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REVIEW

Distance measures and δ-approximations with rough complex fuzzy models

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Abstract
Rough set theory is a key approach to model and apprehend the situations involving uncertainty without additional information and suppositions. Complex fuzzy sets are useful in developing parsimonious models in various fields such as image processing, machine learning and data mining. To manipulate the subjectivity and fluctuation in decision making problems, the integration of rough sets and complex fuzzy sets can provide more objective description of the collected data in terms of upper and lower approximations. In this research, we integrate two powerful mathematical techniques, namely rough sets and complex fuzzy sets, to initiate a hybrid model named rough complex fuzzy sets. We describe some basic properties of rough complex fuzzy set operations using t-norms and s-norms. We study algebraic properties and relations of certain operations of rough complex fuzzy sets. Based on different types of distance measures, we investigate the effects and properties of δ-approximations of complex fuzzy sets under rough information. We demonstrate the feasibility of rough complex fuzzy sets with a pragmatic application regarding the classification of unknown building materials. We illustrate certain relations of δ-approximations with different types of rough complex fuzzy sets. Finally, we compare our rough complex fuzzy set based method with several well-established approaches to show its validity and superiority.

Keywords Rough complex fuzzy set · Triangular norm · Distance measures · t-norm · s-norm · δ-approximations · Decision making

1 Introduction

The theory of fuzzy sets was established by Zadeh (1965) in order to deal with uncertain and vague information. This remarkable notion of fuzziness has shown its power in a number of different fields, including algebra, topology, operational research, decision making, biomedical engineering, information engineering, social networking, artificial intelligence and computer science. Ramot et al. (2002) generalized fuzzy sets to complex fuzzy (CF) sets whose membership functions are complex-valued functions rather than real-valued functions taking values in [0,1]. Note also that CF sets are distinct from Buckley's fuzzy complex numbers initiated in (Buckley 1989). It retains the characterization to describe uncertainty as the membership value of amplitude term lies in [0,1] adding a phase value represented by fuzzy sets. The main feature of CF set is the membership value of phase term which gives wavelike properties to objects resulting in destructive and constructive interference. This unique characteristic distinguishes CF sets from fuzzy sets and other extensions. By taking non-membership functions into consideration, Alkouri and Salleh (2012) put forth complex intuitionistic fuzzy (CIF) sets. Dick et al. (2015) studied operations on CF sets and complex Pythagorean fuzzy (CPyF) sets. Hu et al. (2018) examined the continuity of CF operations and distances of CF sets. Ullah et al. (2020) studied some distance measures

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of CPyF sets and presented several useful characterizations. Ma et al. (2011) explored applications of CF sets to multiple periodic factor prediction. Akram and Bashir (2021) introduced various quadratic averaging operators for CF information aggregation and demonstrated their importance with practical examples.

The notion of rough approximation was first initiated by Pawlak (1982) that extends the idea of set theory for coping with deviations, indiscernibility and inexactness arising in information systems. Considering the real-world application base of rough set models in knowledge retrieval, decision analysis, risk assessment and machine learning, the hybridization of rough approximation with other MCDM models and mathematical structures has turned int an increasingly active research topic. The concept of rough relations was studied by Pawlak (1996) in 1986. The fundamental properties of rough relations were discussed by Nicoletti et al. (2001). Meanwhile, another emerging mathematical tool named soft sets (Molodtsov 1999) has promoted the progress of rough set theory. Rough lower and upper approximations of soft sets were first considered in Feng et al. (2010), which gave rise to a hybrid structure named rough soft sets. Feng et al. (2011) considered the generalization of Pawlak's rough sets by virtue of soft sets, and came up with the concept of soft rough sets. On many occasions, soft rough sets enable to give more flexible approximations than classical rough sets. The application of soft rough sets to multiple criteria group decision-making was discussed in Feng (2011). Akram and Zafar (2020) studied various properties and applications of fuzzy rough graphs, soft rough fuzzy graphs and their applications in real-world phenomenon. Alcantud (2016) proved that certain extensions of fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets can be obtained from soft sets. In Alcantud et al. (2020), the authors derived tolerance rough sets, rough sets and multi-granulation rough sets from *N*-soft sets. Liu et al. (2019) presented two mapping methods to show that every soft set can be regarded as a fuzzy set.

Dubois and Prade (1990) brought forth two important extensions of classical rough models, namely rough fuzzy (RF) sets and fuzzy rough (FR) sets which are helpful for manipulating rough information from the perspective of fuzziness. Feng et al. (2010) investigated lower and upper approximations of a fuzzy set based on a soft approximation space, which put forward soft-rough fuzzy sets. It can be seen that soft-rough fuzzy sets are generalizations of Dubois and Prade's rough fuzzy sets. Zafar and Akram (2018) addressed some mathematical properties of graph products and an MCDM problem under a rough fuzzy setting. Yu et al. (2019) introduced the notion of α-RF sets and discussed related characterizations and applications to decision-making. Zhang et al. (2020) studied operators of FR sets and elaborated on feasibility of these concepts by evaluating fuzzy information in decision making problems. Yu et al. (2020) put forward (γ, δ)-RF sets by means of δ-clusters, and explored their application to predictive analysis. Based on fuzzy information, Zhang et al. (2020) presented various three-way decision methods of rough vague sets. Shaheen et al. (2020) proposed a failure modes and effects analysis (FMEA) approach based on generalized hesitant fuzzy rough sets. Zhan et al. (2017) introduced the notion of soft rough hemi-rings and discussed their importance in solving multiple criteria decision-making problems. Jiang et al. (2020) presented covering-based variable precision FR sets and discussed related application regarding medical diagnosis. The researchers nowadays are actively working on rough set models such as: rough ELECTRE-II method for risk assessment in poer generation plants Sarwar et al. (2021), Sarwar (2022a) introduced the notion of rough hypergraphs based on rough relations of more than two elements. The properties of coloring, duality, isomorphism, linearity, conformality, distributivity, associativity, coloring, Helly property, commutativity, formulae of L2-section, 2-section, and rank, etc, of rough hypergraphs also discussed in (?), fuzzy coverings based on rough sets (D'eer and Cornelis 2018), some results on fuzzy soft sets based on scalar products (Feng and Pedrycz 2015), parametrization of intuitionistic fuzzy sets (Feng et al. 2019), coverings and applications in fuzzy rough environment Ma et al. (2020), color spectrum of rough graphs and rough D-TOPSIS method in networking and agricultural problems (Sarwar 2020), granular variable precision of fuzzy rough sets (Wang and Wan 2021) and operations, δ-approximations of complex fuzzy sets (Zhang et al. 2009) design assessment schemes based on interval rough integrated cloud VIKOR (Sarwar 2022b), normal fuzzy aggregation operators based Q-rung orthopair fuzzy information (Kumar and Gupta 2023), average aggregation operators based on T-spherical fuzzy soft rough information (Akram and Martino 2023), group decision analysis based on complex Pythagorean fuzzy sets (Aldring and Ajay 2023).

1.1 Motivation of the current research

This research is mainly motivated from two different aspects as follows:

1. The concept of CF sets plays a vital role to examine fuzzy and wavelike properties of objects. The uncertainty in CF sets is dealt using two pre-defined functions ignoring approximate and rough features at the same time. To overcome such information loss, another significant way is to extend the notion CF sets in rough environment. Most of existing approaches are

based on pre-defined parameters and additional suppositions when dealing with uncertainty in the decision-making process. Rough approximations can cope with uncertainty using initial data without any biased parameters. Nevertheless, in practical situations, it might be difficult to gather a more relevant and accurate information. Rough sets have the powerful capability to cope with this barricade and provide more constructive framework of the concerned problem. Thus this study focuses on the integration of CF sets and rough sets to introduce rough complex fuzzy (RCF) sets, and study uncertainty in more sophisticated manner.

2. The concept of distance measures and different derivations of δ-approximations of CF sets under rough information is another tool towards decision making problems. Distance measures have successful applications in diverse areas of science and technology. To consolidate the theoretical basis of RCF sets, there is a need to investigate distance measures of RCF sets for better handling uncertainty in data and obtaining more realistic results in relevant applications. RCF sets along with certain distance measures can be utilized for best alternative selection and classification of unknown objects problems.

The significant contributions of this study are threefold:

1. The present study explores rough approximations of CF sets to put forward a novel notion called RCF sets.
2. Several basic operations of RCF sets are introduced by virtue of triangular norms, s-norms and t-norms. Based on different types of distance measures, δ-approximation properties of RCF sets are studied in detail. We also examine the impact of δ-equalities on certain types of RCF sets.
3. The importance of RCF stes is illustrated with a numerical example regrading the classification of unknown building materials.

2 Preliminaries

In the sequel, let T be a non-empty set called the universe of discourse. An *approximation space* (Pawlak 1982) is defined to be an ordered pair (T, X), where X is an equivalence relation on T. For any $Z \subseteq T$, the lower rough approximation $\underline{X}Z$ and the upper rough approximation $\overline{X}Z$ of Z in (T, X) are: $\underline{X}Z = \{\epsilon \in T \mid [\epsilon]_S \subseteq Z\}$, $\overline{X}Z = \{\epsilon \in T \mid [\epsilon]_S \cap Z \neq \varnothing\}$, where $[\epsilon]_X = \{t \in T \mid (t, \epsilon) \in X\}$ is an equivalence (EQ) class of element $\epsilon \in T$. The 2-tuple $(\underline{X}Z, \overline{X}Z)$ is called a *rough set* and Z is said to be *definable* in (T, X) if $\underline{X}Z = \overline{X}Z$.

Definition 1 (Dubois and Prade 1990) Let (T, X) be an approximation space and μ be a fuzzy set in T. The lower and upper rough approximations of μ in (T, X), denoted by $\underline{X}\mu$ and $\overline{X}\mu$ respectively, are fuzzy sets in T such that for all $\epsilon \in T$,

$$(\underline{X}\mu)(\epsilon) = \bigwedge_{t \in [\epsilon]_X} \mu(t),$$
$$(\overline{X}\mu)(\epsilon) = \bigvee_{t \in [\epsilon]_X} \mu(t).$$

The pair $(\underline{X}\mu, \overline{X}\mu)$ is called a *rough fuzzy set*. We say that μ is *definable* in (T, X) if $\overline{X}\mu = \overline{X}\mu$.

A fuzzy set X in $T \times T$ is said to be a fuzzy binary relation on T. We say that X is a *fuzzy similarity relation* on T if it satisfies:

1. $X(t_1, t_1) = 1$ for all $t_1 \in T$;
2. $X(t_1, t_2) = X(t_2, t_1)$ for all $t_1, t_2 \in T$;
3. $X(t_1, t_2) \geq \bigvee_{t \in T} \min\{X(t_1, t), X(t, t_2)\}$ for all $t_1, t_2 \in T$.

When X is a fuzzy similarity relation on T, we also refer to (T, X) as a *fuzzy approximation space*.

Definition 2 (Dubois and Prade 1990) Let (T, X) be a fuzzy approximation space and μ be a fuzzy set in T. The lower and upper fuzzy rough approximations of μ in (T, X), denoted by $\underline{X}\mu$ and $\overline{X}\mu$ respectively, are fuzzy sets in T such that for all $\epsilon \in T$,

$$(\underline{X}\mu)(\epsilon) = \bigwedge_{t \in T} \max\{(1 - X(\epsilon, t)), \mu(t)\},$$
$$(\overline{X}\mu)(\epsilon) = \bigvee_{t \in T} \min\{X(\epsilon, t), \mu(t)\}.$$

The pair $(\underline{X}\mu, \overline{X}\mu)$ is called a *fuzzy rough set*, and μ is said to be *definable* in (T, X) if $\overline{X}\mu = \overline{X}\mu$.

In particular, it is easy to see that fuzzy rough approximations of a fuzzy set μ in a fuzzy approximation space (T, X) reduce to rough approximations if X is an equivalence relation on T.

Definition 3 (Ramot et al. 2002) A *CF set* ρ on T is given by a mapping $\rho : T \to \{\epsilon \in \mathbb{C} \mid |z| \leq 1\}$ that assigns to every element $\epsilon \in T$, a complex grade of membership in the complex unit disk and is thus of the form $\rho(\epsilon) = k_\rho(\epsilon) e^{i \arg_\rho(\epsilon)}$, where $i = \sqrt{-1}$, $k_\rho(\epsilon) \in [0, 1]$ and $\arg_\rho(\epsilon) \in [0, 2\pi]$. The terms $k_\rho(\epsilon)$ and $\arg_\rho(\epsilon)$ are called the *amplitude* and *principal argument* of $\rho(\epsilon)$, respectively. The family of all complex fuzzy sets on T is denoted by $CF(T)$.

Remark 1 Note that the universe T can be viewed as a CF set such that the membership value of each element $\epsilon \in T$ is $k_T(\epsilon) e^{i \arg_T(\epsilon)} = 1 \cdot e^{i2\pi}$. Similarly, the empty set $\varnothing$ can

be also identified with a CF set such that $k_{\varnothing}(\epsilon) = 0$ and $\arg_{\varnothing}(\epsilon) = 0$ for all $\epsilon \in T$.

Definition 4 (Zhang et al. 2009) Let $\alpha, \varphi \in CF(T)$. The *union* of α, φ is a CF set $\rho = \alpha \cup \varphi$ such that

$$\rho(\epsilon) = k_\rho(\epsilon)e^{i\arg_\rho(\epsilon)} = \big(k_\alpha(\epsilon) \vee k_\varphi(\epsilon)\big)e^{i\left(\arg_\alpha(\epsilon)\vee\arg_\varphi(\epsilon)\right)}$$

where $\epsilon \in T$, $\alpha(\epsilon) = k_\alpha(\epsilon)e^{i\arg_\alpha(\epsilon)}$ and $\varphi(\epsilon) = k_\varphi(\epsilon)e^{i\arg_\varphi(\epsilon)}$.

Definition 5 Zhang et al. (2009) Let $\alpha, \varphi \in CF(T)$. The *intersection* of α, φ is a CF set $\rho = \alpha \cap \varphi$ such that

$$\rho(\epsilon) = k_\rho(\epsilon)e^{i\arg_\rho(\epsilon)} = \big(k_\alpha(\epsilon) \wedge k_\varphi(\epsilon)\big)e^{i\left(\arg_\alpha(\epsilon)\wedge\arg_\varphi(\epsilon)\right)}$$

where $\epsilon \in T$, $\alpha(\epsilon) = k_\alpha(\epsilon)e^{i\arg_\alpha(\epsilon)}$ and $\varphi(\epsilon) = k_\varphi(\epsilon)e^{i\arg_\varphi(\epsilon)}$.

Definition 6 (Zhang et al. 2009) Let $\alpha \in CF(T)$. The *complement* of α is a CF set $\rho = \alpha^c$ such that

$$\rho(\epsilon) = k_\rho(\epsilon)e^{i\arg_\rho(\epsilon)} = (1 - k_\alpha(\epsilon))e^{i(2\pi - \arg_\alpha(\epsilon))}$$

where $\epsilon \in T$ and $\alpha(\epsilon) = k_\alpha(\epsilon)e^{i\arg_\alpha(\epsilon)}$.

Definition 7 (Yager and Rybalov 1996) A function $\xi : [0,1] \times [0,1] \to [0,1]$ is called a *uninorm* if it satisfies:

1. $\xi(\epsilon, t) = \xi(t, \epsilon)$ for all $\epsilon, t \in [0,1]$;
2. If $x \le y$ and $u \le v$, then $\xi(x,u) \le \xi(y,v)$ for all $x, y, u, v \in [0,1]$;
3. $\xi(\xi(\epsilon,t),u) = \xi(\epsilon, \xi(t,u))$ for all $\epsilon, t, u \in [0,1]$;
4. There exists an *identity* $e \in [0,1]$ such that $\xi(\epsilon, e) = z$ for all $\epsilon \in [0,1]$.

A uninorm ξ with identity $e = 1$ is called a *triangular norm* (or *t-norm*). Dually, a uninorm ξ is said to be a *triangular conorm* (or *s-norm*) if its identity $e = 0$. The list of abbreviations used in this study are given in Table 1.

3 Rough complex fuzzy sets

In this section, we introduce the concept of RCF set by integrating the technique of rough approximations with CF sets. With the aid of *s*-norms and *t*-norms, we discuss certain operation properties of RCF sets.

Table 1 List of abbreviations

Abbreviation	Description
CF set	Complex fuzzy set
CIF set	Complex intuitionistic fuzzy set
CPyF set	Complex Pythagorean fuzzy set
RF set	Rough fuzzy set
FR set	Fuzzy rough set
FMEA	Failure modes and effects analysis
MCDM	Multiple criteria decision-making
RCF set	Rough complex fuzzy set

Definition 8 Let (T, X) be an approximation space, $CF(T)$ be the family of all complex fuzzy sets on T and $\rho \in CF(T)$. The lower and upper rough approximations of ρ in (T, X), denoted by $\underline{X}\rho$ and $\overline{X}\rho$, are CF sets on T such that for all $\epsilon \in T$,

$$(\underline{X}\rho)(\epsilon) = \wedge_{t\in T}\big[(1 - X(\epsilon,t)) \vee k_\rho(t)\big] \cdot e^{i2\pi\left[\wedge_{t\in T}\left((1-X(\epsilon,t))\vee\frac{\arg_\rho(t)}{2\pi}\right)\right]}, \tag{1}$$

$$(\overline{X}\rho)(\epsilon) = \vee_{t\in T}\big[X(\epsilon,t) \wedge k_\rho(t)\big] \cdot e^{i2\pi\left[\vee_{t\in T}\left(X(\epsilon,t)\wedge\frac{\arg_\rho(t)}{2\pi}\right)\right]}. \tag{2}$$

The pair $X\rho = (\underline{X}\rho, \overline{X}\rho)$ is called a *rough complex fuzzy set*. An RCF set is also denoted by $X\rho = (k_{\underline{X}_\rho}e^{i\arg_{\underline{X}_\rho}}, k_{\overline{X}_\rho}e^{i\arg_{\overline{X}_\rho}})$. We say that ρ is *definable* in (T, X) if $\underline{X}\rho = \overline{X}\rho$.

Remark 2 We would like point out several basic facts regarding the notion of RCF sets.

1. Note first that Eqs. (1) and (2) in Definition 8 can be simplified as follows:

$$(\underline{X}\rho)(\epsilon) = \big[\wedge_{t\in[\epsilon]_X}k_\rho(t)\big] \cdot e^{i2\pi\left[\wedge_{t\in[\epsilon]_X}\frac{\arg_\rho(t)}{2\pi}\right]}, \tag{3}$$

$$(\overline{X}\rho)(\epsilon) = \big[\vee_{t\in[\epsilon]_X}k_\rho(t)\big] \cdot e^{i2\pi\left[\vee_{t\in[\epsilon]_X}\frac{\arg_\rho(t)}{2\pi}\right]}. \tag{4}$$

2. According to Remark 1, we can view the universe T as a CF set. Then for all $\epsilon \in T$,

$$\begin{aligned}(\underline{X}T)(\epsilon) &= \wedge_{t\in T}\big[(1 - X(\epsilon,t)) \vee k_T(t)\big] \cdot e^{i2\pi\left[\wedge_{t\in T}\left((1-X(\epsilon,t))\vee\frac{\arg_T(t)}{2\pi}\right)\right]}\\ &= \big[\wedge_{t\in[\epsilon]_X}k_T(t)\big] \cdot e^{i2\pi\left[\wedge_{t\in[\epsilon]_X}\frac{\arg_T(t)}{2\pi}\right]}\\ &= 1 \cdot e^{i2\pi\left[\wedge_{t\in[\epsilon]_X}\frac{2\pi}{2\pi}\right]}\\ &= e^{i2\pi},\end{aligned}$$

$$\begin{aligned}(\overline{X}T)(\epsilon) &= \vee_{t\in T}\big[X(\epsilon,t) \wedge k_T(t)\big] \cdot e^{i2\pi\left[\vee_{t\in T}\left(X(\epsilon,t)\wedge\frac{\arg_T(t)}{2\pi}\right)\right]}\\ &= \big[\vee_{t\in[\epsilon]_X}k_T(t)\big] \cdot e^{i2\pi\left[\vee_{t\in[\epsilon]_X}\frac{\arg_T(t)}{2\pi}\right]}\\ &= 1 \cdot e^{i2\pi\left[\vee_{t\in[\epsilon]_X}\frac{2\pi}{2\pi}\right]}\\ &= e^{i2\pi}.\end{aligned}$$

Thus, we can see that $\underline{X}T = \overline{X}T$. That is, the universe T is definable.

