E_0=-6,E_1=3,F_0=6,F_1=2.2)$ and observe that amplitude of soliton wave increase and increases the value of fractional parameter $k=0.6,0.7,0.8$.

5 Results and physical interpretation

This section describe the physical meaning of calculated fractional soliton solutions and their dynamics characteristics. The Figs. 1, 2, 3, 4, 5 and 6 represent the polynomial and rational solutions of fractional CKDE and mKdVZKE. Figures 1a, 2a and 3a represent the 3D plots at fix parameter (k = 0.5,y = 0.8) and Figs. 1b, 2b, 3b represent the 2D plots at different parameters (k = 0.5,0.7,0.9) of Eqs. (4.1), (4.15) and (4.20). Similarly, Figs. 4a, 5a and 6a represent the 3D plots at fix parameter (k = 0.6,y = 0.44) and Figs. 4b, 5b and 6b represent the 2D plots at different parameters (k = 0.6,0.7,0.8) of Eqs. (4.29), (4.32) and (4.36). These plotted graphs demonstrate the amplitude of solitary waves. As we increased the impact of fractional parameter then amplitude of solitary wave moves in upward direction that highlighted through (red, blue, and green lines) and vice versa.

6 Conclusion

In this paper, the unified method have applied with conformable derivative on the CKDE and local M fractional derivative on mKdVZKE and derived the exact solution from differential equations. These fractional soliton wave solutions exist in polynomial and rational form as well as their subcases also describes at different value of parameters. Graphs plotted in 3D and 2D form of obtained solutions that expose in Figs. 1, 2, 3, 4, 5 and 6 which describe the physical meaning of fractional models. Finally, we conclude that proposed method is simple and efficient for solving the nonlinear homogenous evolution equations (NLEEs).

Fig. 1 Polynomial form solution of coupled (KD) Eq. (4.13) of Case (1)

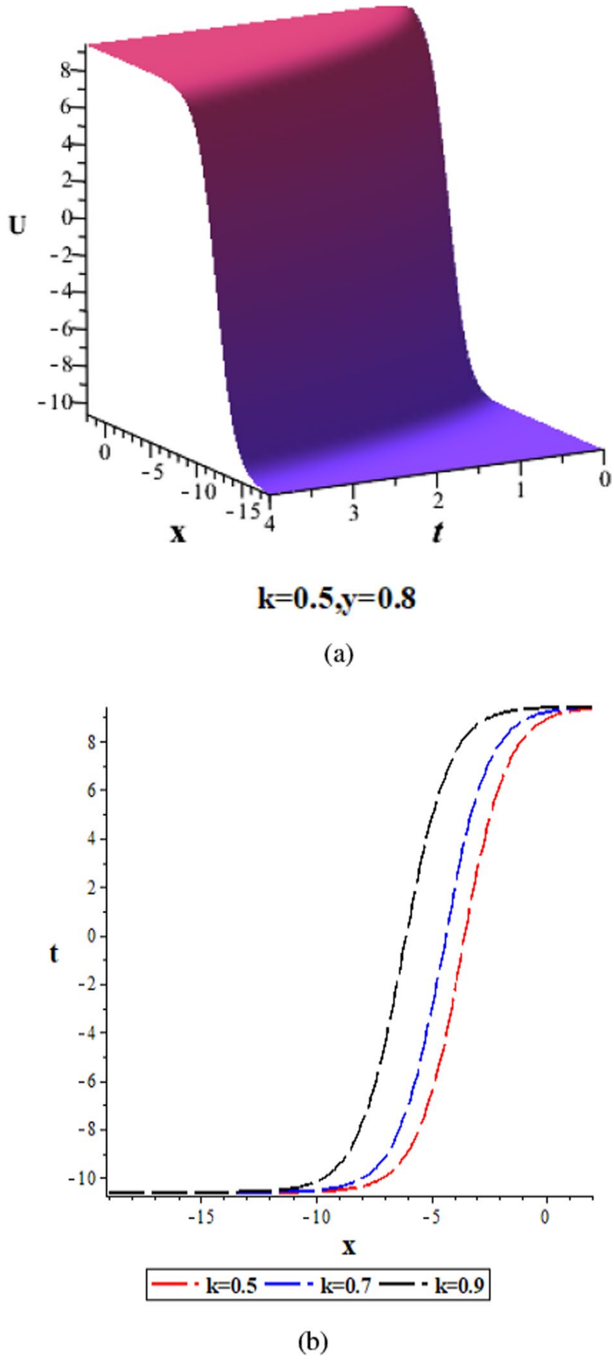


Fig. 2 Polynomial form solution of coupled (KD) Eq. (4.15) Case (2)

