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The modulation instability analysis and generalized fractional propagating patterns of the Peyrard–Bishop DNA dynamical equation

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Abstract
This research examines the fractional Peyrard–Bishop DNA dynamical governing system, which displays the proliferation of optical pulses in field of plasma and the optical fibre. The analytical method is utilized to find travelling wave solutions because the inverse scattering transform cannot solve the Cauchy problem for this equation. The Φ^6-model expansion method is an efficient and dependable technique for generating the generalised solitonic wave profiles with a wide range of soliton families. The main advantage of the offered analytical strategy is that it specifies a constraint for each solution to guarantee its existence. As a result, solitonic wave structures get attributes such as the Jacobi elliptic function, periodicity, brightness, dark-brightness, singularity, exponential, trigonometry, and rational solitonic structures, among others, under existence conditions that have not been explored previously. The results are represented graphically in 2-D, 3-D, and contour glimpses to illustrate the behavioural responses to pulse propagation by inferring the fitting values of system parameters. The stabilty of the considered model is discussed and develope the stability condition. The fractional parameter is responsible for reducing singularity and continuing to increase the smoothness in wave patterns. It is easy to employ the Φ^6-model expansion method to other complicated non-linear systems and acquire solitary waves pattern.

Keywords M-truncated fractional operator · Φ^6-model · Analytical solution · Peyrard–Bishop DNA dynamical governing system · Modulation instability

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1 Introduction

A double helix is a natural structure for a Deoxyribonucleic acid molecule. This signifies that it has two polymeric chains that are corresponding and are entangled (Manafian et al. 2020). The double helix with two strings is the Watson-Crick model of B-shaped DNA. It is possible to consider of the homogeneous crystal structure because the masses of nucleotides are not very different. The harmonic longitudinal length is strong because hydrogen bonds hold strings together, whereas these bonds are weak. The Peyrard–Bishop equation completely ignores any displacements along transverse (Zafar et al. 2020). In the literature, Morse's potential originates before the Hamiltonian formation of Peyrard and Bishop equations as,

$$\mathbb{U}_m(\mathbb{Q}_n - u_n) = D[e^{-v(\mathbb{Q}_n - u_n)} - 1]^2. \tag{1}$$

Here,the Morse's potential is represented by $\mathbb{U}_m(\mathbb{Q}_n - u_n)$, $\mathbb{Q}_n$ and u_n are the displacements of the nucleotides, and D and v are the depth and the width of the Morse's potential. Zdravkovc also proposed the Hamiltonian for the DNA chain (ZdravkoviĆ 2011). Furthermore, Dauxois has suggested an improved version of the Peyrard–Bishop model (Dauxois 1991). The hydrogen bonds' strand apertures can be explained by the following Hamiltonian (Agüero et al. 2008),

$$\mathcal{H}(\mathbb{Q}) = \frac{1}{2m}v_n^2 + \frac{\gamma_1}{2}\Delta^2\mathbb{Q}_n + \frac{\gamma_2}{4}\Delta^4\mathbb{Q}_n + \delta(e^{-v\sqrt{2}\mathbb{Q}_n})^2, \quad \Delta\mathbb{Q}_n = \mathbb{Q}_{n+1} - \mathbb{Q}_n, \tag{2}$$

where γ_1 and γ_2 indicate the linear and nonlinear coupling strengths, respectively. The δ is a constant and $v_n = m\mathbb{Q}_n$ is the momentum for the displacement $\mathbb{Q}_n$. The equation of motion in the continuum limit can be stated by the following form, starting with the Hamiltonian Eq. (2),

$$\mathbb{Q}_{tt} - (\phi_1 + 3\phi_2\mathbb{Q}_{xx}) - 2\sqrt{2}kDe^{-k\mathbb{Q}}(e^{-k\mathbb{Q}} - 1) = 0, \tag{3}$$

with $\phi_1 = \frac{\gamma_1}{m}d_2$, $\phi_2 = \frac{\gamma_2}{m}d_4$, $D = \frac{\delta}{m}$, $k = \sqrt{2}v$ and being d is the inter-site nucleotide distance in the DNA ladder. Consider the following time fractional Peyrard–Bishop DNA dynamic model equation in this paper,

$${}_iD_{M,t}^{2\alpha,\gamma}\mathbb{Q} - (\phi_1 + 3\phi_2(\mathbb{Q}_x)^2)\mathbb{Q}_{xx} - 2k\omega e^{-k\mathbb{Q}}(e^{-k\mathbb{Q}} - 1) = 0. \tag{4}$$

Here, $\phi_1,\ \phi_2,\ k\ and\ \omega\ =\ D$ are constants.

The mathematical Peyrard–Bishop model of DNA describes the nonlinear interaction between neighboring displacements and hydrogen bonds. It is an efficient and simple model to elucidate the mathematical description of mechanical denaturation. The importance of DNA, the molecule that serves as the foundation of the genetic code, cannot be overstated. The mechanical characteristics of the material are particularly important because accessing the genetic information and replicating the molecule for replication both require opening the double helix structure of DNA.

There have been numerous techniques and methodologies developed to guarantee analytical exact solutions for nonlinear PDEs like as, Kudryashov scheme (Zafar et al. 2022; Srivastava et al. 2020), sine-Gordon expansion algorithm (Fahim et al. 2022; Abdul Kayum et al. 2022), bilinear neural network approach (Zhang et al. 2021; Zhang and Bilige 2019), extended simple equation technique (Khater et al. 2021), F-expansion procedure (Karaman 2021; Yıldırım 2021), unified auxiliary equation strategy (Zayed and Shohib 2019; Zayed et al. 2021), $\frac{G'}{G}$–expansion method (Ismael et al. 2020), Hirota bilinear methodology (Abdulkadir Sulaiman and Yusuf 2021), generalized exponential function approach (Khodadad et al. 2021), and many others (Ghanbari et al. 2021; Lü 2005; Baskonus et al. 2018; Rehman et al. 2020; Zhang et al. 2021). Liu et al. (2022) analyzed the dissipative Burgers equation in sense of fractional theory and developed Lie symmetry algebra and constructed the conservation laws. Liu et al. (2022) investigated the generalized diffusion wave equation via κ–Hilfer-Prabhakar ($\kappa - H - P$) fractional operator and acquired the analytical solutions. Liu et al. (2022) has been derived the new Korteweg-de Vries-like equation. Liu et al. (2022) discussed the Kadomtsev–Petviashvili (KP) equation and multiple solitons displayed by using the linear superposition principle.

This considered model (4) is investigated by many researchers in different contexts and established results. Peyrard and Bishop (Peyrard and Bishop 1989; Abazari et al. 2018) have examined the oscillator-chain in order to investigate the appearance of solitonic structures. In the research of Dusuel et al. (1998) and Alvarez et al. (2006), the examination of scientific and physical modelling of DNA element characteristics indicates that they can be reduced to a fundamental non-linear configuration. The non-linearity of DNA energetic model manifests itself as localised waves with numerous outstanding features, such as transferring vitality without dispersing.

Alharthi et al. (2022) examined the Peyrard–Bishop DNA dynamical model recently in (2022) and created four alternative analytical solutions utilising the Kudryashov analytical method. As a result, many solitons-type solutions appear to be mysterious and there is a gap in the literature like Jacobi elliptic function, periodic, dark, bright, singular, dark-bright, exponential, trigonometric, and rational solitonic structures, etc.. The second issue is that no condition for the reported solutions' existence has been stated. In order to fill this gap, an extended approach ϕ^6-model expansion technique is used, which yields more than twenty-eight analytical solutions for the new and more generalised solitary waves that have never been produced previously. It also illustrates the constraint condition corresponding to each solution for their existence.

This study is categorized as, Sect. 2 is providing overview and solutions, Sect. 3 is fixed for graphical illustration, Sect. 4 is presenting dynamical investigation, and at last, concluding remarks.

2 M-truncated fractional derivative

Definition 2.1 The single-parameter truncated Mittag-Leffler function (Sousa and Oliveira 2018) is defined as:

$$ {}_iE_\theta(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(\theta k + 1)}, $$

in which $\theta > 0$ and $z \in C$. It is defined as follows in terms of a non-fuzzy idea.

Definition 2.2 Suppose that

$$ f : [0, \infty) \to \mathbb{R}, $$

and $\omega \in (0,1)$ the M-truncated derivative of f of order ω is defined by:

$$D_M^{\omega,\theta} f(\phi) = \lim_{\varepsilon \to 0} \frac{f(\phi + {}_iE_\theta(\varepsilon \phi^{-\omega})) - f(\phi)}{\varepsilon},$$

for $\phi > 0$ and ${}_iE_\theta(.)$, $\theta > 0$.

Theorem 2.3 *Suppose that g is differentiable function of ω order at $\phi_0 > 0$ with $\omega \in (0,1]$ and $\theta > 0$ then, f is continuous at ϕ_0.*

Theorem 2.4 *If $\omega \in (0,1]$, $\theta > 0$, f, h are differentiable up to ω order at $\phi > 0$, then*:

1- $D_M^{\omega,\theta}(rf(\phi) + fh(\phi)) = rD_M^{\omega,\theta}(f(\phi)) + sD_M^{\omega,\theta}(h(\phi))$, where r, s are real constants.

2- $D_M^{\omega,\theta}(\phi^{\omega}) = \omega\phi^{\omega-\omega}, \qquad \omega \in \mathbb{R}$.

3- $D_M^{\omega,\theta}(f(\phi)h(\phi)) = f(\phi)D_M^{\omega,\theta}(h(\phi)) + h(\phi)D_M^{\omega,\theta}(f(\phi))$.

4- $D_M^{\omega,\theta}(\frac{f(\phi)}{h(\phi)}) = \frac{f(\phi)D_M^{\omega,\theta}(h(\phi)) - h(\phi)D_M^{\omega,\theta}(f(\phi))}{h(\phi)^2}$.

5- $D_M^{\omega,\theta}(f)(\phi) = \frac{\phi^{1-\omega}}{\Gamma(\theta+1)}\frac{df}{d\phi}$.

6- $D_M^{\omega,\theta}(f \circ h)(\phi) = g'(h(\phi))D_M^{\omega,\theta}h(\phi)$.

