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Effect of electric field on string-oriented perfect fluid collapse in $f(R)$ gravity

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This work investigates the nature of singularity formed from string-oriented charged perfect fluid collapse in theory of $f(R)$ gravity. The spherically symmetric spacetime is assumed to be filled with a string-oriented charged perfect fluid. The gravitational collapse of the fluid is analyzed to understand the effects of electric field on the time and radii of apparent horizons and singularity formation. The presence of an electric field was discovered to increase the mass of the collapsing system, the radii of apparent horizon development, and the time gap between the event horizon and singularity. The term $f(R_0)$ acts as an anti-gravity term. However, the presence of electric field accelerates the gravitational collapse.

Keywords: Gravitational collapse; string-oriented charged perfect fluid; apparent horizons; $f(R)$ gravity.

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1. Introduction

Strings are one-dimensional vibrating filaments of energy and are the most fundamental aspects of nature. In the relativistic dynamic spacetime with fluid, the inclusion of strings covers the quantum aspects. The study of string-oriented fluids helps us to understand the quantized gravity and the unification of all the forces of

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nature [1]. A perfect fluid equipped with radially directed strings is a nonhomogeneous and anisotropic matter distribution mathematically expressed as follows:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} - \lambda x_\mu x_\nu. \tag{1}$$

Here, ρ, p and λ are the energy density, pressure and string tension density, respectively. The comoving velocity and position vectors can be expressed mathematically described as

$$u_\mu = \delta^0_\mu, \quad x^\mu = g^{11}\delta^\mu_1.$$

The string-oriented fluids are interesting to study in the scenario of gravitational collapse [2–5]. Gravitational collapse is an important phenomenon leading to creation of objects in this Universe. The creation of stars and compact stars (dwarfs, neutron stars, strange stars) is a well-known outcome of gravitational collapse. The gravitational collapse may lead to the singularities of spacetime. The collapsing string fluid in conjunction with null radiation has piqued the curiosity of general relativity [6–9].

The extended theories of gravity include the well-known theory of $f(R)$ gravity. It is the simplest generalization of general relativity with Lagrangian as a function of Ricci scalar R. The theory is based on the Hilbert–Einstein action principle:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}f(R) + S_M, \tag{2}$$

where G is Newtonian gravitational constant and S_M is the action of matter fields.

In cosmology, the theory of $f(R)$ gravity has broad applications, including dark energy and dark matter [10,11], inflation and cosmic fluctuations [12–15] and cosmic evolution [10, 16, 17]. The theory resolved several cosmological issues and made significant progress consistent with empirical evidence [18–21].

The theory of $f(R)$ gravity discusses the gravitational collapse in structure formation. Researchers discussed several cosmological models in this efficient theory with results matching the observational data [22–25]. It has been recently observed that naked singularities and black holes are not the fate of a gravitational collapse only [26]. The collapse of fluid filled spheres with singularity has been discussed in $f(R)$ gravity and the profound effects of fluid parameters have been estimated [27, 28]. The stability of compact stars is also a topic of interest in $f(R)$ gravity [29,30]. The findings are consistent with those of general relativity as well as astronomical observation data from various compact stars.

It is worth mentioning that for $f(R) = R$, the $f(R)$ theory gives the same results as in general relativity. One of the generalizations is the quadratic $f(R) = R + aR^2$ models. The collapse has been discussed in these models for the sake of compact stars. It was found that the parameter a admits constraints for the physical results. Moreover, the equation of state limits the parametric value. The positive values of the parameter increase the surface curvature, decrease the interior mass, star's cooling and redshift. The negative values of parameter lead to nonphysical results. The zero value of the parameter exhibits the general relativistic result including no surface curvature and maximum interior mass [31, 32]. The perturbative and non-perturbative approaches lead to different physical results for the positive values of a [33, 34].

The dust collapse in $f(R)$ gravity has been discussed resulting in black holes as well as naked singularities [35–37]. It was found that the $f(R)$ constant term slows down the collapse. A perfect fluid in the interior of a star is a convenient model in $f(R)$ gravity [38–40]. The work has been extended to charged perfect fluids [41, 42] where the Coulomb force slows down the gravitational collapse. The gravitational collapse of viscous dissipative, bulk viscous dissipative, charged adiabatic and dissipative charged anisotropic fluids has also been elaborated in $f(R)$ gravity [43–47]. The results agree with those of general relativity.

This work is devoted to analyze the gravitational collapse of a charged perfect fluid with attached unidirectional strings as an anisotropic fluid in $f(R)$ gravity. The next section contains the discussion of spacetime geometry and junction conditions. Section 3 consists of corresponding field equations and their solutions. Section 4 explores the time and radii of the formation of apparent horizons. The last section contains the conclusion.