3. In a similar fashion, we can view the empty set $\varnothing$ as a CF set as well. Then for all $\epsilon \in T$,

$$
\begin{aligned}
(\underline{X}\varnothing)(\epsilon) &= \wedge_{t\in T}\left[(1-X(\epsilon,t))\vee k_{\varnothing}(t)\right]\cdot \\
&\quad e^{i2\pi\left[\wedge_{t\in T}\left((1-X(\epsilon,t))\vee\frac{\arg_{\varnothing}(t)}{2\pi}\right)\right]} \\
&= \left[\wedge_{t\in[\epsilon]_X}k_{\varnothing}(t)\right]\cdot e^{i2\pi\left[\wedge_{t\in[\epsilon]_X}\frac{\arg_{\varnothing}(t)}{2\pi}\right]} \\
&= 0\cdot e^{i2\pi\left[\wedge_{t\in[\epsilon]_X}\frac{0}{2\pi}\right]} \\
&= 0, \\
(\overline{X}\varnothing)(\epsilon) &= \vee_{t\in T}\left[X(\epsilon,t)\wedge k_{\varnothing}(t)\right]\cdot \\
&\quad e^{i2\pi\left[\vee_{t\in T}\left(X(\epsilon,t)\wedge\frac{\arg_{\varnothing}(t)}{2\pi}\right)\right]} \\
&= \left[\vee_{t\in[\epsilon]_X}k_{\varnothing}(t)\right]\cdot e^{i2\pi\left[\vee_{t\in[\epsilon]_X}\frac{\arg_{\varnothing}(t)}{2\pi}\right]} \\
&= 0\cdot e^{i2\pi\left[\vee_{t\in[\epsilon]_X}\frac{0}{2\pi}\right]} \\
&= 0.
\end{aligned}
$$

Thus, we have $\underline{X}\varnothing = \overline{X}\varnothing = \varnothing$. It follows that the empty set $\varnothing$ is definable.

4. Let $\alpha, \varphi \in CF(T)$. The ordered pair (α, φ) is an RCF set in (T, X) if and only if there exists $\rho \in CF(T)$ such that $\underline{X}\rho = \alpha$ and $\overline{X}\rho = \varphi$ (i.e., $X\rho = (\alpha, \varphi) = (\underline{X}\rho, \overline{X}\rho)$ for some CF set ρ).

Example 1 Let $T = \{\epsilon, t, w, s, u, v\}$ be a universe and X be an equivalence relation on T given by

$$
X = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}. \tag{5}
$$

The equivalence relation X can be identified with the corresponding partition $X = \{\{\epsilon, u\}, \{t, w\}, \{s, v\}\}$. Let $\rho = \{(t, 0.6e^{i1.2\pi}), (w, 0.8e^{i1.6\pi}), (u, 0.5e^{i\pi})\}$ be a CF set on T. Then the lower and upper rough approximations of ρ can be calculated as,

$$
\begin{aligned}
k_{\underline{X}\rho}(t) &= (1\vee 0)\wedge(0\vee 0.6)\wedge(0\vee 0.8)\wedge(1\vee 0) \\
&\quad \wedge(1\vee 0.5)\wedge(1\vee 0) = 0.6 \\
\arg_{\underline{X}\rho}(t) &= 2\pi\left[(1\vee 0)\wedge(0\vee\frac{1.2\pi}{2\pi})\wedge(0\vee\frac{1.6\pi}{2\pi})\right. \\
&\quad \left.\wedge(1\vee 0)\wedge(1\vee\frac{\pi}{2\pi})\wedge(1\vee 0)\right] = 1.2\pi \\
k_{\overline{X}\rho}(t) &= (0\wedge 0)\vee(1\wedge 0.6)\vee(1\wedge 0.8)\vee(0\wedge 0) \\
&\quad \vee(0\wedge 0.5)\vee(0\wedge 0) = 0.8 \\
\arg_{\overline{X}\rho}(t) &= 2\pi\left[(0\wedge 0)\vee(1\wedge\frac{1.2\pi}{2\pi})\vee(1\wedge\frac{1.6\pi}{2\pi})\vee(0\wedge 0)\right. \\
&\quad \left.\vee(0\wedge\frac{\pi}{2\pi})\vee(0\wedge 0)\right] = 1.6\pi.
\end{aligned}
$$

Thus, $\underline{X}\rho = \{(t, 0.6e^{i1.2\pi}), (w, 0.6e^{i1.2\pi})\}$ and $\overline{X}\rho = \{(\epsilon, 0.5e^{i\pi}), (t, 0.8e^{i1.6\pi}), (w, 0.8e^{i1.6\pi}), (u, 0.5e^{i\pi})\}$. The pair $X\rho = (\underline{X}\rho, \overline{X}\rho)$ is an RCF set.

Definition 9 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets in an approximation space (T, X). The *union* of $X\rho$ and $X\varphi$, denoted by $X\rho \cup X\varphi = (\underline{X}\rho \cup \underline{X}\varphi, \overline{X}\rho \cup \overline{X}\varphi)$, is an RCF set such that for all $\epsilon \in T$,

$$
\begin{aligned}
(\underline{X}\rho \cup \underline{X}\varphi)(\epsilon) &= k_{\underline{X}\rho\cup\underline{X}\varphi}(\epsilon)e^{i\arg_{\underline{X}\rho\cup\underline{X}\varphi}(\epsilon)} \\
&= \left(k_{\underline{X}\rho}(\epsilon)\vee k_{\underline{X}\varphi}(\epsilon)\right)e^{i\left(\arg_{\underline{X}\rho}(\epsilon)\vee\arg_{\underline{X}\varphi}(\epsilon)\right)}, \\
(\overline{X}\rho \cup \overline{X}\varphi)(\epsilon) &= k_{\overline{X}\rho\cup\overline{X}\varphi}(\epsilon)e^{i\arg_{\overline{X}\rho\cup\overline{X}\varphi}(\epsilon)} \\
&= \left(k_{\overline{X}\rho}(\epsilon)\vee k_{\overline{X}\varphi}(\epsilon)\right)e^{i\left(\arg_{\overline{X}\rho}(\epsilon)\vee\arg_{\overline{X}\varphi}(\epsilon)\right)}.
\end{aligned}
$$

Remark 3 By Definition 4 given by Zhang et al. (2009), $\underline{X}\alpha \cup \underline{X}\varphi$ and $\overline{X}\alpha \cup \overline{X}\varphi$ are both CF sets. Although we have $\overline{X}\alpha \cup \overline{X}\varphi = \overline{X}(\alpha \cup \varphi)$, the equality $\underline{X}\alpha \cup \underline{X}\varphi = \underline{X}(\alpha \cup \varphi)$ does not hold in general. As similar to the case of Pawlak's rough sets, we only have $\underline{X}\alpha \cup \underline{X}\varphi \subseteq \underline{X}(\alpha \cup \varphi)$. Nevertheless, it should be noted that by Definition 8, the ordered pair $(\underline{X}\alpha \cup \underline{X}\varphi, \overline{X}\alpha \cup \overline{X}\varphi)$ is an RCF set in (T, X) if and only if $\underline{X}\alpha \cup \underline{X}\varphi$ and $\overline{X}\alpha \cup \overline{X}\varphi$ are the lower and upper rough approximations of some CF set ρ on T.

In Zhang et al. (2010) Zhang et al. proved that for any two RF sets $X\mu$ and $X\nu$, there exists a fuzzy set λ such that $(\underline{X}\lambda, \overline{X}\lambda) = (\underline{X}\mu \cup \underline{X}\nu, \overline{X}\mu \cup \overline{X}\nu)$. It is clear that any CF set can be captured by two fuzzy sets corresponding to its amplitude and phase terms. Thus, Theorem 6 in Zhang et al. (2010) can be generalized for RCF sets as follows: Let $X\alpha$ and $X\varphi$ be two RCF sets, then there exists $\rho \in CF(T)$ such that the equalities $\underline{X}\rho = \underline{X}\alpha \cup \underline{X}\varphi$ and $\overline{X}\rho = \overline{X}\alpha \cup \overline{X}\varphi$ hold simultaneously. In fact, the CF set ρ can be determined as follows:

$$
\begin{aligned}
\rho = (\cup\{\delta \in RCF(T) \mid \delta \subseteq \alpha,\ supp(\delta) \cap R(\alpha, \varphi) \\
= \varnothing\}) \cup \varphi,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
R(\alpha, \varphi) = \cup\{[t]_X \mid t \in T,\ [t]_X \subseteq supp(\alpha \cup \varphi),\ [t]_X \not\subseteq supp(\alpha), \\
[t]_X \not\subseteq supp(\varphi)\},
\end{aligned} \tag{7}
$$

where $supp(\alpha) = \{t \in T \mid \alpha(t) \neq 0\}$.

Example 2 We now explain Remark 3 with a simple example. Let $T = \{x, y, w, \epsilon\}$, $X = \{\{x\}, \{y\}, \{w, \epsilon\}\}$, $\alpha = \{(x, 0.8e^{i1.2\pi}), (w, 0.7e^{i0.8\pi})\}$ and $\varphi = \{(\epsilon, 1.3e^{i0.9\pi})\}$. By calculation, we have

$$
\begin{aligned}
&\underline{X}\alpha = \{(x, 0.8e^{i1.2\pi})\} \\
&\overline{X}\alpha = \{(x, 0.8e^{i1.2\pi}), (w, 0.7e^{i0.8\pi}), (\epsilon, 0.7e^{i0.8\pi})\} \\
&\underline{X}\varphi = \varnothing \\
&\overline{X}\varphi = \{(w, 1.3e^{i0.9\pi}), (\epsilon, 1.3e^{i0.9\pi})\}.
\end{aligned}
$$

According to Eq. (6) and Eq. (7) in Remark 3, we have $R(\alpha, \varphi) = \{w, \epsilon\}$ and

$$\rho = \{(x, 0.8e^{i1.2\pi})\} \cup \varphi = \{(x, 0.8e^{i1.2\pi}), (\epsilon, 1.3e^{i0.9\pi})\}.$$

Note also that

$$\underline{X}\rho = \{(x, 0.8e^{i1.2\pi})\}$$
$$\overline{X}\rho = \{(x, 0.8e^{i1.2\pi}), (w, 1.3e^{i0.9\pi}), (\epsilon, 1.3e^{i0.9\pi})\}.$$

Thus we conclude that $(\underline{X}\rho, \overline{X}\rho) = X\alpha \cup X\varphi$.

Example 3 Let $X = \{\{\epsilon, u\}, \{t, v\}\}$ be an equivalence relation on $T = \{\epsilon, t, u, v\}$. Let φ and ϑ be two CF sets on T which are defined as,

$$\varphi = \{(\epsilon, 0.8e^{i1.3\pi}), (t, 0.6e^{i1.1\pi}), (u, 0.7e^{i1.2\pi}), (v, 0.5e^{i\pi})\}$$
$$\vartheta = \{(\epsilon, 0.9e^{i1.4\pi}), (t, 0.5e^{i\pi}), (u, 0.8e^{i1.3\pi}), (v, 0.4e^{i0.9\pi})\}.$$

The RCF sets $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ and $X\vartheta = (\underline{X}\vartheta, \overline{X}\vartheta)$ are obtained as,

$$\underline{X}\varphi = \{(\epsilon, 0.7e^{i1.2\pi}), (t, 0.5e^{i\pi}), (u, 0.7e^{i1.2\pi}), (v, 0.5e^{i\pi})\}$$
$$\overline{X}\varphi = \{(\epsilon, 0.8e^{i1.3\pi}), (t, 0.6e^{i1.1\pi}), (u, 0.8e^{i1.3\pi}), (v, 0.6e^{i1.1\pi})\}$$
$$\underline{X}\vartheta = \{(\epsilon, 0.8e^{i1.3\pi}), (t, 0.4e^{i0.9\pi}), (u, 0.8e^{i1.3\pi}), (v, 0.4e^{i0.9\pi})\}$$
$$\overline{X}\vartheta = \{(\epsilon, 0.9e^{i1.4\pi}), (t, 0.5e^{i\pi}), (u, 0.9e^{i1.4\pi}), (v, 0.5e^{i\pi})\}.$$

The union of RCF sets $X\varphi$ and $X\vartheta$ is $X\varphi \cup X\vartheta = (\underline{X}\varphi \cup \underline{X}\vartheta, \overline{X}\varphi \cup \overline{X}\vartheta)$, where

$$\underline{X}\varphi \cup \underline{X}\vartheta = \{(\epsilon, 0.8e^{i1.3\pi}), (t, 0.5e^{i\pi}), (u, 0.8e^{i1.3\pi}), (v, 0.5e^{i\pi})\}$$
$$\overline{X}\varphi \cup \overline{X}\vartheta = \{(\epsilon, 0.9e^{i1.4\pi}), (t, 0.6e^{i1.1\pi}), (u, 0.9e^{i1.4\pi}), (v, 0.6e^{i1.1\pi})\}.$$

According to Eq. (6) and Eq. (7) in Remark 3, we have $R(\varphi, \vartheta) = \varnothing$ and so $\rho = (\cup\{\delta \in RCF(T) \mid \delta \subseteq \varphi\}) \cup \vartheta = \varphi \cup \vartheta = \{(\epsilon, 0.9e^{i1.4\pi}), (t, 0.6e^{i1.1\pi}), (u, 0.8e^{i1.3\pi}), (v, 0.5e^{i\pi})\}$. By calculation, one easily verify that $\underline{X}\rho = \underline{X}\varphi \cup \underline{X}\vartheta$ and $\overline{X}\rho = \overline{X}\varphi \cup \overline{X}\vartheta$.

Remark 4 For any two RCF sets α and φ, if $R(\alpha, \varphi) = \varnothing$, given in Remark 3, then $\underline{X}\alpha \cup \underline{X}\varphi = \underline{X}(\alpha \cup \varphi)$ and $\overline{X}\alpha \cup \overline{X}\varphi = \overline{X}(\alpha \cup \varphi)$.

Definition 10 Let $X\alpha$ and $X\vartheta$ be two RCF sets on T. Then $X\alpha$ is called a *RCF subset* of $X\vartheta$ if $\underline{X}\alpha \subseteq \underline{X}\vartheta$ and $\overline{X}\alpha \subseteq \overline{X}\vartheta$. That is, for all $\epsilon \in T$, we have

$$k_{\underline{X}\alpha}(\epsilon) \le k_{\underline{X}\vartheta}(\epsilon), \quad \arg_{\underline{X}\alpha}(\epsilon) \le \arg_{\underline{X}\vartheta}(\epsilon),$$
$$k_{\overline{X}\alpha}(\epsilon) \le k_{\overline{X}\vartheta}(\epsilon), \quad \arg_{\overline{X}\alpha}(\epsilon) \le \arg_{\overline{X}\vartheta}(\epsilon).$$

Definition 11 Let $\xi : [0, 1] \times [0, 1] \to [0, 1]$ be a uninorm and $RCF(T, X)$ be the set of all RCF sets in an approximation space (T, X). An *RCF uninorm* is a mapping $\Psi : RCF(T, X) \times RCF(T, X) \to RCF(T, X)$ defined as

$$\Psi(X\alpha, X\vartheta) = (\Psi(\underline{X}\alpha, \underline{X}\vartheta), \Psi(\overline{X}\alpha, \overline{X}\vartheta)),$$

where $X\alpha = (\underline{X}\alpha, \overline{X}\alpha), X\vartheta = (\underline{X}\vartheta, \overline{X}\vartheta) \in RCF(T, X)$ and

$$\Psi(\underline{X}\alpha, \underline{X}\vartheta) = \sup_{\epsilon \in T} \xi\left(k_{\underline{X}\alpha}(\epsilon), k_{\underline{X}\vartheta}(\epsilon)\right) e^{i2\pi \sup_{\epsilon \in T} \xi\left(\frac{\arg_{\underline{X}\alpha}(\epsilon)}{2\pi}, \frac{\arg_{\underline{X}\vartheta}(\epsilon)}{2\pi}\right)},$$

$$\Psi(\overline{X}\alpha, \overline{X}\vartheta) = \sup_{\epsilon \in T} \xi\left(k_{\overline{X}\alpha}(\epsilon), k_{\overline{X}\vartheta}(\epsilon)\right) e^{i2\pi \sup_{\epsilon \in T} \xi\left(\frac{\arg_{\overline{X}\alpha}(\epsilon)}{2\pi}, \frac{\arg_{\overline{X}\vartheta}(\epsilon)}{2\pi}\right)}.$$

In particular, an RCF uninorm Ψ is called an *RCF triangular norm* (or *RCF t-norm*) if $\Psi(X\alpha, T) = X\alpha$ for all $X\alpha \in RCF(T, X)$. An RCF uninorm Ψ is called an *RCF triangular conorm* (or *RCF s-norm*) if $\Psi(X\alpha, \varnothing) = X\alpha$ for all $X\alpha \in RCF(T, X)$.

Theorem 1 *The union of RCF sets is an RCF s-norm.*

Proof Since the union over any two RCF sets is symmetric, therefore the first condition in Definition 7 holds. Let $X\alpha$, $X\varphi$, $X\vartheta$ and $X\rho$ be RCF sets on T. To show that union is an RCF *s*-norm, we will prove the following conditions.

1. If $X\alpha \subseteq X\vartheta$ and $X\varphi \subseteq X\rho$, then $X\alpha \cup X\varphi \subseteq X\vartheta \cup X\rho$,
2. $(X\alpha \cup X\vartheta) \cup X\varphi = X\alpha \cup (X\vartheta \cup X\varphi)$,
3. $X\alpha \cup \varnothing = X\alpha$.

1. Let $X\alpha \subseteq X\vartheta$ and $X\varphi \subseteq X\rho$, then for all $\epsilon \in T$
$$k_{\underline{X}\alpha \cup \underline{X}\varphi}(\epsilon) = k_{\underline{X}\alpha}(\epsilon) \vee k_{\underline{X}\varphi}(\epsilon) \le k_{\underline{X}\vartheta}(\epsilon) \vee k_{\underline{X}\rho}(\epsilon)$$
$$= k_{\underline{X}\vartheta \cup \underline{X}\rho}(\epsilon),$$
$$\arg_{\underline{X}\alpha \cup \underline{X}\varphi}(\epsilon) = \arg_{\underline{X}\alpha}(\epsilon) \vee \arg_{\underline{X}\varphi}(\epsilon) \le \arg_{\underline{X}\vartheta}(\epsilon) \vee \arg_{\underline{X}\rho}(\epsilon)$$
$$= \arg_{\underline{X}\vartheta \cup \underline{X}\rho}(\epsilon).$$

It follows that $\underline{X}\alpha \cup \underline{X}\varphi \subseteq \underline{X}\vartheta \cup \underline{X}\rho$. Similarly, it can be shown that $\overline{X}\alpha \cup \overline{X}\varphi \subseteq \overline{X}\vartheta \cup \overline{X}\rho$. Thus, $X\alpha \cup X\varphi \subseteq X\vartheta \cup X\rho$.