3 Application of the NAEM

In this section, we merely implement the proposed technique to identify soliton solutions to the RLW problem using Beta and M-Truncated fractional derivatives, respectively (Table 1). We illustrate the graphs of acquired solutions by adjusting the fractional parameter η to appropriate values.

Table 1 Limiting cases for functions

The Jacobi elliptic functions			
No.	Functions	$s \to 1$	$s \to 0$
1	$sn(\Upsilon, s)$	$\tanh(\Upsilon)$	$\sin(\Upsilon)$
2	$cn(\Upsilon, s)$	sech (Υ)	$\cos(\Upsilon)$
3	$dn(\Upsilon, s)$	sech (Υ)	1
4	$ns(\Upsilon, s)$	$\coth(\Upsilon)$	$\csc(\Upsilon)$
5	$cs(\Upsilon, s)$	csch (Υ)	$\cot(\Upsilon)$
6	$ds(\Upsilon, s)$	csch (Υ)	$\csc(\Upsilon)$
6	$sc(\Upsilon, s)$	$\sinh(\Upsilon)$	$\tan(\Upsilon)$
8	$sd(\Upsilon, s)$	$\sinh(\Upsilon)$	$\sin(\Upsilon)$
9	$nc(\Upsilon, s)$	$\cosh(\Upsilon)$	$\sec(\Upsilon)$
10	$cd(\Upsilon, s)$	1	$\cos(\Upsilon)$

4 Construction of analytical solutions

4.1 The Φ^6-model expansion method

Assume a general differential equation:

$$\mathbb{R}(\mathbb{Q}, \mathbb{Q}_t, \mathbb{Q}_x, \mathbb{Q}_{tt}, \mathbb{Q}_{xx}, ...) = 0. \tag{5}$$

It is possible to convert it into an ordinary differential equation:

$$\mathfrak{M}(\mathfrak{P}, \mathfrak{P}', \mathfrak{P}'', ...) = 0. \tag{6}$$

Have used following transformation:

$$\mathbb{Q}(x,t) = \mathfrak{P}(\Upsilon), \tag{7}$$

where, $\Upsilon = k_1 x + k_2 t$. Assuming that Eq. (6)'s solution has the following form:

$$\mathfrak{P}(\Upsilon) = \sum_{m=0}^{2j} \left[a_m \mathbb{R}^m(\Upsilon)\right], \tag{8}$$

where j is homogeneous balancing constant. The function $\mathbb{R}(\Upsilon)$ satisfies ,

$$\begin{aligned} \mathbb{R}'^2(\Upsilon) &= h_0 + h_2\mathbb{R}^2(\Upsilon) + h_4\mathbb{R}^4(\Upsilon) + h_6\mathbb{R}^6(\Upsilon), \\ \mathbb{R}''(\Upsilon) &= h_2\mathbb{R}(\Upsilon) + 2h_4\mathbb{R}^3(\Upsilon) + 3h_6\mathbb{R}^5(\Upsilon). \end{aligned} \tag{9}$$

The Eq. (9) satisfies,

$$\mathbb{R}(\Upsilon) = \frac{\Phi(\Upsilon)}{\sqrt{f\Phi^2(\Upsilon) + g}}, \tag{10}$$

where $f\Phi^2(\Upsilon) + g > 0$ and $\Phi(\Upsilon)$ is the solution of Jacobi elliptic equation,

$$\Phi'^2(\Upsilon) = l_0 + l_2\Phi^2(\Upsilon) + l_4\Phi^4(\Upsilon), \tag{11}$$

where $l_0, l,$ and l_4 are constants to be determined, while f and g are the given as,

$$f = \frac{h_4(l_2 - h_2)}{3l_0l_4 + (h_2^2 - l_2^2)}, \quad g = \frac{3h_4l_0}{3l_0l_4 + (h_2^2 - l_2^2)}, \tag{12}$$

under the constraint,

$$h_4^2(l_2 - h_2)[9l_0l_4 - (l_2 - h_2)(2l_2 + h_2)] + 3h_6[-l_2^2 + h_2^2 + 3l_0l_4]^2 = 0.$$

4.2 Travelling wave structures of the Eq. (4)

We construct a travelling wave transformation to locate solutions to the Eq. (4):

$$\mathbb{Q}(x,t) = \mathfrak{P}(\Upsilon), \text{ where, } \Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha). \tag{13}$$

We get ODE by employing the next wave transformation (13) to (4).

$$\mu^2\mathfrak{P}'' - (\phi_1 + 3\phi_2(\mathfrak{P}')^2)\mathfrak{P}'' - 2k\omega e^{-k\mathfrak{P}}(e^{-k\mathfrak{P}} - 1) = 0. \tag{14}$$

Let be a hypothesis,

$$\mathcal{H} = e^{-k\mathfrak{P}}, \tag{15}$$

We have the non-linear ordinary differential equation by plugging Eq. (15) into Eq. (14),

$$\begin{aligned} &k^2(\phi_1 - \mu^2)\mathcal{H}^3\mathcal{H}'' - k^2(\phi_1 - \mu^2)\mathcal{H}^2(\mathcal{H}')^2 + 3\phi_2\mathcal{H}(\mathcal{H}')^2\mathcal{H}'' \\ &\quad - 3\phi_2(\mathcal{H}')^4 - 2k^4\omega\mathcal{H}^6 + 2k^4\omega\mathcal{H}^5 = 0. \end{aligned} \tag{16}$$

The homogeneous balancing constant $j = 1$ for Eq. (16), thus,

$$\mathcal{H} = b_0 + b_1\mathbb{R}(\Upsilon) + b_2\mathbb{R}^2(\Upsilon), \tag{17}$$

where,

$$\begin{aligned} &\mathbb{R}'^2(\Upsilon) = h_0 + h_2\mathbb{R}^2(\Upsilon) + h_4\mathbb{R}^4(\Upsilon) + h_6\mathbb{R}^6(\Upsilon), \\ &\mathbb{R}''(\Upsilon) = h_2\mathbb{R}(\Upsilon) + 2h_4\mathbb{R}^3(\Upsilon) + 3h_6\mathbb{R}^5(\Upsilon). \end{aligned} \tag{18}$$