2. Spacetime Geometry

Consider a spherical hypersurface Σ which divides the spacetime into two regions:

- an **Interior Region** (V^-), the Friedmann–Robertson–Walker (FRW) spacetime with line element:

$$ds_-^2 = dt^2 - A^2(t,r)dr^2 - B^2(t,r)d\Omega^2, \tag{3}$$

  where $d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2$, with t, r, θ and ϕ are time, radial, axial and azimuthal coordinates, respectively,

and

- an **Exterior Region** (V^+), the Reissner–Nordstrom spacetime with line element:

$$ds_+^2 = G(R)dT^2 - \frac{1}{G(R)}dR^2 - R^2 d\Omega^2, \tag{4}$$

  where $G(R) = 1 - 2\frac{M}{R} + \frac{Q^2}{R^2}$, M and Q are total mass and charge inside the hypersurface Σ, T and R as time and radial coordinates of the exterior spacetime.

The junction conditions (JCs) at the hypersurface are described as follows:

JC1: The line element is continuous over Σ, i.e.

$$(ds_-^2)_\Sigma = (ds_+^2)_\Sigma = ds_\Sigma^2, \tag{5}$$

JC2: The extrinsic curvature of the space-time K_{ab} is continuous over Σ, i.e.

$$[K_{ab}] = K^{+}_{ab} - K^{-}_{ab} = 0, \quad \text{where } a, b = 0, 2, 3, \tag{6}$$

where

$$K^{\pm}_{cd} = -n^{\pm}_{\zeta}\left(\frac{\partial^2 x^{\zeta}_{\pm}}{\partial\xi^c \partial\xi^d} + \Gamma^{\zeta}_{\mu\nu}\frac{\partial x^{\mu}_{\pm}}{\partial\xi^c}\frac{\partial x^{\nu}_{\pm}}{\partial\xi^d}\right), \quad (\zeta, \mu, \nu = 0, 1, 2, 3).$$

The equations of hypersurface Σ in the coordinates $x^{\gamma}_{\pm}$ are written as

$$g_{-}(r, t) : r - r_{\Sigma} = 0, \quad r_{\Sigma} = \text{constant}, \tag{7}$$

$$g_{+}(R, T) : R - R_{\Sigma}(T) = 0. \tag{8}$$

Using Eq. (7) in Eq. (3) and Eq. (8) in Eq. (4), respectively, imply

$$(ds^2_{-})_{\Sigma} = dt^2 - B^2(t, r_{\Sigma})d\Omega^2, \tag{9}$$

$$(ds^2_{+})_{\Sigma} = \left[G(R_{\Sigma}) - \frac{1}{G(R_{\Sigma})}\left(\frac{dR_{\Sigma}}{dT}\right)^2\right] dT^2 - R^2_{\Sigma}d\Omega^2. \tag{10}$$

In order to preserve the metric signature,

$$G(R_{\Sigma}) - \frac{1}{G(R_{\Sigma})}\left(\frac{dR_{\Sigma}}{dT}\right)^2 > 0.$$

Using Eqs. (9) and (10), the **JC1** (Eq. (5)) provides

$$R_{\Sigma} = B(r_{\Sigma}, t), \tag{11}$$

$$\sqrt{G(R_{\Sigma}) - \frac{1}{G(R_{\Sigma})}\left(\frac{dR_{\Sigma}}{dT}\right)^2}\, dT = dt. \tag{12}$$

The outward unit normal in V^- and V^+ can be inferred from Eqs. (7) and (8) as

$$n^{-}_{\mu} = (0, X(t, r_{\Sigma}), 0, 0), \quad n^{+}_{\mu} = (-\dot{R}_{\Sigma}, \dot{T}, 0, 0), \tag{13}$$

where dots represent differentiation with t.

The **JC2** (Eq. (6)) leads to

$$\left(\dot{R}\ddot{T} - \dot{T}\ddot{R} - \frac{G}{2}\frac{dG}{dR}\dot{T}^3 + \frac{3}{2G}\frac{dG}{dR}\dot{T}\dot{R}^2\right)_{\Sigma} = 0, \tag{14}$$

$$\left(\frac{BB'}{A}\right)_{\Sigma} = (GR\dot{T})_{\Sigma}, \tag{15}$$

where prime represent differentiation with r.

Using Eqs. (14) and (15) in Eqs. (11) and (12) leads to the following results:

$$(A\dot{B}' - \dot{A}B')_{\Sigma} = 0, \tag{16}$$

$$M = \left(\frac{B}{2} - \frac{B\dot{B}^2}{2} - \frac{BB'^2}{2A^2} + \frac{Q^2}{B^2}\right)_{\Sigma}. \tag{17}$$

3. Einstein's Maxwell Field Equations

In the presence of a charged fluid, the field equations become

$$G_{\mu\nu} = \frac{\kappa}{f(R)}(T^{(m)}_{\mu\nu} + T^{(E)}_{\mu\nu}). \tag{18}$$

where $G_{\mu\nu}$ is the Einstein tensor, $T^{(m)}_{\mu\nu}$ is the matter energy momentum tensor and $T^{(E)}_{\mu\nu}$ is the electromagnetic energy momentum tensor. We consider a radially directed string-oriented perfect fluid as $T^{(m)}_{\mu\nu}$ with energy momentum tensor given in Eq. (1). The charged content of the fluid can be expressed by the Maxwell energy momentum tensor represented as

$$T^{(E)}_{\mu\nu} = \frac{1}{4\pi}\left(F^{\lambda}_{\mu}F_{\nu\lambda} - \frac{1}{4}F^{\lambda\gamma}F_{\lambda\gamma}g_{\mu\nu}\right), \tag{19}$$

where the Maxwell field tensor $F_{\mu\nu}$ satisfies the Maxwell field equations

$$F_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu}, \; F^{\mu\nu}_{\mu} = 4\pi J^{\nu} \tag{20}$$

with $\phi_\nu = \phi\delta^0_\nu$ and $J_\mu = \sigma u_\mu$ representing the four-potential and four-current, respectively, $\phi = \phi(t,r)$ is an arbitrary function and $\sigma = \rho_e(t,r)$ is the charge density.