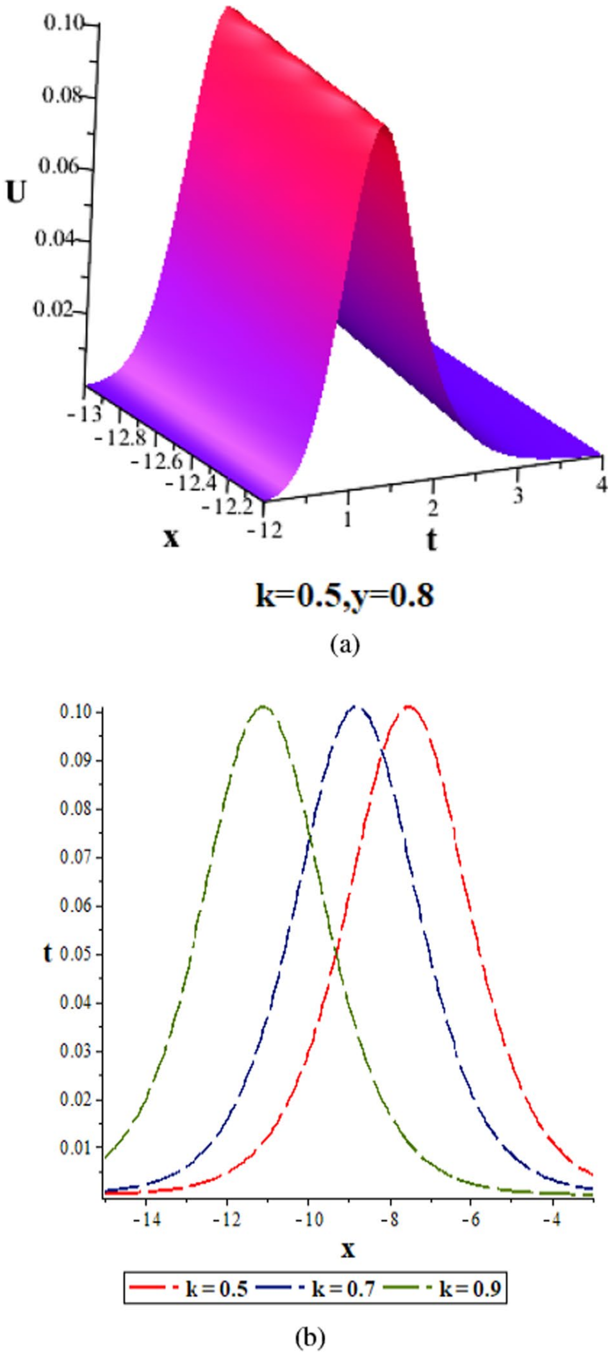
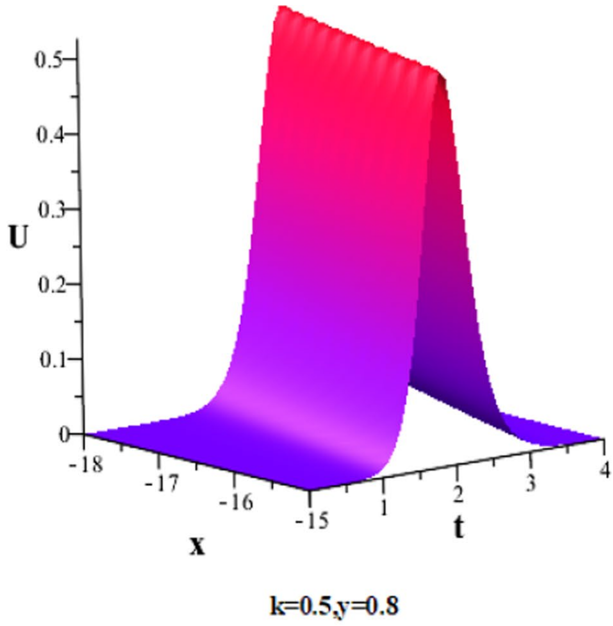
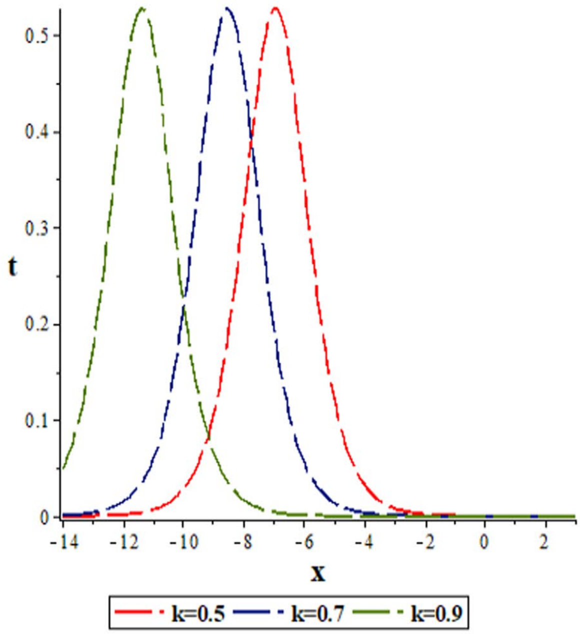