The solution (17) is plugging into Eq. (16) and get a system,

$$\begin{aligned}
&\mathbb{R}^0 : -2k^4\omega {b_0}^6 + 2k^4\omega {b_0}^5 - 2k^2{b_0}^3\mu^2 b_2 h_0 + k^2{b_0}^2\mu^2{b_1}^2 h_0 + 2k^2{b_0}^3\phi_1 b_2 h_0 - k^2{b_0}^2\phi_1{b_1}^2 h_0 \\
&\quad + 6\phi_2 b_0{b_1}^2{h_0}^2 b_2 - 3\phi_2{b_1}^4{h_0}^2 = 0, \\
&\mathbb{R}^1 : -12\,k^4\omega\,{b_0}^5 b_1 + 10\,k^4\omega\,{b_0}^4 b_1 - k^2\mu^2{b_0}^3 b_1 h_2 - 2\,k^2\mu^2{b_0}^2 b_1 b_2 h_0 + 2\,k^2\mu^2 b_0{b_1}^3 h_0 + k^2{b_0}^3 b_1 h_2\phi_1 \\
&\quad + 2\,k^2{b_0}^2 b_1 b_2 h_0\phi_1 - 2\,k^2 b_0{b_1}^3 h_0\phi_1 + 3\,b_0{b_1}^3 h_0 h_2\phi_2 + 24\,b_0 b_1{b_2}^2{h_0}^2\phi_2 - 18\,{b_1}^3 b_2{h_0}^2\phi_2 = 0, \\
&\mathbb{R}^2 : -12\,k^4\omega\,{b_0}^5 b_2 - 30\,k^4\omega\,{b_0}^4{b_1}^2 + 10\,k^4\omega\,{b_0}^4 b_2 \\
&\quad + 20\,k^4\omega\,{b_0}^3{b_1}^2 - 4\,k^2\mu^2{b_0}^3 b_2 h_2 - 2\,k^2\mu^2{b_0}^2{b_1}^2 h_2 \\
&\quad - 2\,k^2\mu^2{b_0}^2{b_2}^2 h_0 + 4\,k^2\mu^2 b_0{b_1}^2 b_2 h_0 + k^2\mu^2{b_1}^4 h_0 \\
&\quad + 4\,k^2{b_0}^3 b_2 h_2\phi_1 + 2\,k^2{b_0}^2{b_1}^2 h_2\phi_1 + 2\,k^2{b_0}^2{b_2}^2 h_0\phi_1 \\
&\quad - 4\,k^2 b_0{b_1}^2 b_2 h_0\phi_1 - k^2{b_1}^4 h_0\phi_1 + 30\,b_0{b_1}^2 b_2 h_0 h_2\phi_2 \\
&\quad + 24\,b_0{b_2}^3{h_0}^2\phi_2 - 3\,{b_1}^4 h_0 h_2\phi_2 - 42\,{b_1}^2{b_2}^2{h_0}^2\phi_2 = 0, \\
&\mathbb{R}^3 : -60\,k^4\omega\,{b_0}^4 b_1 b_2 - 40\,k^4\omega\,{b_0}^3{b_1}^3 + 40\,k^4\omega\,{b_0}^3 b_1 b_2 \\
&\quad + 20\,k^4\omega\,{b_0}^2{b_1}^3 - 2\,k^2\mu^2{b_0}^3 b_1 h_4 - 11\,k^2\mu^2{b_0}^2 b_1 b_2 h_2 \\
&\quad - k^2\mu^2 b_0{b_1}^3 h_2 + 4\,k^2\mu^2 b_0 b_1{b_2}^2 h_0 + 4\,k^2\mu^2{b_1}^3 b_2 h_0 \\
&\quad + 2\,k^2{b_0}^3 b_1 h_4\phi_1 + 11\,k^2{b_0}^2 b_1 b_2 h_2\phi_1 + k^2 b_0{b_1}^3 h_2\phi_1 \\
&\quad - 4\,k^2 b_0 b_1{b_2}^2 h_0\phi_1 - 4\,k^2{b_1}^3 b_2 h_0\phi_1 + 6\,b_0{b_1}^3 h_0 h_4\phi_2 + 3\,b_0{b_1}^3{h_2}^2\phi_2 + 84\,b_0 b_1{b_2}^2 h_0 h_2\phi_2 \\
&\quad - 15\,{b_1}^3 b_2 h_0 h_2\phi_2 - 48\,b_1{b_2}^3{h_0}^2\phi_2 = 0,
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{R}^4: -30\,k^4\omega\,b_0{}^4b_2{}^2-120\,k^4\omega\,b_0{}^3b_1{}^2b_2-30\,k^4\omega\,b_0{}^2b_1{}^4+20\,k^4\omega\,b_0{}^3b_2{}^2+60\,k^4\omega\,b_0{}^2b_1{}^2b_2+10\,k^4\omega\,b_0b_1{}^4\\
&\quad-6\,k^2\mu^2b_0{}^3b_2h_4-5\,k^2\mu^2b_0{}^2b_1{}^2h_4-8\,k^2\mu^2b_0{}^2b_2{}^2h_2-8\,k^2\mu^2b_0b_1{}^2b_2h_2+2\,k^2\mu^2b_0b_2{}^3h_0+7\,k^2\mu^2b_1{}^2b_2{}^2h_0\\
&\quad+6\,k^2b_0{}^3b_2h_4\phi_1+5\,k^2b_0{}^2b_1{}^2h_4\phi_1+8\,k^2b_0{}^2b_2{}^2h_2\phi_1+8\,k^2b_0b_1{}^2b_2h_2\phi_1-2\,k^2b_0b_2{}^3h_0\phi_1-7\,k^2b_1{}^2b_2{}^2h_0\phi_1\\
&\quad+48\,b_0b_1{}^2b_2h_0h_4\phi_2+24\,b_0b_1{}^2b_2h_2{}^2\phi_2+72\,b_0b_2{}^3h_0h_2\phi_2-30\,b_1{}^2b_2{}^2h_0h_2\phi_2-24\,b_2{}^4h_0{}^2\phi_2=0,\\
&\mathbb{R}^5: -120\,k^4\omega\,b_0{}^3b_1b_2{}^2-120\,k^4\omega\,b_0{}^2b_1{}^3b_2-12\,k^4\omega\,b_0b_1{}^5+60\,k^4\omega\,b_0{}^2b_1b_2{}^2+40\,k^4\omega\,b_0b_1{}^3b_2+2\,k^4\omega\,b_1{}^5\\
&\quad-3\,k^2\mu^2b_0{}^3b_1h_6-20\,k^2\mu^2b_0{}^2b_1b_2h_4-4\,k^2\mu^2b_0b_1{}^3h_4-11\,k^2\mu^2b_0b_1b_2{}^2h_2-k^2\mu^2b_1{}^3b_2h_2+6\,k^2\mu^2b_1b_2{}^3h_0\\
&\quad+3\,k^2b_0{}^3b_1h_6\phi_1+20\,k^2b_0{}^2b_1b_2h_4\phi_1+4\,k^2b_0b_1{}^3h_4\phi_1+11\,k^2b_0b_1b_2{}^2h_2\phi_1+k^2b_1{}^3b_2h_2\phi_1-6\,k^2b_1b_2{}^3h_0\phi_1\\
&\quad+9\,b_0b_1{}^3h_0h_6\phi_2+9\,b_0b_1{}^3h_2h_4\phi_2+120\,b_0b_1b_2{}^2h_0h_4\phi_2+60\,b_0b_1b_2{}^2h_2{}^2\phi_2\\
&+6\,b_1{}^3b_2h_0h_4\phi_2+3\,b_1{}^3b_2h_2{}^2\phi_2-36\,b_1b_2{}^3h_0h_2\phi_2=0,\\
&\mathbb{R}^6: -40\,k^4\omega\,b_0{}^3b_2{}^3-180\,k^4\omega\,b_0{}^2b_1{}^2b_2{}^2-60\,k^4\omega\,b_0b_1{}^4b_2\\
&\quad-2\,k^4\omega\,b_1{}^6+20\,k^4\omega\,b_0{}^2b_2{}^3+60\,k^4\omega\,b_0b_1{}^2b_2{}^2\\
&\quad+10\,k^4\omega\,b_1{}^4b_2-8\,k^2\mu^2b_0{}^3b_2h_6-8\,k^2\mu^2b_0{}^2b_1{}^2h_6\\
&\quad-14\,k^2\mu^2b_0{}^2b_2{}^2h_4-20\,k^2\mu^2b_0b_1{}^2b_2h_4-4\,k^2\mu^2b_0b_2{}^3h_2\\
&\quad-k^2\mu^2b_1{}^4h_4-2\,k^2\mu^2b_1{}^2b_2{}^2h_2+2\,k^2\mu^2b_2{}^4h_0+8\,k^2b_0{}^3b_2h_6\phi_1\\
&\quad+8\,k^2b_0{}^2b_1{}^2h_6\phi_1+14\,k^2b_0{}^2b_2{}^2h_4\phi_1\\
&\quad+20\,k^2b_0b_1{}^2b_2h_4\phi_1+4\,k^2b_0b_2{}^3h_2\phi_1+k^2b_1{}^4h_4\phi_1\\
&\quad+2\,k^2b_1{}^2b_2{}^2h_2\phi_1-2\,k^2b_2{}^4h_0\phi_1+66\,b_0b_1{}^2b_2h_0h_6\phi_2\\
&\quad+66\,b_0b_1{}^2b_2h_2h_4\phi_2+96\,b_0b_2{}^3h_0h_4\phi_2+48\,b_0b_2{}^3h_2{}^2\phi_2+3\,b_1{}^4h_0h_6\phi_2+3\,b_1{}^4h_2h_4\phi_2\\
&\quad+24\,b_1{}^2b_2{}^2h_0h_4\phi_2+12\,b_1{}^2b_2{}^2h_2{}^2\phi_2-24\,b_2{}^4h_0h_2\phi_2=0,\\
&\mathbb{R}^7: -120\,k^4\omega\,b_0{}^2b_1b_2{}^3-120\,k^4\omega\,b_0b_1{}^3b_2{}^2-12\,k^4\omega\,b_1{}^5b_2\\
&\quad+40\,k^4\omega\,b_0b_1b_2{}^3+20\,k^4\omega\,b_1{}^3b_2{}^2-29\,k^2\mu^2b_0{}^2b_1b_2h_6\\
&\quad-7\,k^2\mu^2b_0b_1{}^3h_6-26\,k^2\mu^2b_0b_1b_2{}^2h_4-6\,k^2\mu^2b_1{}^3b_2h_4\\
&\quad-k^2\mu^2b_1b_2{}^3h_2+29\,k^2b_0{}^2b_1b_2h_6\phi_1+7\,k^2b_0b_1{}^3h_6\phi_1\\
&\quad+26\,k^2b_0b_1b_2{}^2h_4\phi_1+6\,k^2b_1{}^3b_2h_4\phi_1+k^2b_1b_2{}^3h_2\phi_1\\
&\quad+12\,b_0b_1{}^3h_2h_6\phi_2+6\,b_0b_1{}^3h_4{}^2\phi_2+156\,b_0b_1b_2{}^2h_0h_6\phi_2\\
&\quad+156\,b_0b_1b_2{}^2h_2h_4\phi_2+27\,b_1{}^3b_2h_0h_6\phi_2+27\,b_1{}^3b_2h_2h_4\phi_2+24\,b_1b_2{}^3h_0h_4\phi_2+12\,b_1b_2{}^3h_2{}^2\phi_2=0,\\
&\mathbb{R}^8: -30\,k^4\omega\,b_0{}^2b_2{}^4-120\,k^4\omega\,b_0b_1{}^2b_2{}^3-30\,k^4\omega\,b_1{}^4b_2{}^2+10\,k^4\omega\,b_0b_2{}^4+20\,k^4\omega\,b_1{}^2b_2{}^3-20\,k^2\mu^2b_0{}^2b_2{}^2h_6\\
&\quad-32\,k^2\mu^2b_0b_1{}^2b_2h_6-10\,k^2\mu^2b_0b_2{}^3h_4-2\,k^2\mu^2b_1{}^4h_6\\
&\quad-11\,k^2\mu^2b_1{}^2b_2{}^2h_4+20\,k^2b_0{}^2b_2{}^2h_6\phi_1+32\,k^2b_0b_1{}^2b_2h_6\phi_1\\
&\quad+10\,k^2b_0b_2{}^3h_4\phi_1+2\,k^2b_1{}^4h_6\phi_1+11\,k^2b_1{}^2b_2{}^2h_4\phi_1\\
&\quad+84\,b_0b_1{}^2b_2h_2h_6\phi_2+42\,b_0b_1{}^2b_2h_4{}^2\phi_2+120\,b_0b_2{}^3h_0h_6\phi_2\\
&\quad+120\,b_0b_2{}^3h_2h_4\phi_2+6\,b_1{}^4h_2h_6\phi_2+3\,b_1{}^4h_4{}^2\phi_2+78\,b_1{}^2b_2{}^2h_0h_6\phi_2+78\,b_1{}^2b_2{}^2h_2h_4\phi_2=0,\\
&\mathbb{R}^9: -60\,k^4\omega\,b_0b_1b_2{}^4-40\,k^4\omega\,b_1{}^3b_2{}^3+10\,k^4\omega\,b_1b_2{}^4\\
&\quad-41\,k^2\mu^2b_0b_1b_2{}^2h_6-11\,k^2\mu^2b_1{}^3b_2h_6-8\,k^2\mu^2b_1b_2{}^3h_4\\
&\quad+41\,k^2b_0b_1b_2{}^2h_6\phi_1+11\,k^2b_1{}^3b_2h_6\phi_1+8\,k^2b_1b_2{}^3h_4\phi_1\\
&\quad+15\,b_0b_1{}^3h_4h_6\phi_2+192\,b_0b_1b_2{}^2h_2h_6\phi_2+96\,b_0b_1b_2{}^2h_4{}^2\phi_2\\
&\quad+48\,b_1{}^3b_2h_2h_6\phi_2+24\,b_1{}^3b_2h_4{}^2\phi_2+84\,b_1b_2{}^3h_0h_6\phi_2+84\,b_1b_2{}^3h_2h_4\phi_2=0,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\mathbb{R}^{10}: -12\,k^4\omega\,b_0{b_2}^5-30\,k^4\omega\,{b_1}^2{b_2}^4+2\,k^4\omega\,{b_2}^5-16\,k^2\mu^2b_0{b_2}^3h_6-20\,k^2\mu^2{b_1}^2{b_2}^2h_6-2\,k^2\mu^2{b_2}^4h_4\\
&\quad+16\,k^2b_0{b_2}^3h_6\phi_1+20\,k^2{b_1}^2{b_2}^2h_6\phi_1+2\,k^2{b_2}^4h_4\phi_1+102\,b_0{b_1}^2b_2h_4h_6\phi_2\\
&\quad+144\,b_0{b_2}^3h_2h_6\phi_2+72\,b_0{b_2}^3{h_4}^2\phi_2\\
&\quad+9\,{b_1}^4h_4h_6\phi_2+132\,{b_1}^2{b_2}^2h_2h_6\phi_2+66\,{b_1}^2{b_2}^2{h_4}^2\phi_2+24\,{b_2}^4h_0h_6\phi_2+24\,{b_2}^4h_2h_4\phi_2=0,\\
&\mathbb{R}^{11}: -12\,k^4\omega\,b_1{b_2}^5-15\,k^2\mu^2b_1{b_2}^3h_6+15\,k^2b_1{b_2}^3h_6\phi_1+9\,b_0{b_1}^3{h_6}^2\phi_2+228\,b_0b_1{b_2}^2h_4h_6\phi_2\\
&\quad+69\,{b_1}^3b_2h_4h_6\phi_2+144\,b_1{b_2}^3h_2h_6\phi_2+72\,b_1{b_2}^3{h_4}^2\phi_2=0,\\
&\mathbb{R}^{12}: -2\,k^4\omega\,{b_2}^6-4\,k^2\mu^2{b_2}^4h_6+4\,k^2{b_2}^4h_6\phi_1+60\,b_0{b_1}^2b_2{h_6}^2\phi_2+168\,b_0{b_2}^3h_4h_6\phi_2+6\,{b_1}^4{h_6}^2\phi_2\\
&\quad+186\,{b_1}^2{b_2}^2h_4h_6\phi_2+48\,{b_2}^4h_2h_6\phi_2+24\,{b_2}^4{h_4}^2\phi_2=0,\\
&\mathbb{R}^{13}: 132\,b_0b_1{b_2}^2{h_6}^2\phi_2+45\,{b_1}^3b_2{h_6}^2\phi_2+204\,b_1{b_2}^3h_4h_6\phi_2=0,\\
&\mathbb{R}^{14}: 96\,b_0{b_2}^3{h_6}^2\phi_2+120\,{b_1}^2{b_2}^2{h_6}^2\phi_2+72\,{b_2}^4h_4h_6\phi_2=0,\\
&\mathbb{R}^{15}: 132\,\phi_2b_1{b_2}^3{h_6}^2=0,\\
&\mathbb{R}^{16}: 48\,\phi_2{b_2}^4{h_6}^2=0.
\end{aligned}
\tag{19}
$$