2. For all $\epsilon \in T$,
$$[(\underline{X}\alpha \cup \underline{X}\varphi) \cup \underline{X}\vartheta](\epsilon) = k_{(\underline{X}\alpha \cup \underline{X}\varphi) \cup \underline{X}\vartheta}(\epsilon) e^{i \arg_{(\underline{X}\alpha \cup \underline{X}\varphi) \cup \underline{X}\vartheta}(\epsilon)}$$
$$= \left[(k_{\underline{X}\alpha}(\epsilon) \vee k_{\underline{X}\varphi}(\epsilon)) \vee k_{\underline{X}\vartheta}(\epsilon)\right] e^{i\left[(\arg_{\underline{X}\alpha}(\epsilon) \vee \arg_{\underline{X}\varphi}(\epsilon)) \vee \arg_{\underline{X}\vartheta}(\epsilon)\right]}$$
$$= \left[k_{\underline{X}\alpha}(\epsilon) \vee (k_{\underline{X}\varphi}(\epsilon) \vee k_{\underline{X}\vartheta}(\epsilon))\right] e^{i\left[\arg_{\underline{X}\alpha}(\epsilon) \vee (\arg_{\underline{X}\varphi}(\epsilon) \vee \arg_{\underline{X}\vartheta}(\epsilon))\right]}$$
$$= k_{\underline{X}\alpha \cup (\underline{X}\varphi \cup \underline{X}\vartheta)}(\epsilon) e^{i \arg_{\underline{X}\alpha \cup (\underline{X}\varphi \cup \underline{X}\vartheta)}(\epsilon)}$$
$$= [\underline{X}\alpha \cup (\underline{X}\varphi \cup \underline{X}\vartheta)](\epsilon)$$
$$\Rightarrow \quad (\underline{X}\alpha \cup \underline{X}\varphi) \cup \underline{X}\vartheta = \underline{X}\alpha \cup (\underline{X}\varphi \cup \underline{X}\vartheta).$$

Similarly, it can be proved that $(\overline{X}\alpha \cup \overline{X}\varphi) \cup \overline{X}\vartheta = \overline{X}\alpha \cup (\overline{X}\varphi \cup \overline{X}\vartheta)$. Hence $(X\alpha \cup X\varphi) \cup X\vartheta = X\alpha \cup (X\varphi \cup X\vartheta)$.

3. For all $\epsilon \in T$,

$$
\begin{aligned}
X\alpha \cup \varnothing &= \left(\underline{X}\alpha \cup \varnothing, \overline{X}\alpha \cup \varnothing\right) \\
&= \left(k_{\underline{X}\alpha\cup\varnothing}(\epsilon)e^{i\arg_{\underline{X}\alpha\cup\varnothing}(\epsilon)}, k_{\overline{X}\alpha\cup\varnothing}(\epsilon)e^{i\arg_{\overline{X}\alpha\cup\varnothing}(\epsilon)}\right) \\
&= \left((k_{\underline{X}_\alpha}(\epsilon)\vee 0)e^{i(\arg_{\underline{X}_\alpha}(\epsilon)\vee 0)}, (k_{\overline{X}_\alpha}(\epsilon)\vee 0)e^{i(\arg_{\overline{X}_\alpha}(\epsilon)\vee 0)}\right) \\
&= \left(k_{\underline{X}_\alpha}(\epsilon)e^{i\arg_{\underline{X}_\alpha}(\epsilon)}, k_{\overline{X}_\alpha}(\epsilon)e^{i\arg_{\overline{X}_\alpha}(\epsilon)}\right) \\
\Rightarrow\ & X\alpha \cup \varnothing = X\alpha.
\end{aligned}
$$

□

Corollary 1 *Let* $\{XC_\alpha \in RCF(T,X) \mid \alpha \in I\}$ *be the countable collection of RCF sets on T, then* $\cup_{j=1}^{n} XC_j = \left(\cup_{j=1}^{n}\underline{X}C_j, \cup_{j=1}^{n}\overline{X}C_j\right)$ *is defined as,*

$$
\begin{aligned}
\left(\cup_{\alpha\in I} XC_\alpha\right)(\epsilon) = \Bigg(&\sup_{\alpha\in I} k_{\underline{X}C_\alpha}(\epsilon)e^{i\sup_{\alpha\in I}\arg_{\underline{X}C_\alpha}(\epsilon)}, \\
&\sup_{\alpha\in I} k_{\overline{X}C_\alpha}(\epsilon)e^{i\sup_{\alpha\in I}\arg_{\overline{X}C_\alpha}(\epsilon)}\Bigg).
\end{aligned}
$$

The proof of Corollary 1 is a direct consequence of Theorem 1 and Definition 9.

Definition 12 Let $X\alpha = (\underline{X}\alpha, \overline{X}\alpha)$ be an RCF set in an approximation space (T, X). The *complement* of $X\alpha$, denoted by $(X\alpha)^c = \left((\overline{X}\alpha)^c, (\underline{X}\alpha)^c\right)$, is an RCF set such that for all $\epsilon \in T$,

$$
\begin{aligned}
&(\underline{X}\alpha)^c(\epsilon) = (1 - k_{\underline{X}_\alpha}(\epsilon))e^{i(2\pi - \arg_{\underline{X}_\alpha}(\epsilon))} \quad \text{and} \quad (\overline{X}\alpha)^c(\epsilon) \\
&\quad = (1 - k_{\overline{X}_\alpha}(\epsilon))e^{i(2\pi - \arg_{\overline{X}_\alpha}(\epsilon))}.
\end{aligned}
$$

Remark 5 In contrast to the case discussed in Remark 3, the above-defined complement of RCF sets is indeed a unary operation on $RCF(T, X)$. In fact, we will show in Theorem 6 that $(\overline{X}\alpha)^c = \underline{X}\alpha^c$ and $(\underline{X}\alpha)^c = \overline{X}\alpha^c$. That is, $(\overline{X}\alpha)^c$ and $(\underline{X}\alpha)^c$ are exactly the lower and upper rough approximations of the CF set α^c. In other words, there exists $\alpha^c \in CF(T)$ such that

$$
X\alpha^c = (\underline{X}\alpha^c, \overline{X}\alpha^c) = \left((\overline{X}\alpha)^c, (\underline{X}\alpha)^c\right) = (X\alpha)^c.
$$

Therefore, the ordered pair $\left((\overline{X}\alpha)^c, (\underline{X}\alpha)^c\right)$ is indeed an RCF set.

Example 4 Consider an RCF set $X\rho$ on $T = \{\epsilon, t, w, s, u, v\}$ as defined in Example 1. The complement $(X\rho)^c = ((\overline{X}\rho)^c, (\underline{X}\rho)^c)$ of $X\rho$ is computed as,

$$
\begin{aligned}
(\underline{X}\rho)^c &= \{(t, 0.4e^{i\pi}), (w, 0.4e^{i\pi}), (\epsilon, e^{i\pi}), (s, e^{i\pi}), \\
&\quad (u, e^{i\pi}), (v, e^{i\pi})\} \\
(\overline{X}\rho)^c &= \{(\epsilon, 0.5e^{i\pi}), (t, 0.2e^{i0.4\pi}), (w, 0.2e^{i0.4\pi}), (u, 0.5e^{i\pi}), \\
&\quad (s, e^{i\pi}), (v, e^{i\pi})\}.
\end{aligned}
$$

Theorem 2 *For any RCF set* $X\alpha$ *on a non-empty set T,* $(X\alpha)^{cc} = X\alpha$.

Proof For all $\epsilon \in T$,

$$
\begin{aligned}
(X\alpha)^{cc}(\epsilon) &= \left(k_{(\underline{X}\alpha)^{cc}}(\epsilon)e^{i\arg_{(\underline{X}\alpha)^{cc}}(\epsilon)}, k_{(\overline{X}\alpha)^{cc}}(\epsilon)e^{i\arg_{(\overline{X}\alpha)^{cc}}(\epsilon)}\right) \\
&= \Big((1 - (1 - k_{\underline{X}_\alpha}(\epsilon)))e^{i(2\pi - (2\pi - \arg_{\underline{X}_\alpha}(\epsilon)))}, \\
&\quad (1 - (1 - k_{\overline{X}_\alpha}(\epsilon)))e^{i(2\pi - (2\pi - \arg_{\overline{X}_\alpha}(\epsilon)))}\Big) \\
&= \left(k_{\underline{X}_\alpha}(\epsilon)e^{i\arg_{\underline{X}_\alpha}(\epsilon)}, k_{\overline{X}_\alpha}(\epsilon)e^{i\arg_{\overline{X}_\alpha}(\epsilon)}\right) \\
\Rightarrow\ & (X\alpha)^{cc}(\epsilon) = (X\alpha)(\epsilon).
\end{aligned}
$$

□

Definition 13 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets in an approximation space (T, X). The *intersection* of $X\rho$ and $X\varphi$, denoted by $X\rho \cap X\varphi = (\underline{X}\rho \cap \underline{X}\varphi, \overline{X}\rho \cap \overline{X}\varphi)$, is an RCF set such that for all $\epsilon \in T$,

$$
\begin{aligned}
(\underline{X}\rho \cap \underline{X}\varphi)(\epsilon) &= k_{\underline{X}\rho\cap\underline{X}\varphi}(\epsilon)e^{i\arg_{\underline{X}\rho\cap\underline{X}\varphi}(\epsilon)} \\
&= \left(k_{\underline{X}\rho}(\epsilon) \wedge k_{\underline{X}\varphi}(\epsilon)\right)e^{i\left(\arg_{\underline{X}\rho}(\epsilon)\wedge\arg_{\underline{X}\varphi}(\epsilon)\right)}, \\
(\overline{X}\rho \cap \overline{X}\varphi)(\epsilon) &= k_{\overline{X}\rho\cap\overline{X}\varphi}(\epsilon)e^{i\arg_{\overline{X}\rho\cap\overline{X}\varphi}(\epsilon)} \\
&= \left(k_{\overline{X}\rho}(\epsilon) \wedge k_{\overline{X}\varphi}(\epsilon)\right)e^{i\left(\arg_{\overline{X}\rho}(\epsilon)\wedge\arg_{\overline{X}\varphi}(\epsilon)\right)}.
\end{aligned}
$$

Remark 6 In Theorem 7 by Zhang et al. (2010) and Theorem 2.2 by Zhang et al. (2015), it is proved that for any two RF sets $X\mu$ and $X\nu$, there exists a fuzzy set λ such that $(\underline{X}\lambda, \overline{X}\lambda) = (\underline{X}\mu \cap \underline{X}\nu, \overline{X}\mu \cap \overline{X}\nu)$. As it is clear from Ramot et al. (2002) that any CF set can be captured by two fuzzy sets corresponding to amplitude and phase terms. So, Theorem 6 of Zhang et al. (2010) can be generalized for RCF sets as: Let $X\alpha$ and $X\varphi$ be two RCF sets, then there exists $\rho \in CF(T)$ such that the equalities $\underline{X}\rho = \underline{X}\alpha \cap \underline{X}\varphi$ and $\overline{X}\rho = \overline{X}\alpha \cap \overline{X}\varphi$ hold simultaneously. The CF set ρ can be determined as,

$$
\begin{aligned}
\rho = (&\cup\{\delta \in RCF(T) \mid \delta \subseteq \varphi^c, \\
&supp(\delta) \cap R(\alpha^c, \varphi^c) = \varnothing\}) \cap \alpha^c, \\
R(\alpha^c, \varphi^c) = &\cup\big\{[t]_X \mid t \in T,\ [t]_X \subseteq supp(\alpha \cap \varphi)^c, \\
&[t]_X \not\subseteq supp(\alpha^c),\ [t]_X \not\subseteq supp(\varphi^c)\big\}.
\end{aligned}
$$

Example 5 Consider RCF sets $X\varphi$ and $X\vartheta$ as defined in Example 3. The intersection of $X\varphi$ and $X\vartheta$ is $X\varphi \cap X\vartheta = (\underline{X}\varphi \cap \underline{X}\vartheta, \overline{X}\varphi \cap \overline{X}\vartheta)$, where

$$\underline{X}\varphi \cap \underline{X}\vartheta = \{(\epsilon, 0.7e^{i1.2\pi}), (t, 0.4e^{i0.9\pi}), (u, 0.7e^{i1.2\pi}), (v, 0.4e^{i0.9\pi})\},$$
$$\overline{X}\varphi \cap \overline{X}\vartheta = \{(\epsilon, 0.8e^{i1.3\pi}), (t, 0.5e^{i\pi}), (u, 0.8e^{i1.3\pi}), (v, 0.5e^{i\pi})\}.$$

Theorem 3 *The intersection of RCF sets is an RCF t-norm.*

Proof Since the intersection over any two RCF sets is symmetric, therefore the condition 1 in Definition 7 holds. Let $X\alpha$, $X\varphi$, $X\vartheta$ and $X\rho$ be RCF sets on T. To show that intersection is an RCF t-norm, we will prove the following conditions.

1. If $X\alpha \subseteq X\vartheta$ and $X\varphi \subseteq X\rho$, then $X\alpha \cap X\varphi \subseteq X\vartheta \cap X\rho$,
2. $(X\alpha \cap X\vartheta) \cap X\varphi = X\alpha \cap (X\vartheta \cap X\varphi)$,
3. $X\alpha \cap T = X\alpha$.

1. Let $X\alpha \subseteq X\vartheta$ and $X\varphi \subseteq X\rho$, then for all $\epsilon \in T$

$$k_{\underline{X}\alpha \cap \underline{X}\varphi}(\epsilon) = k_{\underline{X}\alpha}(\epsilon) \wedge k_{\underline{X}\varphi}(\epsilon) \leq k_{\underline{X}\vartheta}(\epsilon) \wedge k_{\underline{X}\rho}(\epsilon) = k_{\underline{X}\vartheta \cap \underline{X}\rho}(\epsilon),$$
$$\arg_{\underline{X}\alpha \cap \underline{X}\varphi}(\epsilon) = \arg_{\underline{X}\alpha}(\epsilon) \wedge \arg_{\underline{X}\varphi}(\epsilon) \leq \arg_{\underline{X}\vartheta}(\epsilon) \wedge \arg_{\underline{X}\rho}(\epsilon) = \arg_{\underline{X}\vartheta \cap \underline{X}\rho}(\epsilon).$$

It follows that $\underline{X}\alpha \cap \underline{X}\varphi \subseteq \underline{X}\vartheta \cap \underline{X}\rho$. Similarly, it can be shown that $\overline{X}\alpha \cap \overline{X}\varphi \subseteq \overline{X}\vartheta \cap \overline{X}\rho$. Thus, $X\alpha \cap X\varphi \subseteq X\vartheta \cap X\rho$.

2. For all $\epsilon \in T$,

$$\begin{aligned}[(\underline{X}\alpha \cap \underline{X}\varphi) \cap \underline{X}\vartheta](\epsilon) &= k_{(\underline{X}\alpha \cap \underline{X}\varphi) \cap \underline{X}\vartheta}(\epsilon) e^{i \arg_{(\underline{X}\alpha \cap \underline{X}\varphi) \cap \underline{X}\vartheta}(\epsilon)} \\ &= \left[(k_{\underline{X}\alpha}(\epsilon) \wedge k_{\underline{X}\varphi}(\epsilon)) \wedge k_{\underline{X}\vartheta}(\epsilon)\right] \\ &\quad e^{i\left[(\arg_{\underline{X}\alpha}(\epsilon) \wedge \arg_{\underline{X}\varphi}(\epsilon)) \wedge \arg_{\underline{X}\vartheta}(\epsilon)\right]} \\ &= \left[k_{\underline{X}\alpha}(\epsilon) \wedge (k_{\underline{X}\varphi}(\epsilon) \wedge k_{\underline{X}\vartheta}(\epsilon))\right] \\ &\quad e^{i\left[\arg_{\underline{X}\alpha}(\epsilon) \wedge (\arg_{\underline{X}\varphi}(\epsilon) \wedge \arg_{\underline{X}\vartheta}(\epsilon))\right]} \\ &= k_{\underline{X}\alpha \cap (\underline{X}\varphi \cap \underline{X}\vartheta)}(\epsilon) e^{i \arg_{\underline{X}\alpha \cap (\underline{X}\varphi \cap \underline{X}\vartheta)}(\epsilon)} \\ &= [\underline{X}\alpha \cap (\underline{X}\varphi \cap \underline{X}\vartheta)](\epsilon) \end{aligned}$$
$$\Rightarrow \quad (\underline{X}\alpha \cap \underline{X}\varphi) \cap \underline{X}\vartheta = \underline{X}\alpha \cap (\underline{X}\varphi \cap \underline{X}\vartheta).$$

Similarly, it can be proved that $(\overline{X}\alpha \cap \overline{X}\varphi) \cap \overline{X}\vartheta = \overline{X}\alpha \cap (\overline{X}\varphi \cap \overline{X}\vartheta)$. Hence $(X\alpha \cap X\varphi) \cap X\vartheta = X\alpha \cap (X\varphi \cap X\vartheta)$.

3. For all $\epsilon \in T$,

$$\begin{aligned} X\alpha \cap T &= (\underline{X}\alpha \cap T, \overline{X}\alpha \cap T) \\ &= \left(k_{\underline{X}\alpha \cap T}(\epsilon) e^{i \arg_{\underline{X}\alpha \cap T}(\epsilon)}, k_{\overline{X}\alpha \cap T}(\epsilon) e^{i \arg_{\overline{X}\alpha \cap T}(\epsilon)}\right) \\ &= \left((k_{\underline{X}\alpha}(\epsilon) \wedge 1) e^{i(\arg_{\underline{X}\alpha}(\epsilon) \wedge 2\pi)}, (k_{\overline{X}\alpha}(\epsilon) \wedge 1) e^{i(\arg_{\overline{X}\alpha}(\epsilon) \wedge 2\pi)}\right) \\ &= \left(k_{\underline{X}\alpha}(\epsilon) e^{i \arg_{\underline{X}\alpha}(\epsilon)}, k_{\overline{X}\alpha}(\epsilon) e^{i \arg_{\overline{X}\alpha}(\epsilon)}\right) \\ \Rightarrow \quad X\alpha \cap T &= X\alpha. \end{aligned}$$

□

Corollary 2 *Let* $\{XC_\alpha \in RCF(T, X) \mid \alpha \in I\}$ *be a countable collection of RCF sets. Then*

$$\left(\cap_{\alpha \in I} XC_\alpha\right)(\epsilon) = \left(\inf_{\alpha \in I} k_{\underline{X}C_\alpha}(\epsilon) e^{i \inf_{\alpha \in I} \arg_{\underline{X}C_\alpha}(\epsilon)}, \inf_{\alpha \in I} k_{\overline{X}C_\alpha}(\epsilon) e^{i \inf_{\alpha \in I} \arg_{\overline{X}C_\alpha}(\epsilon)}\right).$$

The proof of Corollary 2 is a direct consequence of Theorem 3 and Definition 13.

Corollary 3 *Let* $\{XC_{\alpha\varphi} \in RCF(T, X) \mid \alpha \in I_1, \varphi \in I_2\}$ *be the countable collection of RCF sets on T, then*

$$\left(\cap_{\alpha \in I_1} \cup_{\varphi \in I_2} \underline{X}C_\alpha\right)(\epsilon) = \inf_{\alpha \in I_1} \sup_{\varphi \in I_2} k_{\underline{X}C_{\alpha\varphi}}(\epsilon) e^{i \inf_{\alpha \in I_1} \sup_{\varphi \in I_2} \arg_{\underline{X}C_{\alpha\varphi}}(\epsilon)},$$
$$\left(\cap_{\alpha \in I_1} \cup_{\varphi \in I_2} \overline{X}C_\alpha\right)(\epsilon) = \inf_{\alpha \in I_1} \sup_{\varphi \in I_2} k_{\overline{X}C_{\alpha\varphi}}(\epsilon) e^{i \inf_{\alpha \in I_1} \sup_{\varphi \in I_2} \arg_{\overline{X}C_{\alpha\varphi}}(\epsilon)}.$$

The proof of Corollary 3 is trivial from Corollary 1 and 2.