Fig. 3 Rational form solution of coupled (KD) Eq. (4.20)



(a)



(b)

Fig. 4 Polynomial form solution of (mKdVZKE) Eq. (4.29) Case (1)

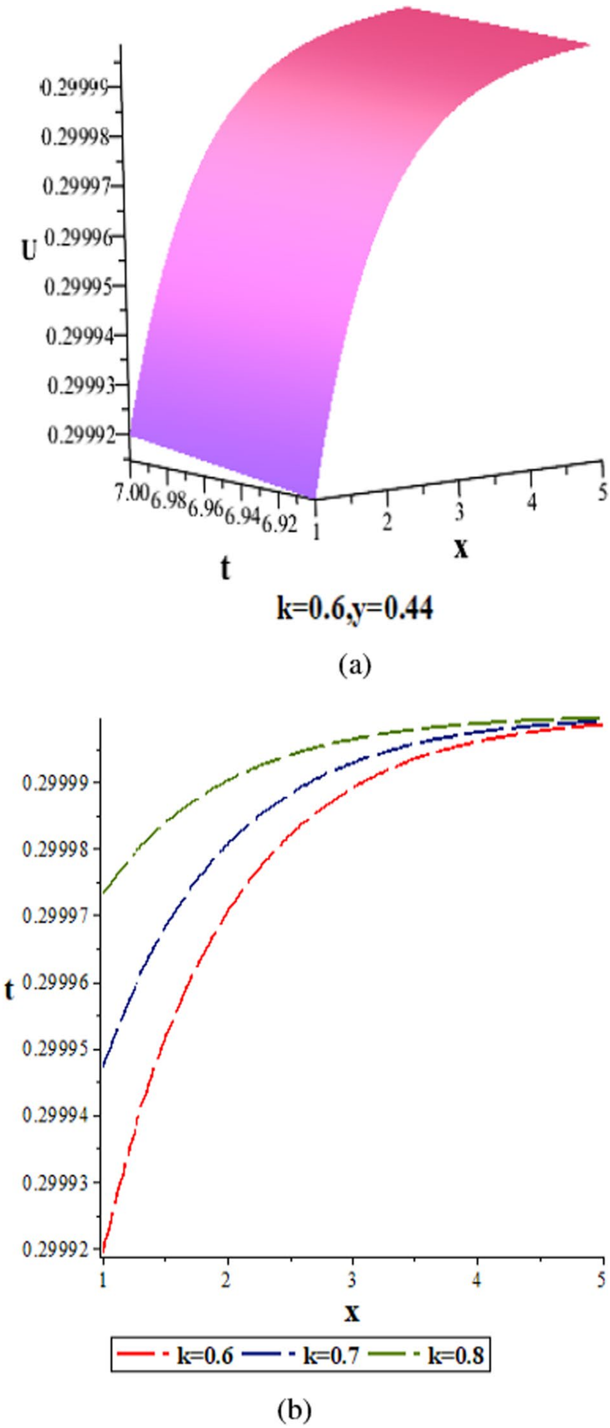


Fig. 5 Polynomial form solution of (mKdVZK) Eq. (4.32) Case (2)