The aforementioned system (19) is solved via `Maple Software`.

Set 1:

$$
\left[b_0=0, b_1=0, b_2=\pm\frac{2\sqrt{3\phi_2}\,h_4}{k^2\sqrt{\omega}},\ \mu=\mu, h_0=0, h_2=\pm\sqrt{3}k^2\frac{\sqrt{3\phi_2}(\mu^2-\phi_1)-6\,\phi_2\sqrt{\omega}}{36\phi_2\sqrt{\phi_2}}, h_6=0.\right]
\tag{20}
$$

Set 2:

$$
\left[b_0=b_0, b_1=0, b_2=\pm 2\,\frac{\sqrt{3}h_4}{k^2}\sqrt{\frac{\phi_2}{\omega}},\ \mu=\pm\sqrt{2\sqrt{3}\sqrt{\frac{\phi_2}{\omega}}\omega\,b_0-2\sqrt{3}\sqrt{\frac{\phi_2}{\omega}}\omega+\phi_1},\ h_0=\frac{k^4\omega\,{b_0}^2}{12\,h_4\phi_2}, h_2=\pm\frac{1}{3}b_0\sqrt{3}k^2\frac{1}{\sqrt{\frac{\phi_2}{\omega}}}, h_6=0.\right]
\tag{21}
$$

Now, we will use only **Set 1** to construct solution for the sake of space. After substituting (20) into (17), we get,

$$
\mathbb{Q}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\mathbb{R}^2(\Upsilon)\right),
\tag{22}
$$

where $\mathcal{A}=\pm\frac{2\sqrt{3\phi_2}\,h_4}{k^2\sqrt{\omega}}$.

Category 1. if $l_0=1, l_2=-1-s^2, l_4=s^2, 0<s<1$, then $\Phi(\Upsilon)=sn(\Upsilon,s)$ or $\Phi(\Upsilon)=cd(\Upsilon,s)$, thus we have,

$$\mathbb{Q}_1(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{sn^2(\Upsilon,s)}{fsn^2(\Upsilon,s)+g}\right) \tag{23}$$

or

$$\mathbb{Q}_2(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{cd^2(\Upsilon,s)}{fcd^2(\Upsilon,s)+g}\right), \tag{24}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^{\alpha})$ and the functions f and g are,

$$f = \frac{h_4\left(-s^2 - 1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{3\,s^2 - \left(-s^2-1\right)^2 + \left(\frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2}$$

$$g = \frac{3\,h_4}{3\,s^2 - \left(-s^2-1\right)^2 + \left(\frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

under the constraint condition,

$$\begin{aligned} 0 = {h_4}^2&\left(-s^2 - 1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\\ &\left(9\,s^2 - \left(-s^2 - 1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-2\,s^2 - 2 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right). \end{aligned}$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{1,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\tanh^2(\Upsilon,s)}{f\tanh^2(\Upsilon,s)+g}\right), \tag{25}$$

under the constraint condition,

$$\begin{aligned} 0 = -\frac{{h_4}^2}{144\,{\phi_2}^2}&\left(-2 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{\sqrt{\frac{\phi_2}{\omega}}}\right)\Bigg[4\sqrt{3}\sqrt{\frac{\phi_2}{\omega}}k^4\mu^2\omega - 4\sqrt{3}\sqrt{\frac{\phi_2}{\omega}}k^4\omega\,\phi_1\\ &- k^4\mu^4 + 2\,k^4\mu^2\phi_1 - 48\sqrt{3}\sqrt{\frac{\phi_2}{\omega}}k^2\omega\,\phi_2 - 12\,k^4\omega\,\phi_2 - k^4{\phi_1}^2\\ &+24\,k^2\mu^2\phi_2 - 24\,k^2\phi_1\phi_2 - 144\,{\phi_2}^2\Bigg]. \end{aligned}$$

It is possible for us to obtain a single wave solution, when $s \to 0$,

$$\mathbb{Q}_{2,1}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\sin^2(\Upsilon,s)}{f\sin^2(\Upsilon,s)+g}\right),\tag{26}$$

or

$$\mathbb{Q}_{2,2}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\cos^2(\Upsilon,s)}{f\cos^2(\Upsilon,s)+g}\right),\tag{27}$$

under the constraint condition,

$$0=-{h_4}^2\left(-1-\frac{k^2 12\mu^2}{\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 2. if $l_0=1-s^2, l_2=2s^2-1, l_4=-s^2, 0<s<1$, then $\Phi(\Upsilon)=cn(\Upsilon,s)$, thus we have,

$$\mathbb{Q}_3(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{cn^2(\Upsilon,s)}{fcn^2(\Upsilon,s)+g}\right),\tag{28}$$

where $\Upsilon=x-\frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f=\frac{h_4\left(2s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{-3\left(-s^2+1\right)s^2-\left(2s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

$$g=\frac{3\left(-s^2+1\right)h_4}{-3\left(-s^2+1\right)s^2-\left(2s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

under the constraint condition,

$$\begin{aligned}0={h_4}^2&\left(2s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\\&\left(-9\left(-s^2+1\right)s^2-\left(2s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right.\\&\left.\left(4s^2-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).\end{aligned}$$

It is possible for us to obtain a single wave solution, when $s\to 1$,

$$\mathbb{Q}_{3,1}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\operatorname{sech}^2(\Upsilon,s)(\Upsilon,s)}{f\operatorname{sech}^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{29}$$

under the constraint condition,

$$0=-{h_4}^2\left(1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 3. if $l_0=s^2-1, l_2=2-s^2, l_4=-1, 0<s<1$, then $\Phi(\Upsilon)=dn(\Upsilon,s)$, thus we have,

$$\mathbb{Q}_4(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{dn^2(\Upsilon,s)}{fdn^2(\Upsilon,s)+g}\right), \tag{30}$$

where $\Upsilon=x-\frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f=\frac{h_4\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{\left(-3\,s^2+3-\left(-s^2+2\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{13\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)},$$

$$g=\frac{3\left(s^2-1\right)h_4}{\left(-3\,s^2+3-\left(-s^2+2\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{13\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)},$$

under the constraint condition,

$$0={h_4}^2\left(-s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(9\,s^2-\left(-s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right.$$
$$\left.\left(-2\,n^2-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s\rightarrow 1$,

$$\mathbb{Q}_{4,1}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\operatorname{sech}^2(\Upsilon,s)(\Upsilon,s)}{f\operatorname{sech}^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{31}$$

under the constraint condition,

$$0=-{h_4}^2\left(1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 4. if $l_0=s^2, l_2=-1-s^2, l_4=1, 0<s<1,$ then $\Phi(\Upsilon)=ns(\Upsilon,s)$ or $\Phi(\Upsilon)=dc(\Upsilon,s)$, thus we have,

$$\mathbb{Q}_5(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{ns^2(\Upsilon,s)}{fns^2(\Upsilon,s)+g}\right),\tag{32}$$

or

$$\mathbb{Q}_6(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{dc^2(\Upsilon,s)}{fdc^2(\Upsilon,s)+g}\right),\tag{33}$$

where $\Upsilon=x-\frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f=\frac{h_4\left(-s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{\left(3\,s^2-\left(-s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)},$$

$$g=\frac{3\,s^2h_4}{\left(3\,s^2-\left(-s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)},$$

under the constraint condition,

$$0={h_4}^2\left(-s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(9\,s^2-\left(-s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-2\,s^2-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s\to 1$,

$$\mathbb{Q}_{5,1}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\coth^2(\Upsilon,s)(\Upsilon,s)}{f\coth^2(\Upsilon,s)(\Upsilon,s)+g}\right),\tag{34}$$