Theorem 4 *Let* $X\alpha, X\varphi$ *and* $X\vartheta$ *be two RCF sets on T, then*

1. $(X\alpha \cap X\vartheta)^c = (X\alpha)^c \cup (X\vartheta)^c$
2. $(X\alpha \cup X\vartheta)^c = (X\alpha)^c \cap (X\vartheta)^c$
3. $(X\alpha \cup X\varphi) \cap X\vartheta = (X\alpha \cap X\vartheta) \cup (X\varphi \cap X\vartheta)$
4. $(X\alpha \cap X\varphi) \cup X\vartheta = (X\alpha \cup X\vartheta) \cap (X\varphi \cup X\vartheta)$.

Proof

1. Using Definition 9, 12 and 13, we have

$$
\begin{aligned}
&(X\alpha \cap X\vartheta)^c(\epsilon)\\
&\quad = \Big(k_{(\underline{X}\alpha\cap\underline{X}\vartheta)^c}(\epsilon)e^{i\arg_{(\underline{X}\alpha\cap\underline{X}\vartheta)^c}(\epsilon)}, k_{(\overline{X}\alpha\cap\overline{X}\vartheta)^c}(\epsilon)e^{i\arg_{(\overline{X}\alpha\cap\overline{X}\vartheta)^c}(\epsilon)}\Big)\\
&\quad = \Big((1-(k_{\underline{X}\alpha}(\epsilon)\wedge k_{\underline{X}\vartheta}(\epsilon)))e^{i(2\pi-(\arg_{\underline{X}\alpha}(\epsilon)\wedge\arg_{\underline{X}\vartheta}(\epsilon)))},\\
&\qquad (1-(k_{\overline{X}\alpha}(\epsilon)\wedge k_{\overline{X}\vartheta}(\epsilon)))e^{i(2\pi-(\arg_{\overline{X}\alpha}(\epsilon)\wedge\arg_{\overline{X}\vartheta}(\epsilon)))}\Big)\\
&\quad = \Big((1-k_{\underline{X}\alpha}(\epsilon))\vee(1-k_{\underline{X}\vartheta}(\epsilon)))e^{i(2\pi-\arg_{\underline{X}\alpha}(\epsilon))\vee(2\pi-\arg_{\underline{X}\vartheta}(\epsilon)))},\\
&\qquad (1-k_{\overline{X}\alpha}(\epsilon))\vee(1-k_{\overline{X}\vartheta}(\epsilon)))e^{i(2\pi-\arg_{\overline{X}\alpha}(\epsilon))\vee(2\pi-\arg_{\overline{X}\vartheta}(\epsilon)))}\Big)\\
&\quad = \Big((k_{(\underline{X}\alpha)^c}(\epsilon)\vee k_{(\underline{X}\vartheta)^c}(\epsilon))e^{i(\arg_{(\underline{X}\alpha)^c}(\epsilon)\vee\arg_{(\underline{X}\vartheta)^c}(\epsilon))},\\
&\qquad (k_{(\overline{X}\alpha)^c}(\epsilon)\vee k_{(\overline{X}\vartheta)^c}(\epsilon))e^{i(\arg_{(\overline{X}\alpha)^c}(\epsilon)\vee\arg_{(\overline{X}\vartheta)^c}(\epsilon))}\Big)\\
&\quad = \Big(k_{(\underline{X}\alpha)^c\cup(\underline{X}\vartheta)^c}(\epsilon)e^{i\arg_{(\underline{X}\alpha)^c\cup(\underline{X}\vartheta)^c}(\epsilon)}, k_{(\overline{X}\alpha)^c\cup(\overline{X}\vartheta)^c}(\epsilon)e^{i\arg_{(\overline{X}\alpha)^c\cup(\overline{X}\vartheta)^c}(\epsilon)}\Big)\\
&\quad \Rightarrow \quad (X\alpha\cap X\vartheta)^c(\epsilon) = [(X\alpha)^c\cup(X\vartheta)^c](\epsilon).
\end{aligned}
$$

It show that $(X\alpha \cap X\vartheta)^c(\epsilon) = (X\alpha)^c \cup (X\vartheta)^c$.

2. On the same lines as above, it can be proved that $(X\alpha \cup X\vartheta)^c(\epsilon) = (X\alpha)^c \cap (X\vartheta)^c$.
3. Using Definition 9 and 13, we have

$$
\begin{aligned}
&[(\underline{X}\alpha\cup\underline{X}\varphi)\cap\underline{X}\vartheta](\epsilon)\\
&= k_{(\underline{X}\alpha\cup\underline{X}\varphi)\cap\underline{X}\vartheta}(\epsilon)e^{i\arg_{(\underline{X}\alpha\cup\underline{X}\varphi)\cap\underline{X}\vartheta}(\epsilon)}\\
&= (k_{\underline{X}\alpha\cup\underline{X}\varphi}(\epsilon)\wedge k_{\underline{X}\vartheta}(\epsilon))e^{i(\arg_{\underline{X}\alpha\cup\underline{X}\varphi}(\epsilon)\wedge\arg_{\underline{X}\vartheta}(\epsilon))}\\
&= ((k_{\underline{X}\alpha}(\epsilon)\vee k_{\underline{X}\varphi}(\epsilon))\wedge k_{\underline{X}\vartheta}(\epsilon))e^{i((\arg_{\underline{X}\alpha}(\epsilon)\vee\arg_{\underline{X}\varphi}(\epsilon))\wedge\arg_{\underline{X}\vartheta}(\epsilon))}\\
&= ((k_{\underline{X}\alpha}(\epsilon)\wedge k_{\underline{X}\vartheta}(\epsilon))\vee(k_{\underline{X}\varphi}(\epsilon)\wedge k_{\underline{X}\vartheta}(\epsilon)))\\
&e^{i((\arg_{\underline{X}\alpha}(\epsilon)\wedge\arg_{\underline{X}\vartheta}(\epsilon))\vee(\arg_{\underline{X}\varphi}(\epsilon)\wedge\arg_{\underline{X}\vartheta}(\epsilon)))}\\
&= (k_{\underline{X}\alpha\cap\underline{X}\vartheta}(\epsilon)\vee k_{\underline{X}\varphi\cap\underline{X}\vartheta}(\epsilon))e^{i(\arg_{\underline{X}\alpha\cap\underline{X}\vartheta}(\epsilon)\vee\arg_{\underline{X}\varphi\cap\underline{X}\vartheta}(\epsilon))}\\
&= k_{(\underline{X}\alpha\cap\underline{X}\vartheta)\cap(\underline{X}\varphi\cap\underline{X}\vartheta)}(\epsilon)e^{i\arg_{(\underline{X}\alpha\cap\underline{X}\vartheta)\cap(\underline{X}\varphi\cap\underline{X}\vartheta)}(\epsilon)}\\
&\Rightarrow\ [(\underline{X}\alpha\cup\underline{X}\varphi)\cap\underline{X}\vartheta](\epsilon) = [(\underline{X}\alpha\cap\underline{X}\vartheta)\cap(\underline{X}\varphi\cap\underline{X}\vartheta)](\epsilon),\ \text{for all}\ \epsilon\in T\\
&\Rightarrow\quad (\underline{X}\alpha\cup\underline{X}\varphi)\cap\underline{X}\vartheta = (\underline{X}\alpha\cap\underline{X}\vartheta)\cap(\underline{X}\varphi\cap\underline{X}\vartheta).
\end{aligned}
$$

Similarly, we can prove that $(\overline{X}\alpha \cup \overline{X}\varphi) \cap \underline{X}\vartheta = (\overline{X}\alpha \cap \overline{X}\vartheta) \cap (\overline{X}\varphi \cap \overline{X}\vartheta)$. Hence the result. The property 4 can be proved on the same lines as above. □

Definition 14 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets on T. The *probabilistic product* of $X\rho$ and $X\varphi$, denoted by $X\rho \circ X\varphi = (\underline{X}\rho \circ \underline{X}\varphi, \overline{X}\rho \circ \overline{X}\varphi)$ is defined as,

$$
\begin{aligned}
(\underline{X}\rho\circ\underline{X}\varphi)(\epsilon) &= k_{\underline{X}\rho\circ\underline{X}\varphi}(\epsilon)e^{i\arg_{\underline{X}\rho\circ\underline{X}\varphi}(\epsilon)}\\
&= \big(k_{\underline{X}\rho}(\epsilon)\cdot k_{\underline{X}\varphi}(\epsilon)\big)e^{i2\pi\left(\frac{\arg_{\underline{X}\rho}(\epsilon)}{2\pi}\cdot\frac{\arg_{\underline{X}\varphi}(\epsilon)}{2\pi}\right)},\\
(\overline{X}\rho\circ\overline{X}\varphi)(\epsilon) &= k_{\overline{X}\rho\circ\overline{X}\varphi}(\epsilon)e^{i\arg_{\overline{X}\rho\circ\overline{X}\varphi}(\epsilon)}\\
&= \big(k_{\overline{X}\rho}(\epsilon)\cdot k_{\overline{X}\varphi}(\epsilon)\big)e^{i2\pi\left(\frac{\arg_{\overline{X}\rho}(\epsilon)}{2\pi}\cdot\frac{\arg_{\overline{X}\varphi}(\epsilon)}{2\pi}\right)}.
\end{aligned}
$$

Example 6 Consider the RCF sets $X\varphi$ and $X\vartheta$ given in Example 3. The probabilistic product $X\varphi \circ X\vartheta = (\underline{X}\varphi \circ \underline{X}\vartheta, \overline{X}\varphi \circ \overline{X}\vartheta)$ is calculated as,

$$
\begin{aligned}
\underline{X}\varphi\circ\underline{X}\vartheta &= \big\{(\epsilon, 0.56e^{i0.78\pi}), (t, 0.20e^{i0.45\pi}),\\
&\qquad (u, 0.56e^{i0.78\pi}), (v, 0.20e^{i0.45\pi})\big\}\\
\overline{X}\varphi\circ\overline{X}\vartheta &= \big\{(\epsilon, 0.72e^{i0.91\pi}), (t, 0.30e^{i0.55\pi}),\\
&\qquad (u, 0.72e^{i0.91\pi}), (v, 0.30e^{i0.55\pi})\big\}.
\end{aligned}
$$

Theorem 5 *The probabilistic product of RCF sets is an RCF t-norm.*

Proof We will only prove properties 3 and 4 of Definition 7. Other properties are trivial from Definition 14. Let $X\alpha$, $X\varphi$, $X\vartheta$ and $X\rho$ be RCF sets on T. To show that probabilistic product is an RCF t-norm, we will prove the following conditions.

1. If $X\alpha \subseteq X\vartheta$ and $X\varphi \subseteq X\rho$, then $X\alpha \circ X\varphi \subseteq X\vartheta \circ X\rho$,
2. $(X\alpha \circ X\vartheta) \circ X\varphi = X\alpha \circ (X\vartheta \circ X\varphi)$,
3. $X\alpha \circ T = X\alpha$.

1. Let $X\alpha \subseteq X\vartheta$ and $X\varphi \subseteq X\rho$, then for all $\epsilon \in T$

$$
\begin{aligned}
k_{\underline{X}\alpha\circ\underline{X}\rho}(\epsilon) &= k_{\underline{X}\alpha}(\epsilon)\cdot k_{\underline{X}\varphi}(\epsilon) \le k_{\underline{X}\vartheta}(\epsilon)\cdot k_{\underline{X}\rho}(\epsilon) = k_{\underline{X}\vartheta\circ\underline{X}\rho}(\epsilon),\\
\arg_{\underline{X}\alpha\circ\underline{X}\rho}(\epsilon) &= 2\pi\Big(\frac{\arg_{\underline{X}\alpha}(\epsilon)}{2\pi}\cdot\frac{\arg_{\underline{X}\varphi}(\epsilon)}{2\pi}\Big) \le\\
&\qquad 2\pi\Big(\frac{\arg_{\underline{X}\vartheta}(\epsilon)}{2\pi}\cdot\frac{\arg_{\underline{X}\rho}(\epsilon)}{2\pi}\Big) = \arg_{\underline{X}\vartheta\circ\underline{X}\rho}(\epsilon).
\end{aligned}
$$

It follows that $\underline{X}\alpha \circ \underline{X}\varphi \subseteq \underline{X}\vartheta \circ \underline{X}\rho$. Similarly, it can be shown that $\overline{X}\alpha \circ \overline{X}\varphi \subseteq \overline{X}\vartheta \circ \overline{X}\rho$. Thus, $X\alpha \circ X\varphi \subseteq X\vartheta \circ X\rho$.

2. For all $\epsilon \in T$,

$$
\begin{aligned}
[(\underline{X}\alpha\circ\underline{X}\varphi)\circ\underline{X}\vartheta](\epsilon) &= k_{(\underline{X}\alpha\circ\underline{X}\varphi)\circ\underline{X}\vartheta}(\epsilon)e^{i\arg_{(\underline{X}\alpha\circ\underline{X}\varphi)\circ\underline{X}\vartheta}(\epsilon)}\\
&= \big[(k_{\underline{X}\alpha}(\epsilon)\cdot k_{\underline{X}\varphi}(\epsilon))\cdot k_{\underline{X}\vartheta}(\epsilon)\big]e^{i2\pi\left[\frac{2\pi\left(\frac{\arg_{\underline{X}\alpha}(\epsilon)}{2\pi}\cdot\frac{\arg_{\underline{X}\varphi}(\epsilon)}{2\pi}\right)}{2\pi}\cdot\frac{\arg_{\underline{X}\vartheta}(\epsilon)}{2\pi}\right]}\\
&= \big[k_{\underline{X}\alpha}(\epsilon)\cdot(k_{\underline{X}\varphi}(\epsilon)\cdot k_{\underline{X}\vartheta}(\epsilon))\big]e^{i2\pi\left[\frac{\arg_{\underline{X}\alpha}(\epsilon)}{2\pi}\cdot\frac{2\pi\left(\frac{\arg_{\underline{X}\varphi}(\epsilon)}{2\pi}\cdot\frac{\arg_{\underline{X}\vartheta}(\epsilon)}{2\pi}\right)}{2\pi}\right]}\\
&= k_{\underline{X}\alpha\circ(\underline{X}\varphi\circ\underline{X}\vartheta)}(\epsilon)e^{i\arg_{\underline{X}\alpha\circ(\underline{X}\varphi\circ\underline{X}\vartheta)}(\epsilon)}\\
&= [\underline{X}\alpha\circ(\underline{X}\varphi\circ\underline{X}\vartheta)](\epsilon)\\
&\Rightarrow\quad (\underline{X}\alpha\circ\underline{X}\varphi)\circ\underline{X}\vartheta = \underline{X}\alpha\circ(\underline{X}\varphi\circ\underline{X}\vartheta).
\end{aligned}
$$

Similarly, it can be proved that $(\overline{X}\alpha \circ \overline{X}\varphi) \circ \overline{X}\vartheta = \overline{X}\alpha \circ (\overline{X}\varphi \circ \overline{X}\vartheta)$. Hence $(X\alpha \circ X\varphi) \circ X\vartheta = X\alpha \circ (X\varphi \circ X\vartheta)$.

3. For all $\epsilon \in T$,

$$
\begin{aligned}
X\alpha \circ T &= \left(\underline{X}\alpha \circ T, \overline{X}\alpha \circ T\right) \\
&= \left(k_{\underline{X}_{\alpha\circ T}}(\epsilon)e^{i\arg_{\underline{X}_{\alpha\circ T}}(\epsilon)}, k_{\overline{X}_{\alpha\circ T}}(\epsilon)e^{i\arg_{\overline{X}_{\alpha\circ T}}(\epsilon)}\right) \\
&= \left((k_{\underline{X}_\alpha}(\epsilon)\cdot 1)e^{i2\pi\left(\frac{\arg_{\underline{X}_\alpha}(\epsilon)}{2\pi}\cdot 1\right)}, (k_{\overline{X}_\alpha}(\epsilon)\cdot 1)e^{i2\pi\left(\frac{\arg_{\overline{X}_\alpha}(\epsilon)}{2\pi}\cdot 1\right)}\right) \\
&= \left(k_{\underline{X}_\alpha}(\epsilon)e^{i\arg_{\underline{X}_\alpha}(\epsilon)}, k_{\overline{X}_\alpha}(\epsilon)e^{i\arg_{\overline{X}_\alpha}(\epsilon)}\right) \\
&\Rightarrow\ X\alpha \circ T = X\alpha.
\end{aligned}
$$

□

Remark 7 Note that probabilistic product is not an RCF *s*-norm since $X\alpha \circ \varnothing = X\alpha$ does not hold.

Corollary 4 *Let* $\{XC_\alpha \in RCF(T,X) \mid \alpha \in I\}$ *be the countable collection of RCF sets on T, then* $\prod_{\alpha\in I} XC_\alpha = XC_1 \circ XC_2 \circ XC_3 \ldots$ *is defined as,*

$$
\begin{aligned}
\Big(\prod_{\alpha\in I} XC_\alpha\Big)(\epsilon) = \Big(&\big(\prod_{\alpha\in I} k_{\underline{X}_{C_\alpha}}(\epsilon)\big)e^{i2\pi\left(\prod_{\alpha\in I}\frac{\arg_{\underline{X}_{C_\alpha}}(\epsilon)}{2\pi}\right)}, \\
&\big(\prod_{\alpha\in I} k_{\overline{X}_{C_\alpha}}(\epsilon)\big)e^{i2\pi\left(\prod_{\alpha\in I}\frac{\arg_{\overline{X}_{C_\alpha}}(\epsilon)}{2\pi}\right)}\Big).
\end{aligned}
$$

The proof of Corollary 4 is a straight consequence of Definition 14.