For the interior of a star (Eq. (3)), the Maxwell equations become

$$\phi'' - \left(\frac{A'}{A} - 2\frac{B'}{B}\right)\phi' = 4\pi\mu A^2, \tag{21}$$

$$\dot{\phi}' - \left(\frac{A'}{A} - 2\frac{B'}{B}\right)\phi' = 0, \tag{22}$$

where dot and prime indicate derivatives with respect to t and r, respectively. Integration of Eq. (21) implies that

$$\phi' = \frac{qA}{B^2}. \tag{23}$$

It is worth mentioning that Eq. (22) satisfies Eq. (23) as well.

The non-zero components of $T^{\mu\nu}_{(em)}$ turn out to be

$$T^{(em)}_{00} = T^{(em)}_{11} = -T^{(em)}_{22} = -T^{(em)}_{33} = \frac{1}{2}pq^2B^2,$$

leading to the following electric field intensity:

$$E^2(r,t) = \frac{q^2}{4\pi B^2}.$$

The operation of $f(R)$ gravity in the existence of matter is characterized by the action of matter fields in Eq. (2). The action leads to the following famous field equations of $f(R)$ gravity:

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = \kappa(T^m_{\mu\nu} + T^E_{\mu\nu}), \tag{24}$$

where $F(R) = \frac{df(R)}{dR}$ and the D'Alambert operator $\Box = \nabla^\mu\nabla_\nu$ with ∇_μ representing the covariant differentiation and κ is the coupling constant with value considered as $\kappa = 8\pi$ in geometric units. Equation (24) can be simplified to

$$R_{\mu\nu} = \frac{1}{F(R)}\left[8\pi\{(\rho+p)u_\mu u_\nu - pg_{\mu\nu} - \lambda x_\mu x_\nu + 2\pi E^2 g_{\mu\nu}\}\right.$$
$$\left. + \frac{1}{2}f(R)g_{\mu\nu} + \nabla_\mu\nabla_\nu F(R) - g_{\mu\nu}\Box F(R)\right]. \tag{25}$$

Thus, the non-zero components of Ricci tensor turn out to be

$$R_{00} = \frac{1}{F(R)}\left[8\pi(\rho + 2\pi E^2) + \frac{1}{2}f(R) + \frac{F''(R)}{A^2} - \frac{\dot{A}\dot{F}(R)}{A}\right.$$
$$\left. - \frac{\dot{A}F'(R)}{A^3} - \frac{2\dot{B}\dot{F}(R)}{B} + \frac{2B'F'(R)}{A^2B}\right],$$

$$R_{11} = \frac{A^2}{F(R)}\left[8\pi(p-\lambda-2\pi E^2) - \frac{1}{2}f(R) + \ddot{F}(R) - \frac{2B'F'(R)}{A^2B} + \frac{2\dot{B}\dot{F}(R)}{B}\right],$$

$$R_{22} = \frac{B^2}{F(R)}\left[8\pi(p-2\pi E^2) - \frac{1}{2}f(R) + \frac{\dot{B}\dot{F}(R)}{B} + \ddot{F}(R) - \frac{F''(R)}{A^2} + \frac{\dot{A}\dot{F}(R)}{A}\right.$$
$$\left. + \frac{\dot{A}F'(R)}{A^3} - \frac{2B'F'(R)}{A^2B}\right],$$

$$R_{33} = \sin^2\theta R_{22},$$

$$R_{01} = \frac{1}{F(R)}\left[\dot{F}' - \frac{A\dot{F}'}{A}\right].$$

It is difficult to solve this set of equations. For convenience, assume that $R = R_0$ leading to $F(R_0) =$ constant. This leads to the simplest case parallel to that of general relativity. Contracting Eq. (25) gives a constant Ricci scalar with the possibility of constant density and pressure, i.e. $\rho = \rho_0,\ p = p_0$. Hence, the field equations become

$$-\frac{\ddot{A}}{A} - 2\frac{\ddot{B}}{B} = \frac{1}{F(R)}[8\pi(\rho_0 + 2\pi E^2) + \frac{1}{2}f(R_0)], \tag{26}$$

$$-A^2\left[-\frac{\ddot{A}}{A} - 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{2}{A^2}\left(\frac{B''}{B} - \frac{A'}{A}\frac{B'}{B}\right)\right]$$
$$= \frac{A^2}{F(R_0)}\left[8\pi(p_0 - 2\pi E^2 - \lambda) - \frac{1}{2}f(R_0)\right], \tag{27}$$