under the constraint condition,

$$0 = {h_4}^2\left(-2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(9-\left(-2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-4+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s \to 0$,

$$\mathbb{Q}_{5,2}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\csc^2(\Upsilon,s)(\Upsilon,s)}{f\csc^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{35}$$

or

$$\mathbb{Q}_{6,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\sec^2(\Upsilon,s)(\Upsilon,s)}{f\sec^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{36}$$

under the constraint condition,

$$0 = -{h_4}^2\left(-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 5. if $l_0 = -s^2, l_2 = -1+2s^2, l_4 = 1-s^2, 0<s<1$, then $\Phi(\Upsilon) = nc(\Upsilon,s)$, thus we have,

$$\mathbb{Q}_7(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{nc^2(\Upsilon,s)}{fnc^2(\Upsilon,s)+g}\right), \tag{37}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f = \frac{h_4\left(2s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{-3\left(-s^2+1\right)s^2-\left(2s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

$$g = \frac{-3s^2h_4}{-3\left(-s^2+1\right)s^2-\left(2s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2}$$

under the constraint condition,

$$0 = {h_4}^2\left(2\,s^2 - 1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$

$$\left[-9\left(-s^2+1\right)s^2 - \left(2\,s^2 - 1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right.$$

$$\left.\left(4\,s^2 - 2 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right].$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{7,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\cosh^2(\Upsilon,s)(\Upsilon,s)}{f\cosh^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{38}$$

under the constraint condition,

$$0 = -{h_4}^2\left(1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

It is possible for us to obtain a single wave solution, when $s \to 0$,

$$\mathbb{Q}_{7,2}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\sec^2(\Upsilon,s)(\Upsilon,s)}{f\sec^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{39}$$

under the constraint condition,

$$0 = -{h_4}^2\left(-1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(-2 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 6. if $l_0 = -1, l_2 = 2 - s^2, l_4 = -1 + s^2, 0 < s < 1$, then $\Phi(\Upsilon) = nd(\Upsilon, s)$, thus we have,

$$\mathbb{Q}_8(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{nd^2(\Upsilon,s)}{fnd^2(\Upsilon,s)+g}\right), \tag{40}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f = \frac{h_4\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{-3\,s^2+3-\left(-s^2+2\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

$$g = -\frac{-3\,h_4}{-3\,s^2+3-\left(-s^2+2\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

under the constraint condition,

$$0 = {h_4}^2\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left[-9\,s^2+9-\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-2\,s^2+4+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right].$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{8,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\cosh^2(\Upsilon,s)(\Upsilon,s)}{f\cosh^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{41}$$

under the constraint condition,

$$0 = -{h_4}^2\left(1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 7. if $l_0 = 1, l_2 = 2 - s^2, l_4 = 1 - s^2, 0 < s < 1$, then $\Phi(\Upsilon) = sc(\Upsilon, s)$, thus we have,

$$\mathbb{Q}_9(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{sc^2(\Upsilon,s)}{fsc^2(\Upsilon,s)+g}\right), \tag{42}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^{\alpha})$ and the functions f and g are,

$$f = \frac{h_4\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{-3\,s^2+3-\left(-s^2+2\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

$$g = \frac{3\,h_4}{-3\,s^2+3-\left(-s^2+2\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

under the constraint condition,

$$\begin{aligned}0 = {h_4}^2&\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\\&\left[-9\,s^2+9-\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right.\\&\left.\left(-2\,s^2+4+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right].\end{aligned}$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{9,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\sinh^2(\Upsilon,s)(\Upsilon,s)}{f\sinh^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{43}$$

under the constraint condition,

$$0 = -{h_4}^2\left(1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

We are able to acquire a periodic wave solution, when $s \to 0$,

$$\mathbb{Q}_{9,2}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\tan^2(\Upsilon,s)(\Upsilon,s)}{f\tan^2(\Upsilon,s)(\Upsilon,s)+g}\right), \tag{44}$$

under the constraint condition,

$$0=h_4{}^2\left(2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(9-\left(2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(4+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

Category 8. if $l_0=1, l_2=2s^2-1, l_4=-s^2(1-s^2), 0<s<1$, then $\Phi(\Upsilon)=sd(\Upsilon,s)$, thus we have,

$$\mathbb{Q}_{10}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{sd^2(\Upsilon,s)}{fsd^2(\Upsilon,s)+g}\right), \tag{45}$$

where $\Upsilon=x-\frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f=\frac{h_4\left(2\,s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)}{-3\left(-s^2+1\right)s^2-\left(2\,s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

$$g=\frac{3\,h_4}{-3\left(-s^2+1\right)s^2-\left(2\,s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2},$$

under the constraint condition,

$$0=h_4{}^2\left(-s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(9\,s^2-\left(-s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-2\,s^2-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s\to 0$,

$$\mathbb{Q}_{10,1}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\sin^2(\Upsilon,s)}{f\sin^2(\Upsilon,s)+g}\right), \tag{46}$$

under the constraint condition,

$$0=-h_4{}^2\left(-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 9. if $l_0 = 1 - s^2, l_2 = 2 - s^2, l_4 = 1, 0 < s < 1$, then $\Phi(\Upsilon) = cs(\Upsilon, s)$, thus we have,

$$\mathbb{Q}_{11}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{cs^2(\Upsilon,s)}{fcs^2(\Upsilon,s)+g}\right), \tag{47}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^{\alpha})$ and the functions f and g are,

$$\begin{aligned} f =& h_4\left(\frac{1}{2}\,s^2+\frac{1}{2}-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{\sqrt{\frac{\phi_2}{\omega}}}\right)\\ &\left(\frac{3}{16}\left(-s^2+1\right)^2-\left(\frac{1}{2}\,s^2+\frac{1}{2}\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1},\\ g =& \frac{3}{4}\,h_4\left(\frac{3}{16}\left(-s^2+1\right)^2-\left(\frac{1}{2}\,s^2+\frac{1}{2}\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1}, \end{aligned}$$

under the constraint condition,

$$\begin{aligned} 0 = {h_4}^2&\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\\ &\left(-9\,s^2+9-\left(-s^2+2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right.\\ &\left.\left(-2\,s^2+4+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right). \end{aligned}$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{11,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\operatorname{csch}^2(\Upsilon,s)}{f\operatorname{csch}^2(\Upsilon,s)+g}\right), \tag{48}$$

under the constraint condition,

$$0 = -{h_4}^2\left(1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

We are able to acquire a periodic wave solution, when $s \to 0$,

$$\mathbb{Q}_{11,2}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\cot^2(\Upsilon,s)}{f\cot^2(\Upsilon,s)+g}\right), \tag{49}$$

under the constraint condition,

$$0=h_4{}^2\left(2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(9-\left(2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(4+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

Category 10. if $l_0=-s^2(1-s^2), l_2=2s^2-1, l_4=1, 0<s<1,$ then $\Phi(\Upsilon)=ds(\Upsilon,s)$, thus we have,

$$\mathbb{Q}_{12}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{ds^2(\Upsilon,s)}{fds^2(\Upsilon,s)+g}\right), \tag{50}$$

where $\Upsilon=x-\frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f=h_4\left(2\,s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}\frac{+k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-3\left(-s^2+1\right)s^2-\left(2\,s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{+k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1},$$

$$g=-3\left(-s^2+1\right)s^2h_4\left(-3\left(-s^2+1\right)s^2-\left(2\,s^2-1\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{+k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1},$$

under the constraint condition,

$$0=h_4{}^2\left(2\,s^2-1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-9\left(-s^2+1\right)s^2-\left(2\,s^2-1-1/12\,\frac{k^2\mu^2}{\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{\sqrt{\frac{\phi_2}{\omega}}}\right)\left(4\,s^2-2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{12,1}(x,t) = \frac{1}{-k} \ln\left(\mathcal{A}\, \frac{\operatorname{csch}^2(\Upsilon, s)}{f \operatorname{csch}^2(\Upsilon, s) + g}\right), \tag{51}$$

under the constraint condition,

$$0 = -{h_4}^2\left(1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{12\sqrt{\frac{\phi_2}{\omega}}}\right).$$

It is possible for us to obtain a single wave solution, when $s \to 0$,

$$\mathbb{Q}_{12,2}(x,t) = \frac{1}{-k} \ln\left(\mathcal{A}\, \frac{\csc^2(\Upsilon, s)}{f \csc^2(\Upsilon, s) + g}\right), \tag{52}$$

under the constraint condition,

$$0 = -{h_4}^2\left(-1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(-2 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

Category 11. if $l_0 = \frac{1-s^2}{4}, l_2 = \frac{1+s^2}{2}, l_4 = \frac{1-s^2}{4}, 0 < s < 1,$ then $\Phi(\Upsilon) = nc(\Upsilon, s) \pm sc(\Upsilon, s)$ or $\Phi(\Upsilon) = \frac{cn(\Upsilon,s)}{1 \pm sn(\Upsilon,s)}$, thus we have,

$$\mathbb{Q}_{13}(x,t) = \frac{1}{-k} \ln\left(\mathcal{A}\, \frac{(nc(\Upsilon, s) \pm sc(\Upsilon, s))^2}{f(nc(\Upsilon, s) \pm sc(\Upsilon, s))^2 + g}\right), \tag{53}$$

or

$$\mathbb{Q}_{14}(x,t) = \frac{1}{-k} \ln\left(\mathcal{A}\, \frac{cn^2(\Upsilon, s)}{fcn^2(\Upsilon, s) + (1 \pm sn^2(\Upsilon, s))g}\right), \tag{54}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$\begin{aligned} f =& h_4\left(\frac{1}{2}s^2 + \frac{1}{2} - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right) \\ &\left(3\left(-\frac{1}{4}s^2 + \frac{1}{4}\right)^2 - \left(\frac{1}{2}s^2 + \frac{1}{2}\right)^2 + \left(\frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1}, \\ g =& \frac{3}{4}h_4(-s^2+1)s^2\left(3\left(-\frac{1}{4}s^2 + \frac{1}{4}\right)^2 - \left(\frac{1}{2}s^2 + \frac{1}{2}\right)^2 + \left(\frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1}, \end{aligned}$$

under the constraint condition,

$$0 = {h_4}^2\left(\frac{1}{2}s^2 + 1/2 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(9\left(-\frac{1}{4}s^2 + \frac{1}{4}\right)^2 - \left(\frac{1}{2}s^2 + \frac{1}{2} - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right.$$
$$\left.\left(s^2 + 1 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{13,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\frac{(\cosh(\Upsilon,s) \pm \sinh(\Upsilon,s))^2}{f(\cosh(\Upsilon,s) \pm \sinh(\Upsilon,s))^2 + g}\right), \tag{55}$$

or

$$\mathbb{Q}_{14,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\frac{\operatorname{sech}^2(\Upsilon,s)}{f\operatorname{sech}^2(\Upsilon,s) + (1 \pm \tanh^2(\Upsilon,s))g}\right),, \tag{56}$$

under the constraint condition,

$$0 = -{h_4}^2\left(1 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

We are able to acquire a periodic wave solution, when $s \to 0$,

$$\mathbb{Q}_{13,2}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\frac{(\sec(\Upsilon,s) \pm \tan(\Upsilon,s))^2}{f(\sec(\Upsilon,s) \pm \tan(\Upsilon,s))^2 + g}\right), \tag{57}$$

or

$$\mathbb{Q}_{14,2}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\frac{\cos^2(\Upsilon,s)}{f\cos^2(\Upsilon,s) + (1 \pm \sin^2(\Upsilon,s))g}\right), \tag{58}$$

under the constraint condition,

$$0 = {h_4}^2\left(1/2 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(\frac{9}{16} - \left(1/2 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(1 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