Definition 15 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets on T. The *probabilistic sum* of $X\rho$ and $X\varphi$, denoted by $X\rho + X\varphi = (\underline{X}\rho + \underline{X}\varphi, \overline{X}\rho + \overline{X}\varphi)$ is defined as,

$$
\begin{aligned}
(\underline{X}\rho + \underline{X}\varphi)(\epsilon) &= \left(k_{\underline{X}_\rho}(\epsilon) + k_{\underline{X}_\varphi}(\epsilon) - k_{\underline{X}_\rho}(\epsilon)\cdot k_{\underline{X}_\varphi}(\epsilon)\right) \\
&\quad e^{i2\pi\left(\frac{\arg_{\underline{X}_\rho}(\epsilon)}{2\pi}+\frac{\arg_{\underline{X}_\varphi}(\epsilon)}{2\pi}-\frac{\arg_{\underline{X}_\rho}(\epsilon)}{2\pi}\cdot\frac{\arg_{\underline{X}_\varphi}(\epsilon)}{2\pi}\right)}, \\
(\overline{X}\rho + \overline{X}\varphi)(\epsilon) &= \left(k_{\overline{X}_\rho}(\epsilon) + k_{\overline{X}_\varphi}(\epsilon) - k_{\overline{X}_\rho}(\epsilon)\cdot k_{\overline{X}_\varphi}(\epsilon)\right) \\
&\quad e^{i2\pi\left(\frac{\arg_{\overline{X}_\rho}(\epsilon)}{2\pi}+\frac{\arg_{\overline{X}_\varphi}(\epsilon)}{2\pi}-\frac{\arg_{\overline{X}_\rho}(\epsilon)}{2\pi}\cdot\frac{\arg_{\overline{X}_\varphi}(\epsilon)}{2\pi}\right)}.
\end{aligned}
$$

Example 7 Consider the RCF sets $X\varphi$ and $X\vartheta$ given in Example 3. The probabilistic sum $X\varphi + X\vartheta = (\underline{X}\varphi + \underline{X}\vartheta, \overline{X}\varphi + \overline{X}\vartheta)$ is calculated as,

$$
\begin{aligned}
\underline{X}\varphi + \underline{X}\vartheta &= \big\{(\epsilon, 0.94e^{i1.72\pi}), (t, 0.70e^{i1.45\pi}), \\
&\qquad (u, 0.94e^{i1.72\pi}), (v, 0.70e^{i1.45\pi})\big\} \\
\overline{X}\varphi + \overline{X}\vartheta &= \big\{(\epsilon, 0.98e^{i1.79\pi}), (t, 0.80e^{i1.55\pi}), \\
&\qquad (u, 0.98e^{i1.79\pi}), \\
&\qquad (v, 0.80e^{i1.55\pi})\big\}.
\end{aligned}
$$

Corollary 5 *Let* $\{XC_1, XC_2, \ldots, XC_n\}$ *be a collection of RCF sets on T, then* $\sum_{j=1}^n XC_j = \left(\sum_{j=1}^n \underline{X}C_j, \sum_{j=1}^n \overline{X}C_j\right)$ *is defined as,*

$$
\begin{aligned}
\Big(\sum_{j=1}^n \underline{X}C_j\Big)(\epsilon) &= \Big(\sum_{j=1}^n k_{\underline{X}_{C_j}}(\epsilon) - \ldots + (-1)^{n-1}\prod_{j=1}^n k_{\overline{X}_{C_j}}(\epsilon)\Big) \\
&\quad e^{i2\pi\left(\sum_{j=1}^n \frac{\arg_{\underline{X}_{C_j}}(\epsilon)}{2\pi} - \ldots + (-1)^{n-1}\prod_{j=1}^n \frac{\arg_{\overline{X}_{C_j}}(\epsilon)}{2\pi}\right)}, \\
\Big(\sum_{j=1}^n \overline{X}C_j\Big)(\epsilon) &= \Big(\sum_{j=1}^n k_{\overline{X}_{C_j}}(\epsilon) - \ldots + (-1)^{n-1}\prod_{j=1}^n k_{\overline{X}_{C_j}}(\epsilon)\Big) \\
&\quad e^{i2\pi\left(\sum_{j=1}^n \frac{\arg_{\overline{X}_{C_j}}(\epsilon)}{2\pi} - \ldots + (-1)^{n-1}\prod_{j=1}^n \frac{\arg_{\overline{X}_{C_j}}(\epsilon)}{2\pi}\right)}.
\end{aligned}
$$

Definition 16 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets on T. The *bounded sum* (*also known as bold union*) of $X\rho$ and $X\varphi$, denoted by $X\rho \oplus X\varphi = (\underline{X}\rho \oplus \underline{X}\varphi, \overline{X}\rho \oplus \overline{X}\varphi)$ is defined as,

$$
\begin{aligned}
(\underline{X}\rho \oplus \underline{X}\varphi)(\epsilon) &= \min\left(1, k_{\underline{X}_\rho}(\epsilon) + k_{\underline{X}_\varphi}(\epsilon)\right)\cdot \\
&\quad e^{\min\left(2\pi, \arg_{\underline{X}_\rho}(\epsilon) + \arg_{\underline{X}_\varphi}(\epsilon)\right)}, \\
(\overline{X}\rho \oplus \overline{X}\varphi)(\epsilon) &= \min\left(1, k_{\underline{X}_\rho}(\epsilon) + k_{\underline{X}_\varphi}(\epsilon)\right)\cdot \\
&\quad e^{\min\left(2\pi, \arg_{\underline{X}_\rho}(\epsilon) + \arg_{\underline{X}_\varphi}(\epsilon)\right)}.
\end{aligned}
$$

Example 8 Consider the RCF sets $X\varphi$ and $X\vartheta$ given in Example 3. The bounded sum $X\varphi \oplus X\vartheta = (\underline{X}\varphi \oplus \underline{X}\vartheta, \overline{X}\varphi \oplus \overline{X}\vartheta)$ is calculated as,

$$
\begin{aligned}
\underline{X}\varphi \oplus \underline{X}\vartheta &= \big\{(\epsilon, 1.0e^{i2\pi}), (t, 0.9e^{i1.9\pi}), (u, 1.0e^{i2\pi}), (v, 0.9e^{i1.9\pi})\big\} \\
\overline{X}\varphi \oplus \overline{X}\vartheta &= \big\{(\epsilon, 1.0e^{i2\pi}), (t, 1.0e^{i2\pi}), (u, 1.0e^{i2\pi}), (v, 1.0e^{i2\pi})\big\}.
\end{aligned}
$$

Remark 8

- The bounded sum of RCF sets is an RCF *s*-norm.
- Let $\{XC_1, XC_2, \ldots, XC_n\}$ be a collection of RCF sets on T, then $XC_1 \oplus XC_2 \oplus XC_3 \ldots XC_n = (\underline{X}C_1 \oplus \underline{X}C_2 \oplus \ldots \oplus \underline{X}C_n, \overline{X}C_1 \oplus \overline{X}C_2 \oplus \ldots \oplus \overline{X}C_n)$ is defined as,

$$
\begin{aligned}
&\big(\underline{X}C_1 \oplus \underline{X}C_2 \oplus \ldots \oplus \underline{X}C_n\big)(\epsilon) \\
&\quad = \min\left(1, \sum_{j=1}^n k_{\underline{X}_{C_j}}(\epsilon)\right)e^{i\min\left(2\pi, \sum_{j=1}^n \arg_{\underline{X}_{C_j}}(\epsilon)\right)}, \\
&\big(\overline{X}C_1 \oplus \overline{X}C_2 \oplus \ldots \oplus \overline{X}C_n\big)(\epsilon) \\
&\quad = \min\left(1, \sum_{j=1}^n k_{\overline{X}_{C_j}}(\epsilon)\right)e^{i\min\left(2\pi, \sum_{j=1}^n \arg_{\overline{X}_{C_j}}(\epsilon)\right)}.
\end{aligned}
$$

Definition 17 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets on T. The *bounded product* (*also known as bold intersection*) of $X\rho$ and $X\varphi$, denoted by $X\rho \odot X\varphi = (\underline{X}\rho \odot \underline{X}\varphi, \overline{X}\rho \odot \overline{X}\varphi)$ is defined as,

$$(\underline{X}\rho \odot \underline{X}\varphi)(\epsilon) = \max\left(0, k_{\underline{X}\rho}(\epsilon) + k_{\underline{X}\varphi}(\epsilon) - 1\right) \cdot e^{\max\left(0, \arg_{\underline{X}\rho}(\epsilon) + \arg_{\underline{X}\varphi}(\epsilon) - 2\pi\right)},$$

$$(\overline{X}\rho \odot \overline{X}\varphi)(\epsilon) = \max\left(0, k_{\underline{X}\rho}(\epsilon) + k_{\underline{X}\varphi}(\epsilon) - 1\right) \cdot e^{\max\left(0, \arg_{\underline{X}\rho}(\epsilon) + \arg_{\underline{X}\varphi}(\epsilon) - 2\pi\right)}.$$

Example 9 Consider the RCF sets $X\varphi$ and $X\vartheta$ given in Example 3. The bounded product $X\varphi \odot X\vartheta = (\underline{X}\varphi \odot \underline{X}\vartheta, \overline{X}\varphi \odot \overline{X}\vartheta)$ is calculated as,

$$\underline{X}\varphi \odot \underline{X}\vartheta = \left\{(\epsilon, 0.5e^{i0.5\pi}), (t, 0), (u, 0.5e^{i0.5\pi}), (v, 0)\right\}$$
$$\overline{X}\varphi \odot \overline{X}\vartheta = \left\{(\epsilon, 0.7e^{i0.7\pi}), (t, 0.1e^{i0.1\pi}), (u, 0.7e^{i0.7\pi}), (v, 0.1e^{i0.1\pi})\right\}.$$

Remark 9

- The bounded product of RCF sets is an RCF t-norm.
- Let $\{XC_1, XC_2, \ldots, XC_n\}$ be a collection of RCF sets on T, then $XC_1 \odot XC_2 \odot XC_3 \ldots XC_n = \left(\underline{X}C_1 \odot \underline{X}C_2 \odot \ldots \odot \underline{X}C_n, \overline{X}C_1 \odot \overline{X}C_2 \odot \ldots \odot \overline{X}C_n\right)$ is defined as,

$$\left(\underline{X}C_1 \odot \underline{X}C_2 \odot \ldots \odot \underline{X}C_n\right)(\epsilon) = \max\left(0, \sum_{j=1}^{n} k_{\underline{X}C_j}(\epsilon) - 1\right) e^{i\max\left(0, \sum_{j=1}^{n} \arg_{\underline{X}C_j}(\epsilon) - 2\pi\right)},$$

$$\left(\overline{X}C_1 \odot \overline{X}C_2 \odot \ldots \odot \overline{X}C_n\right)(\epsilon) = \max\left(0, \sum_{j=1}^{n} k_{\overline{X}C_j}(\epsilon) - 1\right) e^{i\max\left(0, \sum_{j=1}^{n} \arg_{\overline{X}C_j}(\epsilon) - 2\pi\right)}.$$

Definition 18 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets on T. The *disjunctive sum* of $X\rho$ and $X\varphi$ is defined as $X\rho \triangle X\varphi = \left(X\rho \cap (X\varphi)^c\right) \cup \left((X\rho)^c \cap X\varphi\right)$ such that,

$$k_{\underline{X}\rho\triangle\underline{X}\varphi}(\epsilon) = \left(k_{\underline{X}\rho}(\epsilon) \wedge (1 - k_{\underline{X}\varphi}(\epsilon))\right) \vee \left((1 - k_{\underline{X}\rho}(\epsilon)) \wedge k_{\underline{X}\varphi}(\epsilon)\right)$$
$$\arg_{\underline{X}\rho\triangle\underline{X}\varphi}(\epsilon) = \left(\arg_{\underline{X}\rho}(\epsilon) \wedge (2\pi - \arg_{\underline{X}\varphi}(\epsilon))\right) \vee \left((2\pi - \arg_{\underline{X}\rho}(\epsilon)) \wedge \arg_{\underline{X}\varphi}(\epsilon)\right)$$
$$k_{\overline{X}\rho\triangle\overline{X}\varphi}(\epsilon) = \left(k_{\overline{X}\rho}(\epsilon) \wedge (1 - k_{\overline{X}\varphi}(\epsilon))\right) \vee \left((1 - k_{\overline{X}\rho}(\epsilon)) \wedge k_{\overline{X}\varphi}(\epsilon)\right)$$
$$\arg_{\underline{X}\rho\triangle\underline{X}\varphi}(\epsilon) = \left(\arg_{\overline{X}\rho}(\epsilon) \wedge (2\pi - \arg_{\overline{X}\varphi}(\epsilon))\right) \vee \left((2\pi - \arg_{\overline{X}\rho}(\epsilon)) \wedge \arg_{\overline{X}\varphi}(\epsilon)\right).$$

Definition 19 Let $X\rho = (\underline{X}\rho, \overline{X}\rho)$ and $X\varphi = (\underline{X}\varphi, \overline{X}\varphi)$ be two RCF sets on T.

1. The *simple difference* of $X\rho$ and $X\varphi$ is denoted by $X\rho \setminus X\varphi$ and defined as $X\rho \setminus X\varphi = X\rho \cap (X\varphi)^c$.
2. The *symmetrical difference* of $X\rho$ and $X\varphi$ is denoted by $X\rho \nabla X\varphi$ and defined as,

$$(X\rho \nabla X\varphi)(\epsilon) = \left(|k_{\underline{X}\rho}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)| e^{i|\arg_{\underline{X}\rho}(\epsilon) - \arg_{\underline{X}\varphi}(\epsilon)|}, |k_{\overline{X}\rho}(\epsilon) - k_{\overline{X}\varphi}(\epsilon)| e^{i|\arg_{\overline{X}\rho}(\epsilon) - \arg_{\overline{X}\varphi}(\epsilon)|}\right).$$

3. The *bounded difference* of $X\rho$ and $X\varphi$, denoted by $X\rho \Diamond X\varphi$ and is defined as,

$$(\underline{X}\rho \Diamond \underline{X}\varphi)(\epsilon) = \max\left(0, k_{\underline{X}\rho}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right) e^{\max\left(0, \arg_{\underline{X}\rho}(\epsilon) - \arg_{\underline{X}\varphi}(\epsilon)\right)}$$
$$(\overline{X}\rho \Diamond \overline{X}\varphi)(\epsilon) = \max\left(0, k_{\underline{X}\rho}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right) e^{\max\left(0, \arg_{\underline{X}\rho}(\epsilon) - \arg_{\underline{X}\varphi}(\epsilon)\right)}.$$

Theorem 6 *Let ρ and φ be CF sets and (T, X) be an approximation space, then the following properties of lower and upper approximations hold.*

1. $\overline{X}(\rho \cup \varphi) = \overline{X}\rho \cup \overline{\varphi}$ and $\underline{X}(\rho \cup \varphi) \supseteq \underline{X}\rho \cup \underline{\varphi}$,
2. $\underline{X}(\rho \cap \varphi) = \underline{X}\rho \cap \underline{\varphi}$ and $\overline{X}(\rho \cap \varphi) \subseteq \overline{X}\rho \cap \overline{\varphi}$,
3. $\rho \subseteq \varphi \Rightarrow \underline{X}\rho \subseteq \underline{X}\varphi$ and $\overline{X}\rho \subseteq \overline{X}\varphi$,
4. $(\underline{X}\alpha)^c = \overline{X}\alpha^c$ and $(\overline{X}\varphi)^c = \underline{X}\varphi^c$.

Proof 1. By Definition 9, for all $\epsilon \in T$,

$$\begin{aligned}
(\underline{X}\rho \cup \underline{X}\varphi)(\epsilon) &= \left(k_{\underline{X}\rho}(\epsilon) \vee k_{\underline{X}\varphi}(\epsilon)\right) e^{i\left(\arg_{\underline{X}\rho}(\epsilon) \vee \arg_{\underline{X}\varphi}(\epsilon)\right)}, \\
&= \left[\left(\wedge_{t\in[\epsilon]_X} k_\rho(t)\right) \vee \left(\wedge_{t\in[\epsilon]_X} k_\varphi(t)\right)\right] e^{i\left[\left(\wedge_{t\in[\epsilon]_X} \arg_\rho(t)\right) \vee \left(\wedge_{t\in[\epsilon]_X} \arg_\varphi(t)\right)\right]}, \\
&\leq \wedge_{t\in[\epsilon]_X} \left(k_\rho(t) \vee k_\varphi(t)\right) e^{i\wedge_{t\in[\epsilon]_X}\left(\arg_\rho(t) \vee \arg_\varphi(t)\right)}, \\
&= \bigwedge_{t\in[\epsilon]_X} k_{\rho\cup\varphi}(t) e^{i\bigwedge_{t\in[\epsilon]_X} \arg_{\rho\cup\varphi}(t)}, \\
&= \underline{X}(\rho \cup \varphi) \\
&\Rightarrow \quad \underline{X}\rho \cup \underline{X}\varphi \subseteq \underline{X}(\rho \cup \varphi),
\end{aligned}$$

$$\begin{aligned}
(\overline{X}\rho \cup \overline{X}\varphi)(\epsilon) &= (k_{\overline{X}\rho}(\epsilon) \vee k_{\overline{X}\varphi}(\epsilon)) e^{i\left(\arg_{\overline{X}\rho}(\epsilon) \vee \arg_{\overline{X}\varphi}(\epsilon)\right)}, \\
&= \left[\left(\vee_{t\in[\epsilon]_X} k_\rho(t)\right) \vee \left(\vee_{t\in[\epsilon]_X} k_\varphi(t)\right)\right] e^{i\left[\left(\vee_{t\in[\epsilon]_X} \arg_\rho(t)\right) \vee \left(\vee_{t\in[\epsilon]_X} \arg_\varphi(t)\right)\right]}, \\
&= \vee_{t\in[\epsilon]_X} \left(k_\rho(t) \vee k_\varphi(t)\right) e^{i\vee_{t\in[\epsilon]_X}\left(\arg_\rho(t) \vee \arg_\varphi(t)\right)}, \\
&= \bigvee_{t\in[\epsilon]_X} k_{\rho\cup\varphi}(t) e^{i\bigvee_{t\in[\epsilon]_X} \arg_{\rho\cup\varphi}(t)}, \\
&= \overline{X}(\rho \cup \varphi) \\
&\Rightarrow \quad \overline{X}\rho \cup \overline{X}\varphi = \overline{X}(\rho \cup \varphi).
\end{aligned}$$

Condition 2 regarding intersection of RCF sets can be proved on the same lines as above.

3. Since $\rho \subseteq \varphi$, therefore for every $\epsilon \in T$,

$$
\begin{aligned}
(\underline{X}\rho)(\epsilon) &= \bigwedge_{t\in[\epsilon]_X} k_\rho(t)e^{i\bigwedge_{t\in[\epsilon]_X}\arg_\rho(t)} \\
&\leq \bigwedge_{t\in[\epsilon]_X} k_\varphi(t)e^{i\bigwedge_{t\in[\epsilon]_X}\arg_\varphi(t)} = (\underline{X}\varphi)(\epsilon) \\
\Rightarrow \quad & \underline{X}\rho \subseteq \underline{X}\varphi, \\
(\overline{X}\rho)(\epsilon) &= \bigvee_{t\in[\epsilon]_X} k_\rho(t)e^{i\bigvee_{t\in[\epsilon]_X}\arg_\rho(t)} \\
&\leq \bigvee_{t\in[\epsilon]_X} k_\varphi(t)e^{i\bigvee_{t\in[\epsilon]_X}\arg_\varphi(t)} = (\overline{X}\varphi)(\epsilon) \\
\Rightarrow \quad & \overline{X}\rho \subseteq \overline{X}\varphi,
\end{aligned}
$$

4. Using Definitions 8 and 12,

$$
\begin{aligned}
(\underline{X}\alpha)^c(\epsilon) &= (1-k_{\underline{X}\alpha}(\epsilon))e^{i(2\pi-\arg_{\underline{X}\alpha}(\epsilon))} \\
&= (1-\wedge_{t\in[\epsilon]_X}k_\alpha(t))e^{i(2\pi-\wedge_{t\in[\epsilon]_X}\arg_\alpha(t))} \\
&= \vee_{t\in[\epsilon]_X}(1-k_\alpha(t))e^{i\vee_{t\in[\epsilon]_X}(2\pi-\arg_\alpha(t))} \\
&= \vee_{t\in[\epsilon]_X}k_{\alpha^c}(t)e^{i\vee_{t\in[\epsilon]_X}\arg_{\alpha^c}(t)} \\
&= k_{\overline{X}\alpha^c}(\epsilon)e^{i\arg_{\overline{X}\alpha^c}(\epsilon)} \\
&= (\overline{X}\alpha^c)(\epsilon) \\
\Rightarrow \quad (\underline{X}\alpha)^c &= \overline{X}\alpha^c, \\
(\overline{X}\alpha)^c(\epsilon) &= (1-k_{\overline{X}\alpha}(\epsilon))e^{i(2\pi-\arg_{\overline{X}\alpha}(\epsilon))} \\
&= (1-\vee_{t\in[\epsilon]_X}k_\alpha(t))e^{i(2\pi-\vee_{t\in[\epsilon]_X}\arg_\alpha(t))} \\
&= \wedge_{t\in[\epsilon]_X}(1-k_\alpha(t))e^{i\wedge_{t\in[\epsilon]_X}(2\pi-\arg_\alpha(t))} \\
&= \wedge_{t\in[\epsilon]_X}k_{\alpha^c}(t)e^{i\wedge_{t\in[\epsilon]_X}\arg_{\alpha^c}(t)} \\
&= k_{\underline{X}\alpha^c}(\epsilon)e^{i\arg_{\underline{X}\alpha^c}(\epsilon)} \\
&= (\underline{X}\alpha^c)(\epsilon) \\
\Rightarrow \quad (\overline{X}\alpha)^c &= \underline{X}\alpha^c.
\end{aligned}
$$

□

Corollary 6 *Let ρ and φ be CF sets and (T, X) be an approximation space, then the following properties of lower and upper approximations hold.*

1. $\underline{X}(\rho\circ\varphi) \supseteq \underline{X}\rho\circ\underline{\varphi}$ and $\overline{X}(\rho\circ\varphi) \subseteq \overline{X}\rho\circ\overline{\varphi}$,
2. $\underline{X}(\rho+\varphi) \supseteq \underline{X}\rho+\underline{\varphi}$ and $\overline{X}(\rho+\varphi) \subseteq \overline{X}\rho+\overline{\varphi}$,
3. $\underline{X}(\rho\oplus\varphi) \supseteq \underline{X}\rho\oplus\underline{\varphi}$ and $\overline{X}(\rho\oplus\varphi) \subseteq \overline{X}\rho\oplus\overline{\varphi}$.