$$-B^2\left[-\frac{\ddot{B}}{B}-\left(\frac{\dot{B}}{B}\right)^2-\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{1}{A^2}\left\{\frac{B"}{B}+\left(\frac{B'}{B}\right)^2-\frac{A'}{A}\frac{B'}{B}-\left(\frac{A}{B}\right)^2\right\}\right]$$
$$=\frac{B^2}{F(R_0)}\left[8\pi(p_0-2\pi E^2)-\frac{1}{2}f(R_0)\right], \tag{28}$$

$$-2\frac{\dot{B}'}{B}+2\frac{\dot{A}}{A}\frac{B'}{B}=0. \tag{29}$$

Integrating Eq. (29) with respect to time t gives

$$A=\frac{B'}{v(r)}. \tag{30}$$

Substituting the value of A in the field equations (26)–(29) leads to the following relation:

$$2\frac{\ddot{B}}{B}+\left(\frac{\dot{B}}{B}\right)^2+\frac{1}{B^2}(1-v^2)=\frac{1}{F(R_0)}\left[4\pi(p_0-\rho_0+\lambda)-\frac{1}{2}f(R_0)-2\kappa\pi E^2\right]. \tag{31}$$

Integrating Eq. (31) with respect to time gives

$$\dot{B}^2=v^2-1+\frac{2m(r)}{B}+\frac{1}{F(R_0)}\left[4\pi(p_0-\rho_0+\lambda)-\frac{1}{2}f(R_0)-2\kappa\pi E^2\right]\frac{B^2}{3}. \tag{32}$$

Here, $m(r)$ is the mass of the collapsing matter.

Using Eqs. (30) and (32) in Eq. (26) implies

$$m'(r)=\frac{1}{F(R_0)}[4\pi(p_0+\rho_0+\lambda)]B'B^2. \tag{33}$$

Integrating Eq. (33) corresponding to the radius gives the mass of collapsing matter, i.e.

$$m(r)=\frac{1}{F(R_0)}\{4\pi(p_0+\rho_0+\lambda)\}\int_0^r B'B^2dr+c(t), \tag{34}$$

where $c(t)$ is the time-dependent function acting as a constant of integration. The collapsing system admits a positive mass, as negative mass of the system is physically not possible. Using Eqs. (30) and (32) in Eq. (17) leads to

$$M(t,r)=m(r)+\frac{Q^2}{B^2}+S(R_0)\frac{B^3}{6}, \tag{35}$$

where

$$S(R_0):\frac{1}{F(R_0)}\left[4\pi(p_0-\rho_0+\lambda)-\frac{1}{2}f(R_0)-2\kappa\pi E^2\right].$$

Now, we discuss the solution of field equations using the assumptions

$$S(R_0)>0 \tag{36}$$

and

$$v(r) = 1. \tag{37}$$

Substituting Eqs. (30) and (37) in (32), the analytical solution in closed form comes out to be

$$A(t,r) = (6mS(R_0))^{\frac{1}{3}} \left[\frac{m'}{3m}\sinh(\phi(t,r)) + t_0'\sqrt{\frac{S(R_0)}{3}}\cosh(\phi(t,r))\right]\sinh^{\frac{-1}{3}}(\phi(t,r)), \tag{38}$$

$$B(t,r) = (6mS(R_0))^{\frac{1}{3}}\sinh^{\frac{2}{3}}(\phi(t,r)), \tag{39}$$

$$\phi(t,r) = \frac{\sqrt{S(R_0)}}{4}(t_0(r) - t). \tag{40}$$

Here, $t_0(r)$ is a random function of r, indicating the construction of appropriate shell at the distance r. When the limit $F(R_0) \to 8\pi(p_0 - \rho_0 + \lambda)$, the superior result becomes Tolman–Bondi solution [3].

4. Apparent Horizons

The outer boundaries, i.e. apparent horizons are helpful in understanding the singularity type. The apparent horizons are formed when the outer layer of two trapped spheres is fully developed. The formula of finding the outer layer of trapped spheres with null outer normals is given as follows:

$$g^{\mu\nu}B_{,\mu}B_{,\nu} = \dot{B}^2 - \left(\frac{B'}{A}\right)^2 = 0.$$

Equations (30) and (32) give

$$S(R_0)B^3 - 3B + 6m = 0. \tag{41}$$

The value of B from Eq. (41) provides the apparent horizons.

For $f(R_0) = 8\pi(p_0 - \rho_0 + \lambda) - 4\kappa\pi E$, we have $B = 2m$, i.e. Schwarzschild horizon. The Tschirunhaus–Vieta approach gives the roots of Eq. (41) as follows:

- $B_1 = \frac{2}{\sqrt{S(R_0)}}\cos(\frac{\psi}{3}\psi)$,
- $B_2 = \frac{-1}{\sqrt{S(R_0)}}[\cos(\frac{\psi}{3}) + \sqrt{3}\sin(\frac{\psi}{3})]$,
- $B_3 = \frac{-1}{\sqrt{S(R_0)}}[\cos(\frac{\psi}{3}) - \sqrt{3}\sin(\frac{\psi}{3})]$.