Category 12. if $l_0 = -\frac{(1-s^2)^2}{4}, l_2 = \frac{1+s^2}{2}, l_4 = -\frac{1}{4}, /, 0 < s < 1,$ then $\Phi(\Upsilon) = s\, cn(\Upsilon, s) \pm dn(\Upsilon, s)$, thus we have,

$$\mathbb{Q}_{15}(x,t) = \frac{1}{-k} \ln\left(\mathcal{A}\,\frac{(n\, cn(\Upsilon,s) \pm dn(\Upsilon,s))^2}{f(n\, cn(\Upsilon,s) \pm dn(\Upsilon,s))^2 + g}\right), \tag{59}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$\begin{aligned} f =& h_4\left(\frac{1}{2}s^2 + \frac{1}{2} - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right) \\ & \left(\frac{3}{16}\left(-s^2+1\right)^2 - \left(\frac{1}{2}s^2 + \frac{1}{2}\right)^2 + \left(\frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1}, \\ g =& -\frac{3}{4}h_4(-s^2+1)s^2\left(\frac{3}{16}\left(-s^2+1\right)^2 - \left(\frac{1}{2}s^2 + \frac{1}{2}\right)^2 + \left(\frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1}, \end{aligned}$$

under the constraint condition,

$$\begin{aligned} 0 = {h_4}^2&\left(\frac{1}{2}s^2 + \frac{1}{2} - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right) \\ &\left(\frac{9\left(-s^2+1\right)^2}{16} - \left(\frac{1}{2}s^2 + \frac{1}{2} - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right. \\ &\left.\left(s^2 + 1 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right). \end{aligned}$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{15,1}(x,t) = \frac{1}{-k} \ln\left(\mathcal{A}\,\frac{(n\ \mathrm{sech}\,(\Upsilon,s) \pm\ \mathrm{sech}\,(\Upsilon,s))^2}{f(n\ \mathrm{sech}\,(\Upsilon,s) \pm\ \mathrm{sech}\,(\Upsilon,s))^2 + g}\right), \tag{60}$$

under the constraint condition,

$$0=-{h_4}^2\left(1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2$$
$$\left(2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

It is possible for us to obtain a single wave solution, when $s\to 0$,

$$\mathbb{Q}_{15,2}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\frac{(n\ \cos(\Upsilon,s)\pm 1)^2}{f(n\ \cos(\Upsilon,s)\pm 1)^2+g}\right), \tag{61}$$

under the constraint condition,

$$0={h_4}^2\left(1/2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(\frac{9}{16}-\left(1/2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(1+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

Category 13. if $l_0=\frac{1}{4}, l_2=\frac{1-2s^2}{2}, l_4=\frac{1}{4}, 0<s<1$, then $\Phi(\Upsilon)=\frac{sn(\Upsilon,s)}{1\pm cn(\Upsilon,s)}$, thus we have,

$$\mathbb{Q}_{16}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\frac{sn^2(\Upsilon,s)}{fsn^2(\Upsilon,s)+(1\pm cn^2(\Upsilon,s))g}\right), \tag{62}$$

where $\Upsilon=x-\frac{\Gamma(\gamma+1)}{\alpha}(\mu t^\alpha)$ and the functions f and g are,

$$f=h_4\left(-s^2+\frac{1}{2}-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(\frac{3}{16}-\left(-s^2+\frac{1}{2}\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1},$$

$$g=\frac{3}{4}h_4\left(\frac{3}{16}-\left(-s^2+\frac{1}{2}\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1},$$

under the constraint condition,

$$0={h_4}^2\left(-s^2+\frac{1}{2}-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(\frac{9}{16}-\left(-s^2+\frac{1}{2}-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-2\,s^2+1+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s\to 1$,

$$\mathbb{Q}_{16,1}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\tanh^2(\Upsilon,s)}{f\tanh^2(\Upsilon,s)+(1\pm\,\operatorname{sech}^2(\Upsilon,s))g}\right),\tag{63}$$

under the constraint condition,

$$0={h_4}^2\left(-1/2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(\frac{9}{16}-\left(-1/2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(-1+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

We are able to acquire a periodic wave solution, when $s\to 0$,

$$\mathbb{Q}_{16,2}(x,t)=\frac{1}{-k}\ln\left(\mathcal{A}\,\frac{\sin^2(\Upsilon,s)}{f\sin^2(\Upsilon,s)+(1\pm\cos^2(\Upsilon,s))g}\right),\tag{64}$$

under the constraint condition,

$$0={h_4}^2\left(1/2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(\frac{9}{16}-\left(1/2-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(1+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

Category 14. if $l_0=\frac{1}{4}, l_2=\frac{1+s^2}{2}, l_4=\frac{(1-s^2)^2}{4}, 0<s<1$, then $\Phi(\Upsilon)=\frac{sn(\Upsilon,s)}{cn(\Upsilon,s)\pm dn(\Upsilon,s)}$, thus we have,

$$\mathbb{Q}_{17}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\frac{sn^2(\Upsilon,s)}{fsn^2(\Upsilon,s)+g(cn(\Upsilon,s)\pm dn(\Upsilon,s))^2}\right), \tag{65}$$

where $\Upsilon = x - \frac{\Gamma(\gamma+1)}{\alpha}(\mu t^{\alpha})$ and the functions f and g are,

$$f = h_4\left(\frac{1}{2}s^2+\frac{1}{2}-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(\frac{3}{16}\left(-s^2+1\right)^2-\left(\frac{1}{2}s^2+\frac{1}{2}\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1},$$

$$g = \frac{3}{4}h_4\left(\frac{3}{16}\left(-s^2+1\right)^2-\left(\frac{1}{2}s^2+\frac{1}{2}\right)^2+\left(\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\right)^{-1},$$

under the constraint condition,

$$0 = {h_4}^2\left(\frac{1}{2}s^2+\frac{1}{2}-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(\frac{9\left(-s^2+1\right)^2}{16}-\left(\frac{1}{2}s^2+\frac{1}{2}-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right.$$
$$\left.\left(s^2+1+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

It is possible for us to obtain a single wave solution, when $s \to 1$,

$$\mathbb{Q}_{17,1}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\frac{\tanh^2(\Upsilon,s)}{f\tanh^2(\Upsilon,s)+g(\operatorname{sech}(\Upsilon,s)\pm\operatorname{sech}(\Upsilon,s))^2}\right), \tag{66}$$

under the constraint condition,

$$0 = -{h_4}^2\left(1-\frac{k^2\mu^2}{12\phi_2}+\frac{k^2\phi_1}{12\phi_2}+\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)^2\left(2+\frac{k^2\mu^2}{12\phi_2}-\frac{k^2\phi_1}{12\phi_2}-\frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right).$$

We are able to acquire a periodic wave solution, when $s \to 0$,

$$\mathbb{Q}_{17,2}(x,t) = \frac{1}{-k}\ln\left(\mathcal{A}\frac{\sin^2(\Upsilon,s)}{f\sin^2(\Upsilon,s) + g(\cos(\Upsilon,s) \pm 1)^2}\right), \tag{67}$$

under the constraint condition,

$$0 = h_4{}^2\left(1/2 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)$$
$$\left(\frac{9}{16} - \left(1/2 - \frac{k^2\mu^2}{12\phi_2} + \frac{k^2\phi_1}{12\phi_2} + \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\left(1 + \frac{k^2\mu^2}{12\phi_2} - \frac{k^2\phi_1}{12\phi_2} - \frac{k^2\sqrt{3}}{6\sqrt{\frac{\phi_2}{\omega}}}\right)\right).$$

5 Graphical representation and discussion

In this section, we will display 2-D, 3-D, and Contour visualization of the acquired results.

This part visually displays the exact complex analytical solutions of the governing model under consideration (4). We concluded that numerous new solitary wave exact solutions are being developed in the guise of trigonometric function solutions, Jacobi elliptic function solutions, and hyperbolic function solutions by comparing the findings of our work to the results of Alharthi et al. (2022). A suitable collection of parametric values that meet the constraint criteria for solutions is used to display the 2-D surface graph, 3-D surface graph, and contour surface graph.

Figs. 1, 2, 3 and 4 are plotted for $h_4 = 0.9$, $\mu = 0.1$, $\omega = 0.1$, $k = -0.1$, $n = 1$, and $\phi_1 = 23.900455549$. In these figures, it can be observed that the obtained solutions displaying the singular and bright propagating solitary wave solution. It is noticed that the the fractional order is responsible to control the behavior of the soliton. That's why, the scientists and researchers of scientific field can acquire their required results.