The proof of Corollary 6 follow the similar arguments used in conditions 1 and 4 of Theorem 6.

4 δ-approximaiton properties of RCF sets

In this section, different types of distance functions and δ-equalities of RCF sets in studied with certain interesting properties.

Definition 20 The *distance* between two RCF sets $X\rho$ and $X\varphi$ is denoted by $d(X\rho, X\varphi) = \frac{1}{2}\big(d(\underline{X}\rho, \underline{X}\varphi) + d(\overline{X}\rho, \overline{X}\varphi)\big)$ such that

$$
\begin{aligned}
d(\underline{X}\rho, \underline{X}\varphi) &= \max\big(\sup_{\epsilon\in T}|k_{\underline{X}\rho}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)|, \\
&\qquad \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\rho}(\epsilon) - \arg_{\underline{X}\varphi}(\epsilon)|\big), \\
d(\overline{X}\rho, \overline{X}\varphi) &= \max\big(\sup_{\epsilon\in T}|k_{\overline{X}\rho}(\epsilon) - k_{\overline{X}\varphi}(\epsilon)|, \\
&\qquad \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\rho}(\epsilon) - \arg_{\overline{X}\varphi}(\epsilon)|\big).
\end{aligned}
$$

Definition 21 The *Hamming distance* between two RCF sets $X\rho$ and $X\varphi$ is denoted by $d_H(X\rho, X\varphi) = \frac{1}{2}\big(d_H(\underline{X}\rho, \underline{X}\varphi) + d_H(\overline{X}\rho, \overline{X}\varphi)\big)$ such that

$$
\begin{aligned}
d_H(\underline{X}\rho, \underline{X}\varphi) &= \frac{1}{2}\Big(\sum_{\epsilon\in T}|k_{\underline{X}\rho}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)| \\
&\qquad + \frac{1}{2\pi}\sum_{\epsilon\in T}|\arg_{\underline{X}\rho}(\epsilon) - \arg_{\underline{X}\varphi}(\epsilon)|\Big), \\
d_H(\overline{X}\rho, \overline{X}\varphi) &= \frac{1}{2}\Big(\sum_{\epsilon\in T}|k_{\overline{X}\rho}(\epsilon) - k_{\overline{X}\varphi}(\epsilon)| \\
&\qquad + \frac{1}{2\pi}\sum_{\epsilon\in T}|\arg_{\overline{X}\rho}(\epsilon) - \arg_{\overline{X}\varphi}(\epsilon)|\Big).
\end{aligned}
$$

Definition 22 The *Euclidean distance* between two RCF sets $X\rho$ and $X\varphi$ is denoted by $d_E(X\rho, X\varphi) = \frac{1}{2}\big(d_E(\underline{X}\rho, \underline{X}\varphi) + d_E(\overline{X}\rho, \overline{X}\varphi)\big)$ such that

$$
\begin{aligned}
&d_E(\underline{X}\rho, \underline{X}\varphi) \\
&\quad = \sqrt{\frac{1}{2}\Big(\sum_{\epsilon\in T}(k_{\underline{X}\rho}(\epsilon) - k_{\underline{X}\varphi}(\epsilon))^2 + \frac{1}{4\pi^2}\sum_{\epsilon\in T}(\arg_{\underline{X}\rho}(\epsilon) - \arg_{\underline{X}\varphi}(\epsilon))^2\Big)}, \\
&d_E(\overline{X}\rho, \overline{X}\varphi) \\
&\quad = \sqrt{\frac{1}{2}\Big(\sum_{\epsilon\in T}(k_{\overline{X}\rho}(\epsilon) - k_{\overline{X}\varphi}(\epsilon))^2 + \frac{1}{4\pi^2}\sum_{\epsilon\in T}(\arg_{\overline{X}\rho}(\epsilon) - \arg_{\overline{X}\varphi}(\epsilon))^2\Big)}.
\end{aligned}
$$

Definition 23 Let $X\rho$ and $X\varphi$ be two RCF sets on T, where $|T| = r$. The *normalized Hamming distance* between $X\rho$ and $X\varphi$ is denoted by $d_{nH}(X\rho, X\varphi) = \frac{1}{2}\big(d_{nH}(\underline{X}\rho, \underline{X}\varphi) + d_{nH}(\overline{X}\rho, \overline{X}\varphi)\big)$ such that

$$d_{nH}(\underline{X}\rho,\underline{X}\varphi)=\frac{1}{2r}\Big(\sum_{\epsilon\in T}|k_{\underline{X}\rho}(\epsilon)-k_{\underline{X}\varphi}(\epsilon)|+\frac{1}{2\pi}\sum_{\epsilon\in T}|\arg_{\underline{X}\rho}(\epsilon)-\arg_{\underline{X}\varphi}(\epsilon)|\Big),$$

$$d_{nH}(\overline{X}\rho,\overline{X}\varphi)=\frac{1}{2r}\Big(\sum_{\epsilon\in T}|k_{\overline{X}\rho}(\epsilon)-k_{\overline{X}\varphi}(\epsilon)|+\frac{1}{2\pi}\sum_{\epsilon\in T}|\arg_{\overline{X}\rho}(\epsilon)-\arg_{\overline{X}\varphi}(\epsilon)|\Big).$$

Definition 24 Let $X\rho$ and $X\varphi$ be two RCF sets on T, where $|T|=r$. The *normalized Euclidean distance* between $X\rho$ and $X\varphi$ is denoted by $d_{nE}(X\rho,X\varphi)=\frac{1}{2}\big(d_E(\underline{X}\rho,\underline{X}\varphi)+d_E(\overline{X}\rho,\overline{X}\varphi)\big)$ such that

$$d_{nE}(\underline{X}\rho,\underline{X}\varphi)=\sqrt{\frac{1}{2r}\Big(\sum_{\epsilon\in T}(k_{\underline{X}\rho}(\epsilon)-k_{\underline{X}\varphi}(\epsilon))^2+\frac{1}{4\pi^2}\sum_{\epsilon\in T}(\arg_{\underline{X}\rho}(\epsilon)-\arg_{\underline{X}\varphi}(\epsilon))^2\Big)},$$

$$d_{nE}(\overline{X}\rho,\overline{X}\varphi)=\sqrt{\frac{1}{2r}\Big(\sum_{\epsilon\in T}(k_{\overline{X}\rho}(\epsilon)-k_{\overline{X}\varphi}(\epsilon))^2+\frac{1}{4\pi^2}\sum_{\epsilon\in T}(\arg_{\overline{X}\rho}(\epsilon)-\arg_{\overline{X}\varphi}(\epsilon))^2\Big)}.$$

Example 10 The distance between two RCF sets $X\varphi$ and $X\vartheta$, in Example 3, is $d(X\varphi,X\vartheta)=0.1$ and the Hamming distance is $d_H(X\varphi,X\vartheta)=0.075$.

Remark 10

1. The distance functions $d=(\underline{d},\overline{d}):RCF(T,X)\times RCF(T,X)\to[0,1]\times[0,1]$ of RCF sets, given in Definitions 20-24 satisfy the following properties:
   - $d(X\rho,X\varphi)\geq 0$,
   - $d(X\rho,X\varphi)=0\Leftrightarrow X\rho=X\varphi$,
   - $d(X\rho,X\varphi)=d(X\varphi,X\rho)$,
   - $d(X\rho,X\vartheta)\leq d(X\rho,X\varphi)+d(X\varphi,X\vartheta)$.
2. Let $X\rho$ and $X\varphi$ be two RCF sets on T, where $|T|=r$ then the following results hold.
   - $0\leq d(X\rho,X\varphi)\leq 1$,
   - $0\leq d_H(X\rho,X\varphi)\leq r$,
   - $0\leq d_E(X\rho,X\varphi)\leq\sqrt{r}$,
   - $0\leq d_{nH}(X\rho,X\varphi)\leq 1$,
   - $0\leq d_{nE}(X\rho,X\varphi)\leq 1$.

Definition 25 Two RCF sets $X\rho$ and $X\varphi$ are said to be δ-*equal* if $d(\underline{X}\rho,\underline{X}\varphi)\leq 1-\delta$ and $d(\overline{X}\rho,\overline{X}\varphi)\leq 1-\delta$, for $\delta\in[0,1]$. It clearly follows that $d(X\rho,X\varphi)\leq 1-\delta$, for $\delta\in[0,1]$. It is denoted by $X\rho=(\delta)X\varphi$.

Proposition 1 *et $X\rho$ and $X\varphi$ be two RCF sets on T, then*

1. $X\rho=(0)X\varphi$,
2. $X\rho=(1)X\varphi\Longleftrightarrow X\rho=X\varphi$,
3. $X\rho=(\delta)X\varphi\Longleftrightarrow X\varphi=(\delta)X\rho$,
4. If $X\rho=(\delta_1)X\varphi$ and $\delta_2\leq\delta_1\Rightarrow X\rho=(\delta_2)X\varphi$,
5. Let $X\rho=(\delta_\alpha)X\varphi$ for all $\alpha\in I$, then $X\rho=(\sup_{\alpha\in I}\delta_\alpha)X\varphi$,
6. For any two RCF sets $X\rho$ and $X\varphi$, there exists a δ such that $X\rho=(\delta)X\varphi$. If $X\rho=(\delta')X\varphi$, then $\delta'\leq\delta$.

Proof

1. By Remark 10, for any two RCF sets $X\rho$ and $X\varphi$, $d(X\rho,X\varphi)\leq 1-0\Rightarrow X\rho=(0)X\varphi$.
2. Assume that $X\rho=(1)X\varphi$, it follows that $d(X\rho,X\varphi)\leq 0$. But by definition $d(X\rho,X\varphi)\geq 0$, hence the only possibility is $d(X\rho,X\varphi)=0\Leftrightarrow X\rho=X\varphi$.
3. Since $X\rho=(\delta)X\varphi$, we have
$$\begin{aligned}&d(X\rho,X\varphi)\leq 1-\delta\\ \Leftrightarrow\quad &d(X\varphi,X\rho)\leq 1-\delta\\ \Leftrightarrow\quad &X\varphi=(\delta)X\rho.\end{aligned}$$
4. Since $X\rho=(\delta_1)X\varphi$ and $\delta_2\leq\delta_1$, we have
$$\begin{aligned}&d(X\rho,X\varphi)\leq 1-\delta_1\leq 1-\delta_2\\ \Rightarrow\quad &d(X\varphi,X\rho)\leq 1-\delta_2\\ \Rightarrow\quad &X\rho=(\delta_2)X\varphi.\end{aligned}$$
5. Since $X\rho=(\delta_\alpha)X\varphi$, for all $\alpha\in I$ therefore
$$\begin{aligned}&d(X\rho,X\varphi)\leq 1-\delta_\alpha,\quad\text{for all }\alpha\in I\\ \Rightarrow\quad &d(X\varphi,X\rho)\leq 1-\sup_{\alpha\in I}\delta_\alpha,\\ \Rightarrow\quad &X\rho=\big(\sup_{\alpha\in I}\delta_\alpha\big)X\varphi.\end{aligned}$$
6. For any two RCF sets, let $\delta=1-d(X\rho,X\varphi)$ then clearly $X\rho=(\delta)X\varphi$. If $X\rho=(\delta')X\varphi$, then
$$1-\delta=d(X\rho,X\varphi)\leq 1-\delta'\Rightarrow\delta'\leq\delta.$$

□

Proposition 2 *For any two RCF sets, if $X\rho=(\delta_1)X\varphi$ and $X\varphi=(\delta_2)X\vartheta$, then $X\rho=(\delta)X\vartheta$, where $\delta=\max(0,\delta_1+\delta_2-1)$.*

Proof Since $X\rho=(\delta_1)X\varphi$ and $X\varphi=(\delta_2)X\vartheta$, therefore $d(X\rho,X\varphi)\leq 1-\delta_1$ and $d(X\varphi,X\vartheta)\leq 1-\delta_2$. By triangular inequality,

$$\begin{aligned}&d(X\rho,X\vartheta)\leq d(X\rho,X\varphi)+d(X\varphi,X\vartheta)\\ \Rightarrow\quad &d(X\rho,X\vartheta)\leq 1-\delta_1+1-\delta_2\\ \Rightarrow\quad &d(X\rho,X\vartheta)\leq 1-(\delta_1+\delta_2-1).\end{aligned}$$

By Remark 10, $d(X\rho,X\vartheta)\leq 1$, therefore

$$d(X\rho,X\vartheta)\leq 1-\max(0,\delta_1+\delta_2-1).$$

Hence $X\rho = (\delta)X\vartheta$, where $\delta = \max(0, \delta_1 + \delta_2 - 1)$. □

Theorem 7 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi \cup X\vartheta = (\delta_1 \wedge \delta_2)(X\varphi' \cup X\vartheta')$.

Proof Since $X\varphi = (\delta_1)X\varphi'$ and $X\vartheta = (\delta_2)X\vartheta'$, therefore

$$d(X\varphi, X\varphi') \leq 1 - \delta_1 \quad \text{and} \quad d(X\vartheta, X\vartheta') \leq 1 - \delta_2.$$

By Definition 20,

$$\begin{aligned}
&d(\underline{X}\varphi \cup \underline{X}\vartheta, \underline{X}\varphi' \cup \underline{X}\vartheta') = \\
&\quad \max\Big(\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)|, \\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)|\Big), \\
&d(\overline{X}\varphi \cup \overline{X}\vartheta, \overline{X}\varphi' \cup \overline{X}\vartheta') = \\
&\quad \max\Big(\sup_{\epsilon\in T} |k_{\overline{X}\varphi\cup\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\cup\overline{X}\vartheta'}(\epsilon)|, \\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\overline{X}\varphi\cup\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\cup\overline{X}\vartheta'}(\epsilon)|\Big).
\end{aligned}$$

So we have,

$$\begin{aligned}
\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| &= \sup_{\epsilon\in T} \big|(k_{\underline{X}\varphi}(\epsilon) \vee k_{\underline{X}\vartheta}(\epsilon)) \\
&\quad - (k_{\underline{X}\varphi'}(\epsilon) \vee k_{\underline{X}\vartheta'}(\epsilon))\big|.
\end{aligned}$$

There are four cases.

Case I: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) \geq k_{\underline{X}\vartheta}(\epsilon)$ and $k_{\underline{X}\varphi'}(\epsilon) \geq k_{\underline{X}\vartheta'}(\epsilon)$, then

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T} \big|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\big| \\
&\quad \leq 1 - \delta_1 \leq 1 - (\delta_1 \wedge \delta_2).
\end{aligned}$$

Case II: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) \geq k_{\underline{X}\vartheta}(\epsilon)$ and $k_{\underline{X}\varphi'}(\epsilon) \leq k_{\underline{X}\vartheta'}(\epsilon)$, then

$$\begin{aligned}
\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| &= \sup_{\epsilon\in T} \big|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big| \\
&\leq \sup_{\epsilon\in T} \big|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\big| \\
&\leq 1 - \delta_1 \\
\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| &\leq 1 - (\delta_1 \wedge \delta_2).
\end{aligned}$$

Case III: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) \leq k_{\underline{X}\vartheta}(\epsilon)$ and $k_{\underline{X}\varphi'}(\epsilon) \geq k_{\underline{X}\vartheta'}(\epsilon)$, then

$$\begin{aligned}
\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| &= \sup_{\epsilon\in T} \big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\big| \\
&\leq \sup_{\epsilon\in T} \big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big| \\
&\leq 1 - \delta_2 \\
\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| &\leq 1 - (\delta_1 \wedge \delta_2).
\end{aligned}$$

Case IV: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) \leq k_{\underline{X}\vartheta}(\epsilon)$ and $k_{\underline{X}\varphi'}(\epsilon) \leq k_{\underline{X}\vartheta'}(\epsilon)$, then

$$\begin{aligned}
\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| &= \sup_{\epsilon\in T} \big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big| \\
&\leq 1 - \delta_2 \\
\sup_{\epsilon\in T} |k_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| &\leq 1 - (\delta_1 \wedge \delta_2).
\end{aligned}$$

On the same lines, it can be proved that

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\overline{X}\varphi\cup\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\cup\overline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 \wedge \delta_2) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\underline{X}\varphi\cup\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\cup\underline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 \wedge \delta_2) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\overline{X}\varphi\cup\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\cup\overline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 \wedge \delta_2).
\end{aligned}$$

Hence, by Definition 20,

$$\begin{aligned}
&d(X\varphi \cup X\vartheta, X\varphi' \cup X\vartheta') \\
&\quad = \frac{1}{2}(d(\underline{X}\varphi \cup \underline{X}\vartheta, \underline{X}\varphi' \cup \underline{X}\vartheta') \\
&\quad + d(\overline{X}\varphi \cup \overline{X}\vartheta, \overline{X}\varphi' \cup \overline{X}\vartheta')) \\
&\quad \leq \tfrac{1}{2}(1 - (\delta_1 \wedge \delta_2) + 1 - (\delta_1 \wedge \delta_2)) \\
&\quad \Rightarrow \quad d(X\varphi \cup X\vartheta, X\varphi' \cup X\vartheta') \\
&\quad \leq 1 - (\delta_1 \wedge \delta_2),
\end{aligned}$$

and so $X\varphi \cup X\vartheta = (\delta_1 \wedge \delta_2)(X\varphi' \cup X\vartheta')$.

□

Corollary 7 *If* $X\varphi_\alpha = (\delta_\alpha)X\vartheta_\alpha$, *for all* $\alpha \in I$, *then* $\cup_{\alpha\in I} X\varphi_\alpha = (\inf_{\alpha\in I} \delta_\alpha)(\cup_{\alpha\in I} X\vartheta_\alpha)$.

Proof Let $X\varphi_\alpha = (\delta_\alpha)X\vartheta_\alpha$, for all $\alpha \in I$, then continuing the process of Theorem 7, it can be easily proved that for a finite collection I, $\cup_{\alpha\in I} X\varphi_\alpha = (\inf_{\alpha\in I} \delta_\alpha)(\cup_{\alpha\in I} X\vartheta_\alpha)$. □

Theorem 8 *If* $X\varphi = (\delta)X\vartheta$, *then* $(X\varphi)^c = (\delta)(X\vartheta)^c$.