For $S(R_0) > 0$ in Eq. (41), three cases arise for the sake of solution:

Case 1: $3m < \frac{1}{\sqrt{S(R_0)}}$,

Case 2: $3m = \frac{1}{\sqrt{S(R_0)}}$,

Case 3: $3m < \frac{1}{\sqrt{S(R_0)}}$.

The solutions obtained for the three mentioned cases are given as follows:

Case 1: When $3m < \frac{1}{\sqrt{S(R_0)}}$, Eq. (41) gives two positive roots and hence there form two apparent horizons, namely cosmological horizon B_c and black hole horizon $B_{\rm bh}$, i.e.

$$B_c = \frac{2}{\sqrt{S(R_0)}} \cos\left(\frac{\psi}{3}\right), \tag{42}$$

$$B_{\rm bh} = \frac{-1}{\sqrt{S(R_0)}} \left[\cos\left(\frac{\psi}{3}\right) - \sqrt{3}\sin\left(\frac{\psi}{3}\right)\right], \tag{43}$$

where

$$\cos\psi = -3m\sqrt{S(R_0)}. \tag{44}$$

For $m = 0$, it follows from Eqs. (42) and (43) that

$$B_c = \sqrt{\frac{3}{S(R_0)}} \quad \text{and} \quad B_{\rm bh} = 0.$$

B_c and $B_{\rm bh}$ may be generalized for $m \neq 0$ and $f(R_0) \neq 8\pi(p_0 - \rho_0 + \lambda) - 4\kappa\pi E^2$, respectively, [27].

Case 2: For $3m = \frac{1}{\sqrt{S(R_0)}}$, we observe from Eqs. (42) and (43) that both the horizons coincide, i.e.

$$B_c = B_{\rm bh} = \frac{1}{S(R_0)} = B. \tag{45}$$

Equation (41) leads to a single horizon with limits defined as follows:

$$0 \leq B_{\rm bh} \leq \frac{1}{S(R_0)} \leq B_c \leq \frac{3}{\sqrt{S(R_0)}}. \tag{46}$$

The surface area of black hole horizon is defined as

$$4\pi B^2 = \frac{4\pi}{S(R_0)}. \tag{47}$$

The cosmological horizon ranges between $\frac{4\pi}{S(R_0)}$ and $\frac{6\pi}{S(R_0)}$.

Case 3: For $3m > \frac{1}{\sqrt{S(R_0)}}$, there are no positive roots of Eq. (41) and correspondingly no apparent horizon will occur. Equations (39) and (41) give the time of horizon formation as

$$t_n = t_0 - \frac{2}{3S(R_0)} \sinh^{-1} \sqrt{\frac{B}{2m} - 1}, \quad n = 1, 2. \tag{48}$$

Here, t_1, t_2 and t_0 express the time of formation of cosmological horizon, black hole horizon and singularity, respectively. In limit $F(R_0) \to 8\pi(p_0 - \rho_0 + \lambda)$, we get the result corresponding to Tolman–Bondi in general relativity as

$$t_{\rm bh} = t_0 - \frac{4}{3}m. \tag{49}$$

It can be observed from Eq. (48) that the cosmological horizon forms earlier than the black hole horizon. From Eq. (48) we can write

$$\frac{L_n}{2m} = (\cosh\beta_n)^2. \tag{50}$$

Equation (46) shows that $B_c \geq B_{\rm bh}$ and Eq. (48) yields $t_1 \leq t_2$ which is the time of cosmological horizon formation that is less than the time for black hole horizon.

We can calculate the time lapse between the construction of cosmological horizon and singularity and the construction of black hole horizon and singularity, respectively, using Eqs. (42) to (44) as

$$\frac{d\frac{B_c}{2m}}{dm} = \frac{1}{m}\left[\frac{3\cos(\frac{\psi}{3})}{\cos\psi} - \frac{\sin(\frac{\psi}{3})}{\sin\psi}\right] < 0, \tag{51}$$

$$\frac{d\frac{B_{\rm bh}}{2m}}{dm} = \frac{1}{m}\left[\frac{3\cos(\frac{\psi+4\pi}{3})}{\cos\psi} - \frac{\sin(\frac{\psi+4\pi}{3})}{\sin\psi}\right] > 0. \tag{52}$$

Defining τ as the time variation between the singularity and apparent horizons formation as

$$\tau_n = t_0 - t_n. \tag{53}$$

From Eq. (50), we have

$$\frac{d\tau_1}{d(\frac{B_c}{2m})} = \frac{1}{\sinh\phi_n\cosh\phi_n\sqrt{3S(R_0)}}. \tag{54}$$

Equations (52) and (54) lead to the following result:

$$\frac{d\tau_1}{dm} = \frac{d\tau_1}{d\frac{L_1}{2m}}\frac{d\frac{B_c}{2m}}{dm} = \frac{\left[\frac{3\cos\frac{\psi}{3}}{\cos\psi} - \frac{\sin\frac{\psi}{3}}{\sin\psi}\right]}{3m\sqrt{S(R_0)}\sinh\phi_1\cosh\phi_1} < 0. \tag{55}$$

Equation (55) implies that the time variation between the construction of cosmological horizon and singularity decreases in comparison to the mass variations.