6 Modulation instability analysis of the Peyrard–Bishop DNA dynamical equation

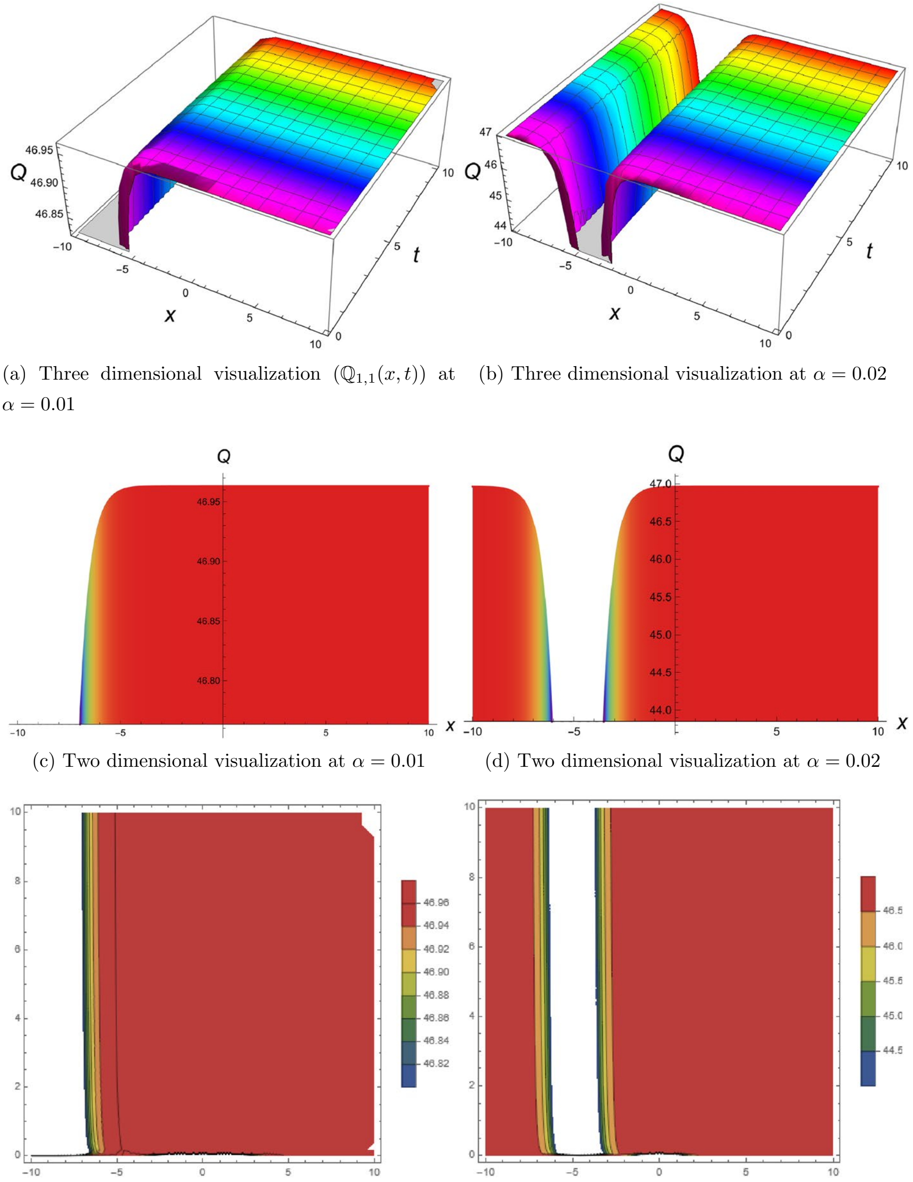
6.1 Linear stability analysis

In this section of paper, the objective is to develop the modulation instability (MI) gain of steady state solution of the governing system (4) by the virtue of the linear stability analysis. The MI can be consisted in exponential growth of the small perturbations in the phase of optical waves or the amplitude. It is significant to investigate it in the nonlinear wave physics.

Let us assume a steady state solutions in order to perform a stability analysis,

$$\mathbb{Q} = \mathfrak{M}_0, \tag{68}$$

where, $\mathfrak{M}_0$ is the initial incidence power (real constant-amplitudes). Furthermore, the solutions (68) is transforming to perturbed stationary solutions as,



(a) Three dimensional visualization $(\mathbb{Q}_{1,1}(x,t))$ at $\alpha = 0.01$

(b) Three dimensional visualization at $\alpha = 0.02$

(c) Two dimensional visualization at $\alpha = 0.01$

(d) Two dimensional visualization at $\alpha = 0.02$

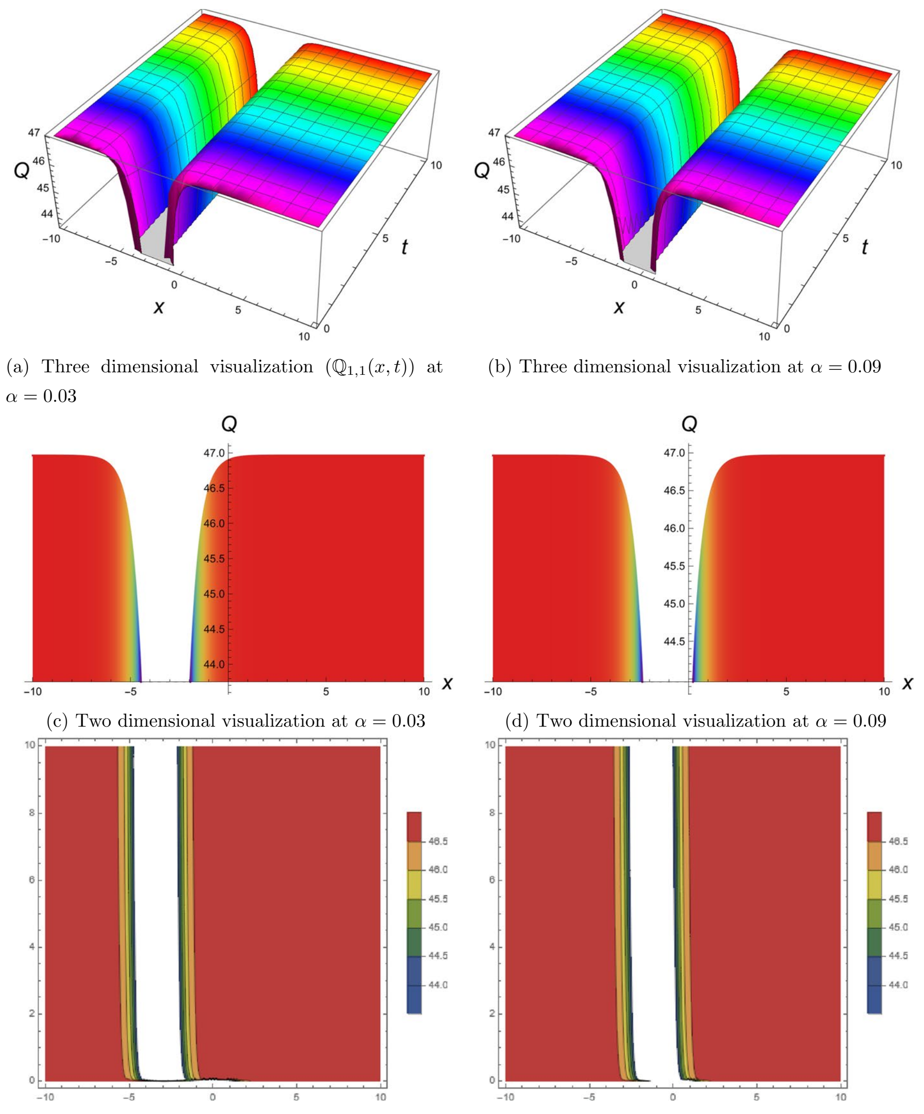
(e) Contour visualization at $\alpha = 0.01$

(f) Contour visualization at $\alpha = 0.02$

Fig. 1 Graphical solitary wave propagation of $(\mathbb{Q}_{1,1}(x, t))$ at different fractional order

$$\mathbb{Q} = \mathfrak{M}_0 + \varrho\, \Xi(x, t), \tag{69}$$

where Ξ is real function of x, t and the perturbation coefficient parameter is $\varrho << 1$. The perturbed stationary solutions are plugging into system (4) and develop the disturbance equations,



(a) Three dimensional visualization $(\mathbb{Q}_{1,1}(x,t))$ at $\alpha = 0.03$

(b) Three dimensional visualization at $\alpha = 0.09$

(c) Two dimensional visualization at $\alpha = 0.03$

(d) Two dimensional visualization at $\alpha = 0.09$

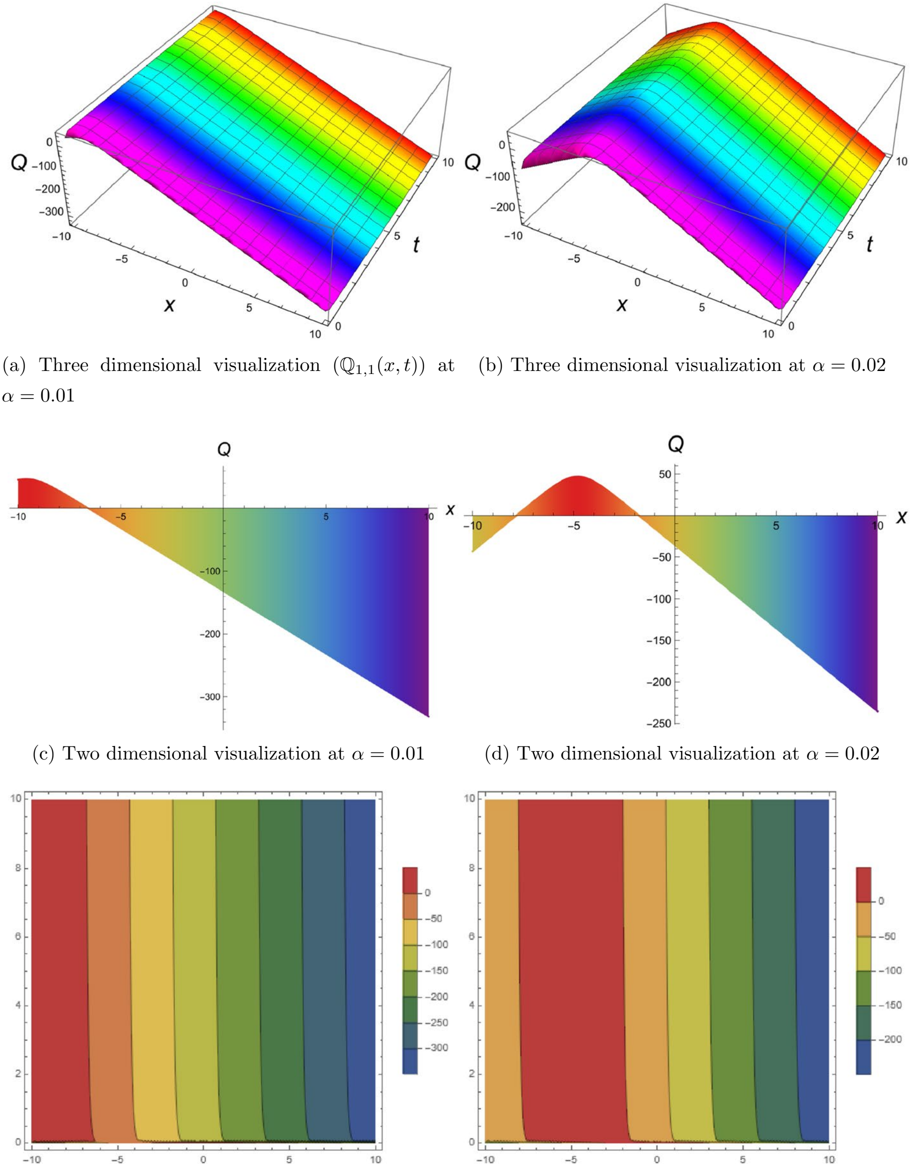
(e) Contour visualization at $\alpha = 0.03$