Proof By Definition 20,

$$
\begin{aligned}
d\big((\underline{X}\varphi)^c,(\underline{X}\vartheta)^c\big) &= \max\Big(\sup_{\epsilon\in T}|k_{(\underline{X}\varphi)^c}(\epsilon)-k_{(\underline{X}\vartheta)^c}(\epsilon)|,\\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{(\underline{X}\varphi)^c}(\epsilon)-\arg_{(\underline{X}\vartheta)^c}(\epsilon)|\Big)\\
&= \max\Big(\sup_{\epsilon\in T}|1-k_{(\underline{X}\varphi)^c}(\epsilon)-(1-k_{(\underline{X}\vartheta)^c}(\epsilon))|,\\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T}|1-\arg_{(\underline{X}\varphi)^c}(\epsilon)-(1-\arg_{(\underline{X}\vartheta)^c}(\epsilon))|\Big)\\
&= \max\Big(\sup_{\epsilon\in T}|k_{(\underline{X}\vartheta)^c}(\epsilon)-k_{(\underline{X}\varphi)^c}(\epsilon)|,\frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{(\underline{X}\vartheta)^c}(\epsilon)\\
&\quad -\arg_{(\underline{X}\varphi)^c}(\epsilon)|\Big)\\
&= d\big(\underline{X}\vartheta,\underline{X}\varphi\big)\\
&= d\big(\underline{X}\varphi,\underline{X}\vartheta\big)\\
&\le 1-\delta\\
\Rightarrow\quad & d\big((\underline{X}\varphi)^c,(\underline{X}\vartheta)^c\big)\le 1-\delta.
\end{aligned}
$$

Similarly, $d\big((\overline{X}\varphi)^c,(\overline{X}\vartheta)^c\big)\le 1-\delta$. Hence $d\big((X\varphi)^c,(X\vartheta)^c\big)\le 1-\delta$ and so $(X\varphi)^c=(\delta)(X\vartheta)^c$. □

Theorem 9 *For any two RCF sets, if* $X\varphi=(\delta_1)X\varphi'$ *and* $X\vartheta=(\delta_2)X\vartheta'$, *then* $X\varphi\cap X\vartheta=(\delta_1\wedge\delta_2)(X\varphi'\cap X\vartheta')$.

Proof By Theorem 8,

$$
\begin{aligned}
& X\varphi=(\delta_1)X\varphi' \quad\text{and}\quad X\vartheta=(\delta_2)X\vartheta'\\
\Rightarrow\quad & (X\varphi)^c=(\delta_1)(X\varphi')^c \quad\text{and}\quad (X\vartheta)^c=(\delta_2)(X\vartheta')^c\\
\Rightarrow\quad & (X\varphi)^c\cup(X\vartheta)^c=(\delta_1\wedge\delta_2)(X\varphi')^c\cup(X\vartheta')^c\\
\Rightarrow\quad & (X\varphi\cap X\vartheta)^c=(\delta_1\wedge\delta_2)(X\varphi'\cap X\vartheta')^c\\
\Rightarrow\quad & (X\varphi\cap X\vartheta)^{cc}=(\delta_1\wedge\delta_2)(X\varphi'\cap X\vartheta')^{cc}\\
\Rightarrow\quad & X\varphi\cap X\vartheta=(\delta_1\wedge\delta_2)(X\varphi'\cap X\vartheta').
\end{aligned}
$$

□

Corollary 8 *If* $X\vartheta_\alpha=(\delta_\alpha)X\varphi_\alpha$, *for all* $\alpha\in I$, *then* $\cap_{\alpha\in I}X\vartheta_\alpha=(\inf_{\alpha\in I})(\cap_{\alpha\in I}X\varphi_\alpha)$.

Proof By Corollary 7 and Theorem 8,

$$
\begin{aligned}
& X\vartheta_\alpha\\
& (X\vartheta_\alpha)^c\\
\Rightarrow\ & \cup_{\alpha\in I}\,(X\vartheta_\alpha)^c\\
\Rightarrow\ & \big(\cap_{\alpha\in I}\,X\vartheta_\alpha\big)^c\\
\Rightarrow\ & \big(\cap_{\alpha\in I}\,X\vartheta_\alpha\big)^{cc}\\
\Rightarrow\ & \cap_{\alpha\in I}\,X\vartheta_\alpha
\end{aligned}
$$

□

Corollary 9 *If* $X\vartheta_{\alpha\varphi}=(\delta_{\alpha\varphi})X\varphi_{\alpha\varphi}$, *for all* $\alpha\in I_1$ *and* $\varphi\in I_2$, *then*

$$\bigcup_{\alpha\in I_1}\bigcap_{\varphi\in I_2}X\vartheta_{\alpha\varphi}=\big(\inf_{\alpha\in I_1}\inf_{\varphi\in I_2}\delta_{\alpha\varphi}\big)\big(\bigcup_{\alpha\in I_1}\bigcap_{\varphi\in I_2}X\varphi_{\alpha\varphi}\big).$$

Proof By Corollary 7 and 8,

$$
\begin{aligned}
& X\vartheta_{\alpha\varphi}=(\delta_{\alpha\varphi})X\varphi_{\alpha\varphi},\quad \text{for all}\ \ \alpha\in I_1,\ \varphi\in I_2\\
\Rightarrow\ & \bigcap_{\varphi\in I_2}X\vartheta_{\alpha\varphi}=\big(\inf_{\varphi\in I_2}\delta_{\alpha\varphi}\big)\big(\bigcap_{\varphi\in I_2}X\varphi_{\alpha\varphi}\big),\quad \text{for all}\quad \alpha\in I_1\\
\Rightarrow\ & \bigcup_{\alpha\in I_1}\bigcap_{\varphi\in I_2}X\vartheta_{\alpha\varphi}=\big(\inf_{\alpha\in I_1}\inf_{\varphi\in I_2}\delta_{\alpha\varphi}\big)\big(\bigcup_{\alpha\in I_1}\bigcap_{\varphi\in I_2}X\varphi_{\alpha\varphi}\big).
\end{aligned}
$$

□

Theorem 10 *For any two RCF sets, if* $X\varphi=(\delta_1)X\varphi'$ *and* $X\vartheta=(\delta_2)X\vartheta'$, *then* $X\varphi\circ X\vartheta=(\delta_1*\delta_2)(X\varphi'\circ X\vartheta')$, *where* $\delta_1*\delta_2=\max(0,\delta_1+\delta_2-1)$.

Proof By Definition 20,

$$
\begin{aligned}
d(\underline{X}\varphi\circ\underline{X}\vartheta,\underline{X}\varphi'\circ\underline{X}\vartheta') &= \max\big(\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)|,\\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon)-\arg_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)|\big),\\
d(\overline{X}\varphi\circ\overline{X}\vartheta,\overline{X}\varphi'\circ\overline{X}\vartheta') &= \max\big(\sup_{\epsilon\in T}|k_{\overline{X}\varphi\circ\overline{X}\vartheta}(\epsilon)-k_{\overline{X}\varphi'\circ\overline{X}\vartheta'}(\epsilon)|,\\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi\circ\overline{X}\vartheta}(\epsilon)-\arg_{\overline{X}\varphi'\circ\overline{X}\vartheta'}(\epsilon)|\big).
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)| &= \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon)k_{\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\varphi'}(\epsilon)k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&= \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon)k_{\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\varphi}(\epsilon)k_{\underline{X}\vartheta'}(\epsilon)+k_{\underline{X}\varphi}(\epsilon)k_{\underline{X}\vartheta'}(\epsilon)-k_{\underline{X}\varphi'}(\epsilon)k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&= \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon)(k_{\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\vartheta'}(\epsilon))+(k_{\underline{X}\varphi}(\epsilon)-k_{\underline{X}\varphi'}(\epsilon))k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\le \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon)\big|\sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\vartheta'}(\epsilon)\big|+\sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon)-k_{\underline{X}\varphi'}(\epsilon)\big|\sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\le \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\vartheta'}(\epsilon)\big|+\sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon)-k_{\underline{X}\varphi'}(\epsilon)\big|\\
&\le (1-\delta_2)+(1-\delta_1)\\
\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)| &\le 1-(\delta_1+\delta_2-1).
\end{aligned}
$$

Since $d(\underline{X}\varphi\circ\underline{X}\vartheta,\underline{X}\varphi'\circ\underline{X}\vartheta')\le 1$, therefore

$$
\begin{aligned}
\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon)-k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)| &\le 1-\max(0,\delta_1+\delta_2-1)\\
&= 1-\delta_1*\delta_2.
\end{aligned}
$$

Similarly, it can be proved that

$$
\begin{aligned}
\frac{1}{2\pi}\sup_{\epsilon\in T}\big|\arg_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon)-\arg_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)\big| &\le 1-\delta_1*\delta_2\\
\sup_{\epsilon\in T}|k_{\overline{X}\varphi\circ\overline{X}\vartheta}(\epsilon)-k_{\overline{X}\varphi'\circ\overline{X}\vartheta'}(\epsilon)| &\le 1-\delta_1*\delta_2\\
\frac{1}{2\pi}\sup_{\epsilon\in T}\big|\arg_{\overline{X}\varphi\circ\overline{X}\vartheta}(\epsilon)-\arg_{\overline{X}\varphi'\circ\overline{X}\vartheta'}(\epsilon)\big| &\le 1-\delta_1*\delta_2.
\end{aligned}
$$

Hence $d(\underline{X}\varphi\circ\underline{X}\vartheta,\underline{X}\varphi'\circ\underline{X}\vartheta')\le 1-\delta_1*\delta_2$, that is, $X\varphi\circ X\vartheta=(\delta_1*\delta_2)(X\varphi'\circ X\vartheta')$. □

Corollary 10 *If* $X\varphi_i=(\delta_i)X\vartheta_i$, *for all* $1\le i\le n$, *then* $X\varphi_1\circ X\varphi_2\circ\ldots\circ X\varphi_n=(\delta_1*\delta_2*\ldots *\delta_n)(X\vartheta_1\circ X\vartheta_2\circ\ldots\circ X\vartheta_n)$.

Proof Let $X\varphi_i = (\delta_i)X\vartheta_i$, for all $1 \le i \le n$, then continuing the process of Theorem 10, it can be easily proved that for a finite collection $I = \{1, 2, \ldots, n\}$, $X\varphi_1 \circ X\varphi_2 \circ \ldots \circ X\varphi_n = (\delta_1 * \delta_2 * \ldots * \delta_n)(X\vartheta_1 \circ X\vartheta_2 \circ \ldots \circ X\vartheta_n)$. □

Theorem 11 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi + X\vartheta = (\delta_1 * \delta_2)(X\varphi' + X\vartheta')$.

Proof By Definition 20,

$$d(\underline{X}\varphi + \underline{X}\vartheta, \underline{X}\varphi' + \underline{X}\vartheta') = \max\Big(\sup_{\epsilon\in T}|k_{\underline{X}\varphi+\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'+\underline{X}\vartheta'}(\epsilon)|, \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi+\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'+\underline{X}\vartheta'}(\epsilon)|\Big),$$

$$d(\overline{X}\varphi + \overline{X}\vartheta, \overline{X}\varphi' + \overline{X}\vartheta') = \max\Big(\sup_{\epsilon\in T}|k_{\overline{X}\varphi+\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'+\overline{X}\vartheta'}(\epsilon)|, \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi+\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'+\overline{X}\vartheta'}(\epsilon)|\Big).$$

Thus we have,

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)|\\
&= \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)k_{\underline{X}\vartheta}(\epsilon)\\
&\quad -k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) + k_{\underline{X}\varphi'}(\epsilon)k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&= \sup_{\epsilon\in T}\big|(1 - k_{\underline{X}\vartheta}(\epsilon))(k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon))\\
&\quad -(1 - k_{\underline{X}\varphi'}(\epsilon))(k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon))\big|\\
&= \sup_{\epsilon\in T}\big|1 - k_{\underline{X}\vartheta}(\epsilon)\big|\big|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\big|\\
&\quad + \sup_{\epsilon\in T}\big|1 - k_{\underline{X}\varphi'}(\epsilon)\big|\big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\le \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\big| + \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\le (1 - \delta_1) + (1 - \delta_2)\\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Since $d(\underline{X}\varphi + \underline{X}\vartheta, \underline{X}\varphi' + \underline{X}\vartheta') \le 1$, therefore

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi+\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'+\underline{X}\vartheta'}(\epsilon)| \le 1 - \max(0, \delta_1 + \delta_2 - 1)\\
&\quad = 1 - \delta_1 * \delta_2.
\end{aligned}$$

Similarly, it can be proved that

$$\begin{aligned}
&\frac{1}{2\pi}\sup_{\epsilon\in T}\big|\arg_{\underline{X}\varphi+\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'+\underline{X}\vartheta'}(\epsilon)\big| \le 1 - \delta_1 * \delta_2\\
&\sup_{\epsilon\in T}|k_{\overline{X}\varphi+\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'+\overline{X}\vartheta'}(\epsilon)| \le 1 - \delta_1 * \delta_2\\
&\frac{1}{2\pi}\sup_{\epsilon\in T}\big|\arg_{\overline{X}\varphi+\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'+\overline{X}\vartheta'}(\epsilon)\big| \le 1 - \delta_1 * \delta_2.
\end{aligned}$$

Hence $d(\underline{X}\varphi + \underline{X}\vartheta, \underline{X}\varphi' + \underline{X}\vartheta') \le 1 - \delta_1 * \delta_2$, that is, $X\varphi + X\vartheta = (\delta_1 * \delta_2)(X\varphi' + X\vartheta')$. □

Corollary 11 *If* $X\varphi_i = (\delta_i)X\vartheta_i$, *for all* $1 \le i \le n$, *then* $X\varphi_1 + X\varphi_2 + \ldots + X\varphi_n = (\delta_1 * \delta_2 * \ldots * \delta_n)(X\vartheta_1 + X\vartheta_2 + \ldots + X\vartheta_n)$.

Proof Let $X\varphi_i = (\delta_i)X\vartheta_i$, for all $1 \le i \le n$, then continuing the process of Theorem 11, it can be easily proved that for a finite collection $I = \{1, 2, \ldots, n\}$, $X\varphi_1 + X\varphi_2 + \ldots + X\varphi_n = (\delta_1 * \delta_2 * \ldots * \delta_n)(X\vartheta_1 + X\vartheta_2 + \ldots + X\vartheta_n)$. □

Theorem 12 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi \oplus X\vartheta = (\delta_1 * \delta_2)(X\varphi' \oplus X\vartheta')$.

Proof Since $X\varphi = (\delta_1)X\varphi'$ and $X\vartheta = (\delta_2)X\vartheta'$, therefore

$$d(X\varphi, X\varphi') \le 1 - \delta_1 \quad \text{and} \quad d(X\vartheta, X\vartheta') \le 1 - \delta_2.$$

By Definition 20,

$$d(\underline{X}\varphi \oplus \underline{X}\vartheta, \underline{X}\varphi' \oplus \underline{X}\vartheta') = \max\Big(\sup_{\epsilon\in T}|k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)|, \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)|\Big),$$

$$d(\overline{X}\varphi \oplus \overline{X}\vartheta, \overline{X}\varphi' \oplus \overline{X}\vartheta') = \max\Big(\sup_{\epsilon\in T}|k_{\overline{X}\varphi\oplus\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\oplus\overline{X}\vartheta'}(\epsilon)|, \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi\oplus\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\oplus\overline{X}\vartheta'}(\epsilon)|\Big).$$

So we have,

$$\sup_{\epsilon\in T}|k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| = \sup_{\epsilon\in T}\big|\min(1, k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon)) - \min(1, k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon))\big|.$$

There are four cases.

Case I: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) \ge 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) \ge 1$, then

$$\sup_{\epsilon\in T}|k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| = 0 \le 1 - (\delta_1 + \delta_2 - 1).$$

Case II: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) \ge 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) < 1$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)|\\
&\quad = \sup_{\epsilon\in T}\big|1 - k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\quad \le \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\quad \le \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\big| + \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\quad \le (1 - \delta_1) + (1 - \delta_2)\\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Case III: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) < 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) \geq 1$, then

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) + (-1)\right| \\
&\quad \leq \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) + (-k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon))\right| \\
&\quad \leq \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right| + \sup_{\epsilon\in T} \left|k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \leq (1-\delta_1) + (1-\delta_2) \\
&\quad \sup_{\epsilon\in T} |k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Case IV: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) < 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) < 1$, then

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\right| \\
&\quad \leq \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right| + \sup_{\epsilon\in T} \left|k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \leq (1-\delta_1) + (1-\delta_2) \\
&\quad \sup_{\epsilon\in T} |k_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

On the same lines, it can be proved that

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\overline{X}\varphi\oplus\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\oplus\overline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 + \delta_2 - 1) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\underline{X}\varphi\oplus\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\oplus\underline{X}\vartheta'}(\epsilon)| \\
&\leq 1 - (\delta_1 + \delta_2 - 1) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\overline{X}\varphi\oplus\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\oplus\overline{X}\vartheta'}(\epsilon)| \\
&\leq 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Hence, by Definition 20,

$$\begin{aligned}
&d(X\varphi \oplus X\vartheta, X\varphi' \oplus X\vartheta') \\
&\quad = \frac{1}{2}(d(\underline{X}\varphi \oplus \underline{X}\vartheta, \underline{X}\varphi' \oplus \underline{X}\vartheta') \\
&\quad + d(\overline{X}\varphi \oplus \overline{X}\vartheta, \overline{X}\varphi' \oplus \overline{X}\vartheta')) \\
&\quad \leq \tfrac{1}{2}(1 - (\delta_1 + \delta_2 - 1)) + 1 - (\delta_1 + \delta_2 - 1)) \\
&\quad \Rightarrow \quad d(X\varphi \oplus X\vartheta, X\varphi' \oplus X\vartheta') \leq 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Since $d(X\varphi \oplus X\vartheta, X\varphi' \oplus X\vartheta') \leq 1$, therefore $d(X\varphi \oplus X\vartheta, X\varphi' \oplus X\vartheta') \leq 1 - (\delta_1 * \delta_2)$. Hence $X\varphi \oplus X\vartheta = (\delta_1 * \delta_2)(X\varphi' \oplus X\vartheta')$.

□

Theorem 13 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi \odot X\vartheta = (\delta_1 * \delta_2)(X\varphi' \odot X\vartheta')$.

Proof Since $X\varphi = (\delta_1)X\varphi'$ and $X\vartheta = (\delta_2)X\vartheta'$, therefore

$$d(X\varphi, X\varphi') \leq 1 - \delta_1 \quad \text{and} \quad d(X\vartheta, X\vartheta') \leq 1 - \delta_2.$$

By Definition 20,

$$\begin{aligned}
d(\underline{X}\varphi \odot \underline{X}\vartheta, \underline{X}\varphi' \odot \underline{X}\vartheta') &= \max\big(\sup_{\epsilon\in T} |k_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)|, \\
&\qquad \frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)|\big), \\
d(\overline{X}\varphi \odot \overline{X}\vartheta, \overline{X}\varphi' \odot \overline{X}\vartheta') &= \max\big(\sup_{\epsilon\in T} |k_{\overline{X}\varphi\odot\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\odot\overline{X}\vartheta'}(\epsilon)|, \\
&\qquad \frac{1}{2\pi}\sup_{\epsilon\in T} |\arg_{\overline{X}\varphi\odot\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\odot\overline{X}\vartheta'}(\epsilon)|\big).
\end{aligned}$$

So we have,

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T} \big|\max(0, k_{\underline{X}\varphi}(\epsilon) \\
&\qquad + k_{\underline{X}\vartheta}(\epsilon) - 1) - \max(0, k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) - 1)\big|.
\end{aligned}$$

There are four cases.

Case I: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) \leq 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) \leq 1$, then

$$\sup_{\epsilon\in T} |k_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)| = 0 \leq 1 - (\delta_1 + \delta_2 - 1).$$

Case II: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) \leq 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) > 1$, then

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) - 1\right| \\
&\quad \leq \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \leq \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right| + \sup_{\epsilon\in T} \left|k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \leq (1-\delta_1) + (1-\delta_2) \\
&\quad \sup_{\epsilon\in T} |k_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Case III: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) > 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) \leq 1$, then

$$\begin{aligned}
&\sup_{\epsilon\in T} |k_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) - 1\right| \\
&\quad \leq \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) + (-k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon))\right| \\
&\quad \leq \sup_{\epsilon\in T} \left|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\right| + \sup_{\epsilon\in T} \left|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\right| \\
&\quad \leq (1-\delta_1) + (1-\delta_2) \\
&\quad \sup_{\epsilon\in T} |k_{\underline{X}\varphi\odot\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\odot\underline{X}\vartheta'}(\epsilon)| \leq 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Case IV: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) > 1$ and $k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon) > 1$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon) - 1 - k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) + 1\right| \\
&\quad \le \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right| + \sup_{\epsilon\in T}\left|k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \le (1-\delta_1) + (1-\delta_2) \\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

On the same lines, it can be proved that

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\overline{X}\varphi\circ\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\circ\overline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi\circ\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\circ\underline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi\circ\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\circ\overline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Hence, by Definition 20, $d(X\varphi \odot X\vartheta, X\varphi' \odot X\vartheta') \le 1 - (\delta_1 + \delta_2 - 1)$. Since $d(X\varphi \odot X\vartheta, X\varphi' \odot X\vartheta') \le 1$, therefore $d(X\varphi \odot X\vartheta, X\varphi' \odot X\vartheta') \le 1 - (\delta_1 * \delta_2)$. Hence $X\varphi \odot X\vartheta = (\delta_1 * \delta_2)(X\varphi' \odot X\vartheta')$.