Now from Eqs. (53) and (54), we have

$$\frac{d\tau_2}{dm} = \frac{\left[\frac{3\cos\frac{\psi+3\phi}{3}}{\cos\psi} - \frac{\sin\frac{\psi+3\phi}{3}}{\sin\psi}\right]}{m\sqrt{3S(R_0)}\sinh\phi_2\cosh\phi_2} > 0. \tag{56}$$

Equation (56) implies that the time variation between the construction of cosmological horizon and singularity increases in comparison to the mass variations.

5. Conclusion

This work describes the final fate of gravitational collapse of charged string-oriented perfect fluid in $f(R)$ gravity. We have taken the function $f(R) = R$ which is the general relativistic case. Since the work has not been done in general relativity before, therefore it is interesting to discuss this function. A string-oriented charged perfect fluid is considered to collapse within the hypersurface, a junction between the non-static exterior and the static interior spacetimes. The source of the exterior spacetime (the fluid) is non-static. The Maxwell equations lead to the components of electromagnetic energy momentum tensor. The field equations of $f(R)$ gravity are solved to get a closed analytic solution. It is found that a black hole is formed as the event horizon is formulated before the occurrence of apparent horizons. The effect of charge on the collapsing fluid is elaborated. The summary of the results is as follows:

- The electric field decreases the mass of the system.
- The term $f(R_0)$ acts like anti-gravity term and slows down the gravitational collapse. However, the presence of electric force E diminishes the effects of $f(R_0)$.
- The collapse slows down due to the presence of term $f(R_0)$ which acts like a repulsive force to the gravitational collapse. The term $f(R_0)$ alters the Newtonian potential ($\phi = \frac{1}{2}(1 - g_{00})$) or

$$\phi = \frac{m}{R} + \frac{R^2}{6} S(R_0).$$

  Differentiating corresponding to R gives the Newtonian force as

$$F = -\frac{m}{R^2} + \frac{RS(R_0)}{3}.$$

  This force becomes ineffective for

$$m = \frac{1}{3\sqrt{S(R_0)}}, \quad R = \frac{1}{\sqrt{S(R_0)}}. \tag{57}$$

- Without encountering the electric field, a larger mass and radius are predicted for the resulting collapsed object. The presence of electric force E decreases the values of m and R. Thus, E accelerates the gravitational collapse.
- The presence of electric field increases the interval between formation of apparent horizons and singularity.
- The results can be validated by the following comparison:
  - When $E = 0, \lambda = 0$ the results reduce to [38].
  - When $E = 0$ the results reduce to [48].
  - When $\lambda = 0$ the results reduce to [49].

It is well known that the three famous equations of state for the perfect fluid are dust, radiation and stiff fluid. Accordingly, the string-oriented perfect fluid admits the states of string dust, stringy radiation and stringy stiff fluid. The results of our research leads to the following interesting key points concerning these states as follows:

- For the string dust (SD) case $p = 0$ in string-oriented perfect fluid (SPF). Due to vanishing pressure, the term

$$S_{\mathrm{SD}}(R_0) : \frac{1}{F(R_0)}\left[4\pi(-\rho_0 + \lambda) - \frac{1}{2}f(R_0) - 2\kappa\pi E^2\right] < S(R_0)_{\mathrm{SPF}}.$$

— The collapse slows down due to the presence of term $f(R_0)$ which acts like a repulsive force to the gravitational collapse. Equation (57) indicates that the Newtonian force becomes ineffective at a higher masses and larger radii as compared to those in string-oriented perfect fluid case.
— Time of apparent (cosmological and black hole) horizons' formation (Eqs. (42) and (43)) would be longer than it is for the string-oriented perfect fluid.
— The surface area of black hole horizon (Eq. (47)) would be more than that of the surface area of black hole formed due to string-oriented perfect fluid collapse.
— Time lag between horizon formation and singularity (Eq. (48)) would be longer than that of in the case of string-oriented perfect fluid.

- For the stringy radiation (SR) case $p = \frac{1}{4}\rho$. The term

$$S_{\mathrm{SR}}(R_0) : \frac{1}{F(R_0)}\left[4\pi(-\frac{3}{4}\rho_0 + \lambda) - \frac{1}{2}f(R_0) - 2\kappa\pi E^2\right] > S_{\mathrm{SD}}(R_0).$$

— The collapse slows down due to the presence of term $f(R_0)$ which acts like a repulsive force to the gravitational collapse. Equation (57) indicates that the Newtonian force becomes ineffective at smaller masses and smaller radii as compared to that in string dust case.
— Time of apparent (cosmological and black hole) horizons' formation (Eqs. (42) and (43)) in stringy radiation would be shorter than it is for the string dust.
— The surface area of black hole horizon (Eq. (47)) would be less than that of the surface area of black hole formed due to string dust.
— Time lag between horizon formation and singularity (Eq. (48)) would be shorter than that of in the case of string dust.