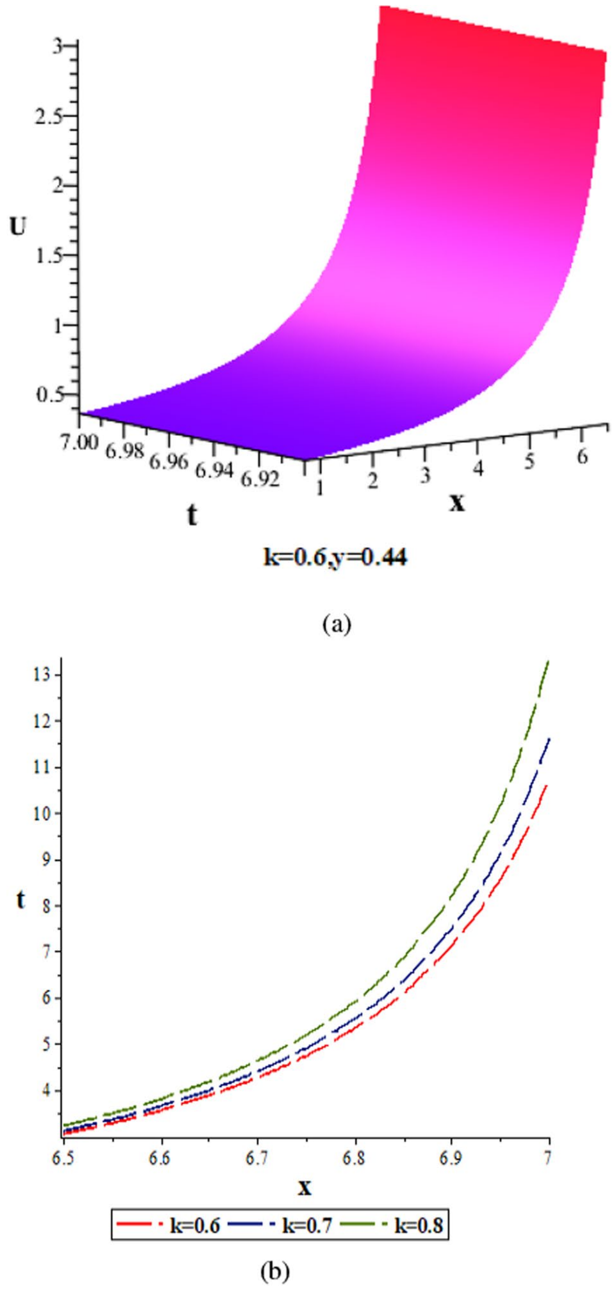
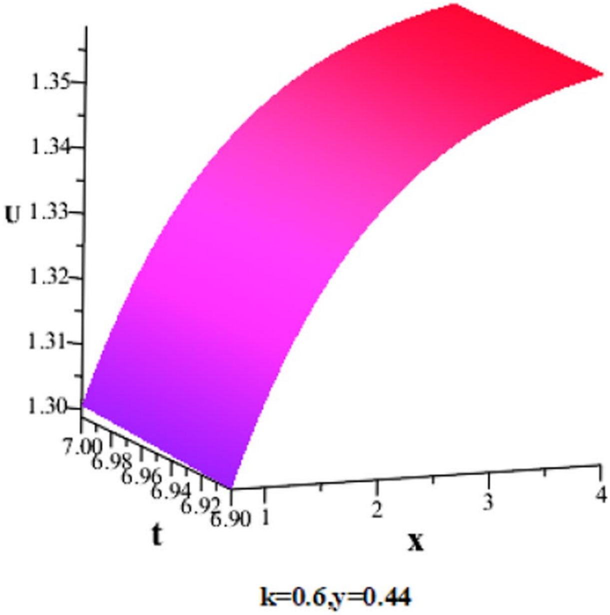
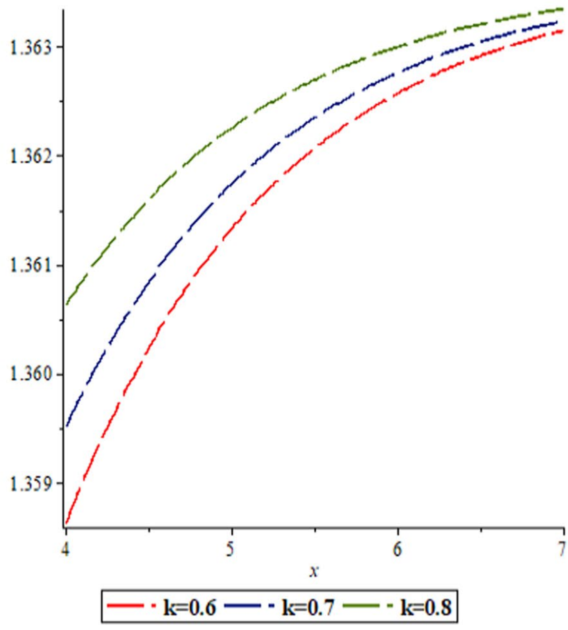


Fig. 6 Rational form solution of (mKdVZK) Eq. (4.36)



(a)



(b)

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Data availability The data sets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest There are no conflict of interest regarding the publication of this manuscript.

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