(f) Contour visualization at $\alpha = 0.09$

Fig. 2 Graphical solitary wave propagation of $(\mathbb{Q}_{1,1}(x,t))$ at different fractional order

$$\begin{aligned}&\varrho\,\frac{\partial^2}{\partial t^2}\Xi(x,t) - 3\,\varrho^3\left(\frac{\partial^2}{\partial x^2}\Xi(x,t)\right)\left(\frac{\partial}{\partial x}\Xi(x,t)\right)^2\phi_2 - \varrho\left(\frac{\partial^2}{\partial x^2}\Xi(x,t)\right)\phi_1\\&\quad - 2\,\frac{\omega\,k}{\left(\mathrm{e}^{k\mathfrak{M}_0}\right)^2\left(\mathrm{e}^{k\varrho\,\Xi(x,t)}\right)^2} + 2\,\frac{\omega\,k}{\mathrm{e}^{k\,\mathfrak{M}_0}\mathrm{e}^{k\varrho\,\Xi(x,t)}} = 0.\end{aligned} \tag{70}$$

After linearization, the disturbance equations (70) can be written as,



(a) Three dimensional visualization ($\mathbb{Q}_{1,1}(x,t)$) at $\alpha = 0.01$

(b) Three dimensional visualization at $\alpha = 0.02$

(c) Two dimensional visualization at $\alpha = 0.01$

(d) Two dimensional visualization at $\alpha = 0.02$

(e) Contour visualization at $\alpha = 0.01$

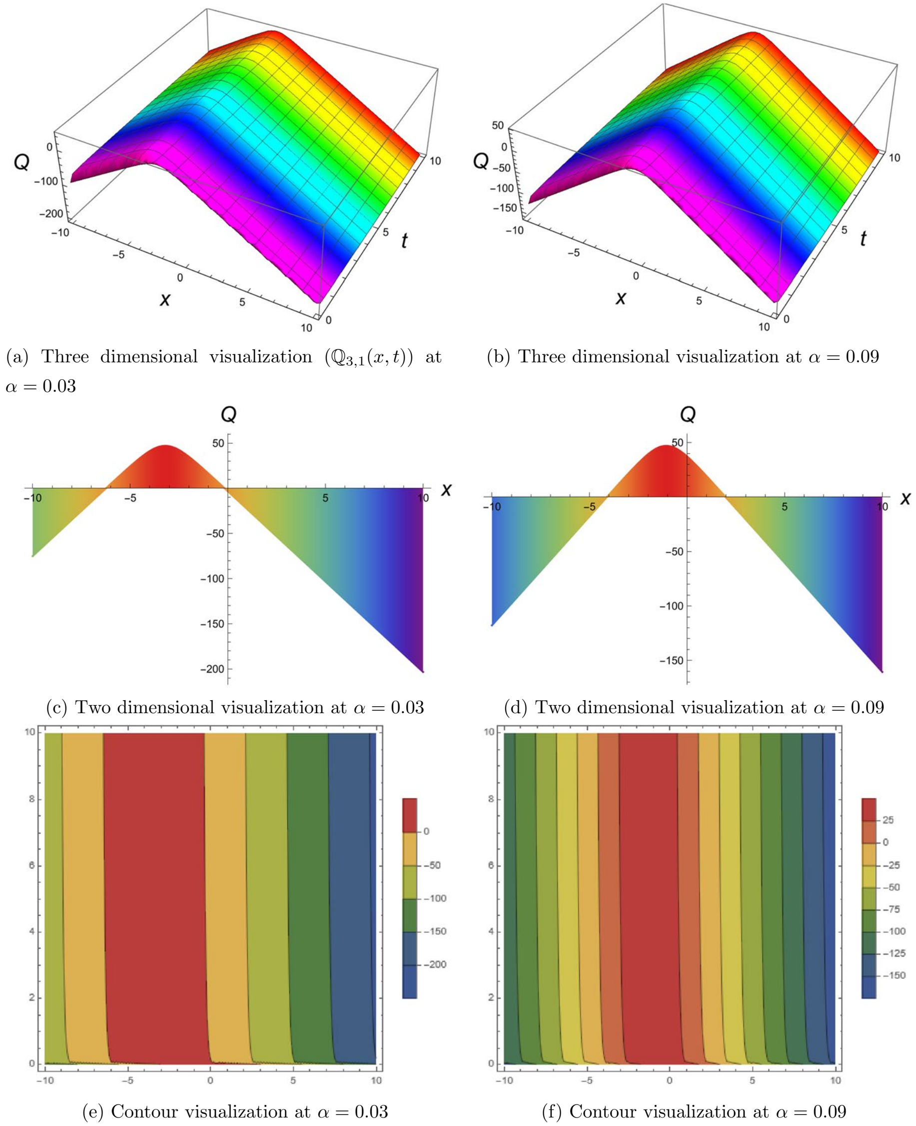
(f) Contour visualization at $\alpha = 0.02$

Fig. 3 Graphical solitary wave propagation of ($\mathbb{Q}_{1,1}(x,t)$) at different fractional order

$$\varrho \frac{\partial^2}{\partial t^2}\Xi(x,t) - \varrho \left(\frac{\partial^2}{\partial x^2}\Xi(x,t) \right)\phi_1 = 0. \tag{71}$$

Now, we have to introduce the $\Xi(x,t)$, such as,

$$\Xi(x,t) = a \ \exp^{i(\mathfrak{K}x - \mathfrak{W}t)} + b \ \exp^{-i(\mathfrak{K}x - \mathfrak{W}t)}. \tag{72}$$



(a) Three dimensional visualization ($\mathbb{Q}_{3,1}(x,t)$) at $\alpha = 0.03$

(b) Three dimensional visualization at $\alpha = 0.09$

(c) Two dimensional visualization at $\alpha = 0.03$

(d) Two dimensional visualization at $\alpha = 0.09$

(e) Contour visualization at $\alpha = 0.03$

(f) Contour visualization at $\alpha = 0.09$

Fig. 4 Graphical solitary wave propagation of ($\mathbb{Q}_{3,1}(x,t)$) at different fractional order

The function (72) substitute into (71) and get system of homogeneous equations,

$$\begin{aligned} &a\mathfrak{K}^2\varrho\,\phi_1 - a\varrho\,\mathfrak{W}^2,\\ &b\mathfrak{K}^2\varrho\,\phi_1 - b\varrho\,\mathfrak{W}^2. \end{aligned} \tag{73}$$

The coefficient matrix of system (73) can be written as following for a and b,

$$\begin{pmatrix} \mathfrak{K}^2\varrho\,\phi_1 - \varrho\,\mathfrak{W}^2 & 0 \\ 0 & \mathfrak{K}^2\varrho\,\phi_1 - \varrho\,\mathfrak{W}^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{74}$$

The non-trivial solutions exist for the coefficient matrix (74), when the determinant vanishes. The dispersion relation derive by expanding the determinant of above coefficient matrix,

$$\varrho^2\left(\mathfrak{K}^2\phi_1 - \mathfrak{W}^2\right)^2 = 0. \tag{75}$$

The Peyrard–Bishop DNA dynamical equation is modulationally stable for any wave number $\mathfrak{K}$ if and only if the four roots of (76) are all positive real values. The roots of (76) are not so simple to acquire, though, as we must use the current effective analytical formulae and the appropriate criteria for the roots of a fourth-order polynomial. The dispersion relation's solution is obtained (76),

$$\mathfrak{W} = \pm\sqrt{\phi_1}\,\mathfrak{K}, \tag{76}$$

Thus, one can observe that, the modulation instability of the Peyrard–Bishop DNA dynamical equation is modulationally stable (4) occurs when,

$$\sqrt{\phi_1} < 0,$$

7 Conclusion

In this study, by applying one of the generalised expansion techniques, multiple new soliton solutions to the Peyrard–Bishop DNA dynamical equation are developed. The new expanded Φ^6-model expansion technique generally given fourteen families of differing soliton patterns.

- The analysis yields twenty-eight solutions.
- The acquired solitary wave patterns are based on Jacobi elliptic functions, with hyperbolic solutions in the limiting case $s \to 1$, and trigonometric solutions in the limiting case $s \to 0$.
- In order to ensure the existence of solutions, the constraint conditions connected to each result are provided.
- The solutions are shown in 2-D, 3-D, and contour real and imaginary profiles.
- The developed constraints are satisfied by the parametric values used.
- The Peyrard–Bishop DNA dynamical equation is modulationally stable under the established constraint.

In order to explain the graphical behaviour of optical pulses employing the generated analytical solutions, some relevant values are taken for the related free parameters. The acceptable solutions can be relied upon to comprehend the nonlinear model's physical perspective. The Φ^6-model procedure is a strong and efficient mathematical tool that can be used to provide the analytical answers to a variety of other challenging mathematical phenomena.

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Conflict of interest The authors declare no conflict of interest.

Ethical approval Not Applicable

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