- For the stringy stiff matter (SSM) case $p = \rho$,

$$S_{\mathrm{SSM}}(R_0) : \frac{1}{F(R_0)}\left[4\pi(\lambda) - \frac{1}{2}f(R_0) - 2\kappa\pi E^2\right] > S_{\mathrm{SR}}(R_0) > S_{\mathrm{SD}}(R_0).$$

— The collapse slows down due to the presence of term $f(R_0)$ which acts like a repulsive force to the gravitational collapse. Equation (57) indicates that the Newtonian force becomes ineffective at smaller masses and smaller radii as compared to that in stringy radiation and string dust cases.
— Time of apparent (cosmological and black hole) horizons' formation (Eqs. (42) and (43)) would be shorter than it is for the stringy radiation and string dust.

— The surface area of black hole horizon (Eq. (47)) would be less than that of the surface area of black hole formed due to stringy radiation and string dust collapse.
— Time lag between horizon formation and singularity (Eq. (48)) would be shorter than that of in the cases of stringy radiation and string dust.

- It is worth mentioning that in all the above-mentioned cases, the electric field and the constant $f(R_0)$ term effects are the same as that in the case of string-oriented perfect fluid.

It is interesting to discuss the dynamical collapse of the same fluid taking higher order models in $f(R)$ gravity.

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References

[1] A. Z. Jones, *String Theory for Dummies* (John Wiley and Sons, 2009).
[2] U. Sheikh, S. Liaqut, Z. Yousaf and M. Z. Bhatti, Information paradox for a collapsing string cloud in rainbow gravity, *Int. J. Geom. Methods Mod. Phys.* **18**(06) (2021) 2150084.
[3] U. Sheikh, M. Z. Bhatti and Z. Yousaf, Gravitational collapse of a self-gravitating unidirectional fluid flow, *New Astron.* **84** (2021) 101533.
[4] U. Sheikh, K. Mahmood and M. Arshad, Analysis of a charged unidirectional perfect fluid gravitational collapse in cylindrical spacetime, *Chin. J. Phys.* **73** (2021) 503–511.
[5] U. Sheikh and S. Arshad, Electromagnetic effects on dynamics of string fluid and information paradox in rainbow gravity, *Ann. Phys.* **441** (2021) 168855.
[6] S. G. Ghosh and N. N. Saste, Gravitational collapse of spherical string fluids, *Int. J. Mod. Phys.* **13**(02) (2004) 263–272.
[7] M. Govender, Nonadiabatic spherical collapse with a two-fluid atmosphere, *Int. J. Mod. Phys.* **22**(07) (2013) 1350049.
[8] S. D. Maharaj and B. P. Brassel, Radiating composite stars with electromagnetic fields, *Eur. Phys. J. C* **81**(9) (2021) 783.
[9] S. D. Maharaj and B. P. Brassel, Radiating stars with composite matter distributions, *Eur. Phys. J. C* **81**(4) (2021) 366.
[10] S. Capozziello and M. De Laurentis, Extended theories of gravity, *Phys. Rep.* **509**(4–5) (2011) 167-321.
[11] V. K. Oikonomou, Unifying inflation with early and late dark energy epochs in axion $F(R)$ gravity, *Phys. Rev. D* **103**(4) (2021) 044036.
[12] V. K. Oikonomou, Rescaled Einstein-Hilbert gravity from $f(R)$ gravity: Inflation, dark energy, and the swampland criteria, *Phys. Rev. D* **103**(12) (2021) 124028.
[13] S. Aziz, S. K. Jha and A. Rahaman, The inflationary scenario in the $f(R)$ gravity model with a R^4 term, *Class. Quantum Grav.* **38**(22) (2021) 225008.

[14] Z. Haghani and T. Harko, Effects of quantum metric fluctuations on the cosmological evolution in Friedmann-Lemaitre-Robertson-Walker geometries, *Physics* **3**(3) (2021) 689–714.
[15] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution, *Phys. Rep.* **692** (2017) 1–104.
[16] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, k-essence $f(R)$ gravity inflation, *Nucl. Phys. B.* **941** (2019) 11–27.
[17] A. S. Agrawal, S. K. Tripathy and B. Mishra, Gravitational baryogenesis models comparison in $f(R)$ Gravity, *Chin. J. Phys.* **71** (2021) 333–340.
[18] S. Tsujikawa, Observational signatures of $f(R)$ dark energy models that satisfy cosmological and local gravity constraints, *Phys. Rev. D* **77** (2008) 023507.
[19] S. Nojiri S. D. Odintsov and D. Saez-Gomez, Cosmological reconstruction of realistic modified $F(R)$ gravities, *Phys. Lett. B* **681** (2009) 74–80.
[20] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Unifying inflation with early and late-time dark energy in $F(R)$ gravity, *Phys. Dark Univ.* **29** (2020) 100602.
[21] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Modified gravity and cosmology, *Phys. Rep.* **513**(1–3) (2012) 1–189.
[22] G. Abbas and H. Nazar, Hybrid star model with quark matter and baryonic matter in minimally coupled $f(R)$ gravity, *Ann. Phys.* **424** (2021) 168336.
[23] S. H. Mazharimousavi, M. Halilsoy and K. Kianfar, Thin-shell wormholes in $f(R)$-gravity coupled with nonlinear electrodynamics, *Eur. Phys. J. Plus* **135** (2020) 440.
[24] T. Tangphati, A. Chatrabhuti, D. Samart and P. Channuie, Traversable wormholes in $f(R)$ massive gravity, *Phys. Rev. D* **102**(8) (2020) 084026.
[25] G. Mustafa, I. Hussain and M. F. Shamir, Stable wormholes in the background of an exponential $f(R)$ gravity, *Universe* **6**(4) (2020) 48.
[26] A. Usman and M. F. Shamir, Collapsing stellar structures in $f(R)$ gravity using Karmarkar condition, *New Astron.* **91** (2022) 101691.
[27] M. Sharif and Z. Yousaf, Stability analysis of cylindrically symmetric self-gravitating systems in $R + \epsilon R^2$ gravity, *Mon. Not. R. Astron. Soc.* **440**(4) (2014) 3479–3490.
[28] M. Sharif and Z. Yousaf, Effects of CDTT model on the stability of spherical collapse in Palatini $f(R)$ gravity, *Eur. Phys. J. C* **73** (2013) 2633.
[29] M. F. Shamir and I. Fayyaz, Effect of $f(R)$-gravity models on compact stars, *Theor. Math. Phys.* **202** (2020) 112–125.
[30] M. F. Shamir and I. Fayyaz, Analysis of compact stars in logarithmic-corrected R^2 gravity, *Mod. Phys. Lett. A* **35**(2) (2020) 1950354.
[31] A. V. Astashenok, S. D. Odintsov and A. Cruz-Dombriz, The realistic models of relativistic stars in $f(R) = R + \alpha R^2$ gravity, *Class. Quantum Grav.* **34**(20) (2017) 205008.
[32] A. Cooney, S. DeDeo and D. Psaltis, Neutron stars in $f(R)$ gravity with perturbative constraints, *Phys. Rev. D* **82**(6) (2010) 064033.
[33] S. S. Yazadjiev, D. D. Doneva, K. D. Kokkotas and K. V. Staykov, Non-perturbative and self-consistent models of neutron stars in $R-$squared gravity, *J. Cosmol. Astropart. Phys.* **2014**(06) (2014) 003.
[34] S. Arapoglu, C. Deliduman and K. Y. Eksi, Constraints on perturbative $f(R)$ gravity via neutron stars, *J. Cosmol. Astropart. Phys.* **2011**(07) (2011) 020.
[35] S. G. Ghosh and S. D. Maharaj, Gravitational collapse of null dust in $f(R)$ gravity, *Phys. Rev. D* **85** (2012) 124064.
[36] M. F. Shamir, Z. Ahmad and Z. Raza, Gravitational dust collapse in $f(R)$ gravity, *Int. J. Theor. Phys.* **54**(5) (2015) 1450–1460.

[37] R. A. Sussman and L. G. Jaime, Lemaître-Tolman-Bondi dust solutions in $f(R)$ gravity, *Class. Quantum Grav.* **34** (2017) 245004.
[38] M. Sharif and H. R. Kausar, Gravitational perfect fluid collapse in $f(R)$ gravity, *Astrophys. Space Sci.* **331** (2011) 281–288.
[39] H. Nazar and G. Abbas, Gravitational collapse in $f(R)$ theories of gravity with non-minimal coupling, *Mod. Phys. Lett. A* **34**(03) (2019) 1950025.
[40] A. V. Astashenok, K. Mosani, S. D. Odintsov and G. C. Samanta, Gravitational collapse in general relativity and in R^2-gravity: A comparative study, *Int. J. Geom. Methods Mod. Phys.* **16**(03) (2019) 1950035.
[41] M. Sharif and N. Farooq, Charged cylindrically symmetric collapse in $f(R)$ gravity, *Eur. Phys. J. Plus* **132**(8) (2017) 1–11.
[42] M. Sharif and N. Farooq, Study of the charged spherical stellar model in $f(R)$ gravity, *Eur. Phys. J. Plus* **133**(2) (2018) 1–9.
[43] G. Abbas and H. Nazar, Dynamics of dissipative gravitational collapse with full causal approach in $f(R)$ gravity, *Can. J. Phys.* **98**(7) (2020) 613–621.
[44] M. Sharif and N. Farooq, Charged bulk viscous cylindrical collapse in $f(R)$ theory, *Int. J. Mod. Phys.* **27**(02) (2018) 1850013.
[45] M. Sharif and Z. Yousaf, Charged adiabatic LTB gravitational collapse in $f(R)$ gravity, *Int. J. Theor. Phys.* **55** (2016) 470–480.
[46] M. F. Shamir and M. A. Fayyaz, Dissipative cylindrical collapse of a charged anisotropic fluid in $f(R)$ gravity, *Can. J. Phys.* **95**(12) (2017) 1278–1284.
[47] M. Z. Bhatti, Z. Yousaf and A. Yousaf, Tolman mass of spherical fluids with electromagnetic field, *Mod. Phys. Lett. A* **34**(02) (2019) 1950012.
[48] M. Zahid, *A Study of String Fluid Collapse in $f(R)$ Gravity* (National Textile University, Faisalabad, 2020).
[49] N. Chakrabarty and A. Banerjee, Charged perfect fluid in general relativity, *Prog. Theor. Phys.* **60**(4) (1978) 1226–1227.