□

Theorem 14 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi \diamondsuit X\vartheta = (\delta_1 * \delta_2)(X\varphi' \diamondsuit X\vartheta')$.

Proof By Definition 20,

$$\begin{aligned}
d(\underline{X}\varphi\diamondsuit\underline{X}\vartheta, \underline{X}\varphi'\diamondsuit\underline{X}\vartheta') &= \max\Big(\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)|, \\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)|\Big), \\
d(\overline{X}\varphi\diamondsuit\overline{X}\vartheta, \overline{X}\varphi'\diamondsuit\overline{X}\vartheta') &= \max\Big(\sup_{\epsilon\in T}|k_{\overline{X}\varphi\diamondsuit\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\diamondsuit\overline{X}\vartheta'}(\epsilon)|, \\
&\quad \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi\diamondsuit\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\diamondsuit\overline{X}\vartheta'}(\epsilon)|\Big).
\end{aligned}$$

So we have,

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T}\big|\max(0, k_{\underline{X}\varphi}(\epsilon) \\
&\qquad - k_{\underline{X}\vartheta}(\epsilon)) - \max(0, k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon))\big|.
\end{aligned}$$

There are four cases.

Case I: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) \le 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) \le 0$, then

$$\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| = 0 \le 1 - (\delta_1 + \delta_2 - 1).$$

Case II: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) \le 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) > 0$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\right| \\
&\quad \le \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon) + k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \le \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right| + \sup_{\epsilon\in T}\left|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\right| \\
&\quad \le (1-\delta_1) + (1-\delta_2) \\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Case III: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) > 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) \le 0$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \le \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) + (-k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon))\right| \\
&\quad \le \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\right| + \sup_{\epsilon\in T}\left|k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \le (1-\delta_1) + (1-\delta_2) \\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Case IV: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) > 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) > 0$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \\
&\quad = \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon)\right| \\
&\quad \le \sup_{\epsilon\in T}\left|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\right| + \sup_{\epsilon\in T}\left|k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\right| \\
&\quad \le (1-\delta_1) + (1-\delta_2) \\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

On the same lines, it can be proved that

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\overline{X}\varphi\diamondsuit\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\diamondsuit\overline{X}\vartheta'}(\epsilon)| \le 1 - (\delta_1 + \delta_2 - 1) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi\diamondsuit\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\diamondsuit\underline{X}\vartheta'}(\epsilon)| \\
&\quad \le 1 - (\delta_1 + \delta_2 - 1) \\
&\frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi\diamondsuit\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\diamondsuit\overline{X}\vartheta'}(\epsilon)| \\
&\quad \le 1 - (\delta_1 + \delta_2 - 1).
\end{aligned}$$

Hence, by Definition 20, $d(X\varphi\diamondsuit X\vartheta, X\varphi'\diamondsuit X\vartheta') \le 1 - (\delta_1 + \delta_2 - 1)$. Since $d(X\varphi \diamondsuit X\vartheta, X\varphi'\diamondsuit X\vartheta') \le 1$, therefore $d(X\varphi\diamondsuit X\vartheta, X\varphi'\diamondsuit X\vartheta') \le 1 - (\delta_1 * \delta_2)$. Hence $X\varphi\diamondsuit X\vartheta = (\delta_1 * \delta_2)(X\varphi'\diamondsuit X\vartheta')$.

□

Theorem 15 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi\nabla X\vartheta = (\delta_1 * \delta_2)(X\varphi'\nabla X\vartheta')$.

Proof By Definition 20,

$$d(\underline{X}\varphi\nabla\underline{X}\vartheta, \underline{X}\varphi'\nabla\underline{X}\vartheta') = \max\Big(\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)|, \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)|\Big),$$

$$d(\overline{X}\varphi\nabla\overline{X}\vartheta, \overline{X}\varphi'\nabla\overline{X}\vartheta') = \max\Big(\sup_{\epsilon\in T}|k_{\overline{X}\varphi\nabla\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\nabla\overline{X}\vartheta'}(\epsilon)|, \frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi\nabla\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\nabla\overline{X}\vartheta'}(\epsilon)|\Big).$$

So we have,

$$\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)| = \sup_{\epsilon\in T}\Big|\,|\,k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\,| - |\,k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\,|\Big|.$$

There are four cases.

Case I: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) \geq 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) \geq 0$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)|\\
&= \sup_{\epsilon\in T}\big|(k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)) - (k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon))\big|\\
&\leq \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\big| + \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\leq (1-\delta_1) + (1-\delta_2)\\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)| \leq 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Case II: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) \geq 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) < 0$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)|\\
&= \sup_{\epsilon\in T}\big|(k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)) - (k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon))\big|\\
&= \sup_{\epsilon\in T}\big|(k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)) + (-k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon))\big|\\
&\leq \sup_{\epsilon\in T}\big|(k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)) + (k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon))\big|\\
&\leq \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\big| + \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\leq (1-\delta_1) + (1-\delta_2)\\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)| \leq 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Case III: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) > 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) \leq 0$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)|\\
&= \sup_{\epsilon\in T}\big|(k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)) - (k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon))\big|\\
&= \sup_{\epsilon\in T}\big|(k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)) + (k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon))\big|\\
&\leq \sup_{\epsilon\in T}\big|(k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)) + (-k_{\underline{X}\varphi'}(\epsilon) + k_{\underline{X}\vartheta'}(\epsilon))\big|\\
&\leq \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon)\big| + \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon)\big|\\
&\leq (1-\delta_1) + (1-\delta_2)\\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)| \leq 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Case IV: If for all $\epsilon \in T$, $k_{\underline{X}\varphi}(\epsilon) - k_{\underline{X}\vartheta}(\epsilon) < 0$ and $k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon) < 0$, then

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)|\\
&= \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi}(\epsilon) - (k_{\underline{X}\vartheta'}(\epsilon) - k_{\underline{X}\varphi'}(\epsilon))\big|\\
&\leq \sup_{\epsilon\in T}\big|k_{\underline{X}\varphi'}(\epsilon) - k_{\underline{X}\varphi}(\epsilon)\big| + \sup_{\epsilon\in T}\big|k_{\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\vartheta'}(\epsilon)\big|\\
&\leq (1-\delta_1) + (1-\delta_2)\\
&\sup_{\epsilon\in T}|k_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - k_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)| \leq 1-(\delta_1+\delta_2-1).
\end{aligned}$$

On the same lines, it can be proved that

$$\begin{aligned}
&\sup_{\epsilon\in T}|k_{\overline{X}\varphi\nabla\overline{X}\vartheta}(\epsilon) - k_{\overline{X}\varphi'\nabla\overline{X}\vartheta'}(\epsilon)| \leq 1-(\delta_1+\delta_2-1)\\
&\frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\underline{X}\varphi\nabla\underline{X}\vartheta}(\epsilon) - \arg_{\underline{X}\varphi'\nabla\underline{X}\vartheta'}(\epsilon)|\\
&\quad\leq 1-(\delta_1+\delta_2-1)\\
&\frac{1}{2\pi}\sup_{\epsilon\in T}|\arg_{\overline{X}\varphi\nabla\overline{X}\vartheta}(\epsilon) - \arg_{\overline{X}\varphi'\nabla\overline{X}\vartheta'}(\epsilon)|\\
&\quad\leq 1-(\delta_1+\delta_2-1).
\end{aligned}$$

Hence, by Definition 20, $d(X\varphi\nabla X\vartheta, X\varphi'\nabla X\vartheta') \leq 1-(\delta_1+\delta_2-1)$. Since $d(X\varphi\nabla X\vartheta, X\varphi'\nabla X\vartheta') \leq 1$, therefore $d(X\varphi\nabla X\vartheta, X\varphi'\nabla X\vartheta') \leq 1-(\delta_1 * \delta_2)$. Hence $X\varphi\nabla X\vartheta = (\delta_1 * \delta_2)(X\varphi'\nabla X\vartheta')$.

□

Theorem 16 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi\triangle X\vartheta = (\delta_1 \wedge \delta_2)(X\varphi'\triangle X\vartheta')$.

Proof By Theorem 7 and 8,

$$\Rightarrow \quad X\varphi = (\delta_1)X\varphi' \quad \text{and} \quad X\vartheta = (\delta_2)X\vartheta'$$
$$\Rightarrow \quad (X\varphi)^c = (\delta_1)(X\varphi')^c \quad \text{and} \quad (X\vartheta)^c = (\delta_2)(X\vartheta')^c$$
$$\Rightarrow \quad X\varphi \cap (X\vartheta)^c = (\delta_1 \wedge \delta_2)(X\varphi' \cap (X\vartheta')^c) \quad \text{and}$$
$$(X\varphi)^c \cap X\vartheta = (\delta_1 \wedge$$
$$\Rightarrow \quad (X\varphi \cap (X\vartheta)^c) \cup ((X\varphi)^c \cap X\vartheta)$$
$$= (\delta_1 \wedge \delta_2)(X\varphi' \cap (X\vartheta')^c) \cup$$
$$((X\varphi')^c \cap X\vartheta')$$
$$\Rightarrow \quad X\varphi \triangle X\vartheta = (\delta_1 \wedge \delta_2)(X\varphi' \triangle X\vartheta').$$

□

Corollary 12 *For any two RCF sets, if* $X\varphi = (\delta_1)X\varphi'$ *and* $X\vartheta = (\delta_2)X\vartheta'$, *then* $X\varphi \backslash X\vartheta = (\delta_1 \wedge \delta_2)(X\varphi' \backslash X\vartheta')$.

The proof of Corollary 12 is a direct consequence of Theorem 8 and 9.

5 Application of rough complex fuzzy information

In this section, we discuss the identification of class of unknown building material using RCF sets. Different building materials including cement, bricks, novel composites, green authorized and other approved materials are used for the construction of mega projects/structures, buildings and others. There is an increasing demand for improving stainability in construction environment. There is additionally more demand to change the current structure to be more green to obtain more effective usage of water and energy while reducing pollution. Such construction processes are complicated and may create negative effects on people. There is limited data on contents of building, handling techniques, toxicity; for instance the existence of toxic and hazardous materials. In the construction of a building, if some unknown material contains toxic substances, will the building be considered as "sustainable construction"? CF sets can be used effectively because of their wavelike properties resulting in destructive and constructive intrusion depending on the argument of membership value. The uncertainty due to the limitation of available information can studied more effectively using rough sets. Rough approximations can convert the complex fuzzy data into RCF information to handle subjective uncertainty in experts' evaluations. The novel tool of RCF sets is implemented in this section to identify the class of unknown building materials. This application can be extended on a more wider level using various decision support algorithms.

Consider the example of recognition of class of unknown building material among known building materials using the distance measures of RCF sets. The procedure of identifying the class of unknown material is given in Algorithm 1.

Algorithm 1 Method to Identify the Class of Unknown Building Materials

1. Let $Y = \{\rho_1, \rho_2, \ldots, \rho_r\}$ be the collection of known materials against attributes $\{y_1, y_2, \ldots, y_m\}$. Let $\rho = \rho_{r+1}$ be an unknown material to be identified among the known materials $\alpha_1, \alpha_2, \ldots, \alpha_r$. Let $A = [k_{\rho_i}(t_j)e^{i\arg_{\rho_i}(t_j)}]_{(r+1)\times m}$ be the matrix of complex membership values of building materials against given criteria.
2. Define an EQ relation X on the collection (set) of building materials $Y \cup \{\alpha\}$.
3. For each ρ_i, $1 \le i \le r$, compute the equivalence classes $[y_{ij}]_X = \{y_{ik} \mid X(t_{ij}, y_{ik}) = 1,\ 1 \le k \le m\}$.
4. For each ρ_i, $1 \le i \le r$, compute the RCF set $(\underline{X}\rho_i, \overline{X}\rho_i)$ using Definition 8.
5. For the weight vector $W = (w_1, w_2, \ldots, w_m)$, compute the distance measure

$$Wd(\rho_i, \rho) = \frac{1}{2}\big(Wd(\underline{X}\rho_i, \underline{X}\rho_i) + Wd(\overline{X}\rho_i, \overline{X}\rho_i)\big), \qquad (8)$$

for each $1 \le i \le r$, where

$$Wd(\underline{X}\rho, \underline{X}\varphi) = \sqrt{\frac{1}{2\sum_{k=1}^{m} w_k}\Big(\sum_{i=1}^{m} w_i(k_{\underline{X}\rho}(t_i) - k_{\underline{X}\varphi}(t_i))^2 + \frac{1}{4\pi^2}\sum_{i=1}^{m} w_i(\arg_{\underline{X}\rho}(t_i) - \arg_{\underline{X}\varphi}(t_i))^2\Big)},$$
$$Wd(\overline{X}\rho, \overline{X}\varphi) = \sqrt{\frac{1}{2\sum_{k=1}^{m} w_k}\Big(\sum_{i=1}^{m} w_i(k_{\overline{X}\rho}(t_i) - k_{\overline{X}\varphi}(t_i))^2 + \frac{1}{4\pi^2}\sum_{i=1}^{m} w_i(\arg_{\overline{X}\rho}(t_i) - \arg_{\overline{X}\varphi}(t_i))^2\Big)}.$$

6. The distance measure shows the closeness of unknown material with each known material.

We now elaborate this phenomenon with an application example of a number of building materials such that only an information table is given discussing the attributes of the known and unknown materials. Consider the four building materials PVC flooring (ρ_1), wall paint (ρ_2), floor varnish (ρ_3) and sealant (ρ) with attributes $Y = \{y_1, y_2, \ldots, y_7\}$. Let ρ be an unknown material whose class needs to be identified. The information about building materials in the form CF values is given in Table 2. The membership value of each material ρ or ρ_k, $k \in \{1, 2, 3, 4\}$ describes the percentage of having attribute y_j. As the phase term represents the wavelike properties of materials, it shows the fluctuation from respective attribute, that is, the variation from actual property.

Consider the weight vector $W = (0.16, 0.1, 0.21, 0.18, 0.1, 0.14, 0.11)$ and an equivalence relation X on Y as given in Equation (9).

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

The RCF sets $(\underline{X}\rho_1, \overline{X}\rho_1)$, $(\underline{X}\rho_2, \overline{X}\rho_2)$, $(\underline{X}\rho_3, \overline{X}\rho_3)$, $(\underline{X}\rho_4, \overline{X}\rho_4)$ and $(\underline{X}\rho, \overline{X}\rho)$ are computed in Tables 3 and 4. The distance measures calculated using Formula (8) and the ranking of building materials is given Table 5. Table 5 clearly shows that the unknown material is close to the class ρ_1 (sealant) and farthest from the class of ρ_3 (wall paint). Thus the unknown material lies in the class of ρ_1 and can never belong to the family of ρ_1. In the given application examples, we are only given a table of CF information of material; no additional information in given. Thus, uncertainty can be discussed more subjectively by incorporating rough sets with given information and the obtained results are more accurate as compared to other approaches.

6 Comparison and discussion

In this section, a detailed comparison of proposed RCF sets is given to show its out-performance than existing models such as of CF sets, CIF (complex intuitionistic fuzzy) sets and CPyF (complex pythagorean fuzzy) sets. The results of identification of class on unknown material with three different approaches of RCF sets (proposed concept), CIF sets and CPyF sets (Ullah et al. 2020) are given in Table 6.

There are similarities and differences between the rankings of three approaches. In all the approaches, it is concluded that the unknown material is close to the class of ρ_1 which shows the effectiveness of proposed RCF sets. ρ_3 is ranked second in other approaches and forth in the proposed method. The other materials have different rank in all the approaches. In Ullah et al. (2020), the distance measures are computed using CIF and CPyF information. These distance measures can be applied when non-membership function is given with certain conditions. We have used lower and upper rough approximations to compute the distance measures to discuss the uncertainty not with a single value but in an interval between the membership values of two approximations. This approach is more valid when we have limited information.

The existing mathematical approaches based on sets use two membership values 1 and 0 considering no objective and accurate information in decision assessment process. The data and information and information are usually evaluated by a team of decision makers having variety of thinking, experience and knowledge increasing the existence of uncertainty in the evaluations or judgements. All the rankings cannot be taken as 0 or 1. Some objects have partial membership in the [0,1] with respect to given criteria. To study the wavelike properties of objects, CF sets play a vital role. In the proposed research, we have studied possible vaguness in data using RCF sets to get more prior and accurate results.

1. The CIF sets (Ramot et al. 2002), CPyF sets (Ullah et al. 2020) and other related extensions (Ma et al.

Table 2 CF information about building materials

	ρ_1	ρ_2	ρ_3	ρ_4	ρ
y_1	$0.7e^{i0.4\pi}$	$0.1e^{i1.2\pi}$	$0.2e^{i\pi}$	$0.7e^{i1.4\pi}$	$e^{i2\pi}$
y_2	$0.4e^{i1.2\pi}$	$0.2e^{i1.4\pi}$	$0.8e^{i1.2\pi}$	$0.6e^{i1.6\pi}$	$e^{i2\pi}$
y_3	$0.5e^{i0.6\pi}$	$0.6e^{i0.6\pi}$	$0.6e^{i1.4\pi}$	$0.7e^{i1.2\pi}$	$e^{i2\pi}$
y_4	$0.6e^{i1.2\pi}$	$0.8e^{i1.8\pi}$	$0.5e^{i1.4\pi}$	$0.4e^{i1.6\pi}$	$e^{i2\pi}$
y_5	$0.9e^{i0.8\pi}$	$0.8e^{i1.6\pi}$	$0.3e^{i1.2\pi}$	$0.7e^{i1.2\pi}$	$e^{i2\pi}$
y_6	$0.7e^{i1.2\pi}$	$0.6e^{i1.4\pi}$	$0.6e^{i\pi}$	$0.8e^{i\pi}$	$e^{i2\pi}$
y_7	$0.8e^{i1.4\pi}$	$0.7e^{i\pi}$	$0.5e^{i1.2\pi}$	$0.7e^{i1.2\pi}$	$e^{i2\pi}$

Table 3 Lower approximations of CF information given in Table 2

	ρ_1	ρ_2	ρ_3	ρ_4	ρ
y_1	$0.7e^{i0.4\pi}$	$0.1e^{i1.2\pi}$	$0.2e^{i\pi}$	$0.7e^{i1.2\pi}$	$e^{i2\pi}$
y_2	$0.4e^{i0.6\pi}$	$0.2e^{i0.6\pi}$	$0.6e^{i1.2\pi}$	$0.6e^{i1.2\pi}$	$e^{i2\pi}$
y_3	$0.4e^{i0.6\pi}$	$0.2e^{i0.6\pi}$	$0.6e^{i1.2\pi}$	$0.6e^{i1.2\pi}$	$e^{i2\pi}$
y_4	$0.6e^{i1.2\pi}$	$0.6e^{i1.4\pi}$	$0.5e^{i\pi}$	$0.4e^{i\pi}$	$e^{i2\pi}$
y_5	$0.7e^{i0.4\pi}$	$0.1e^{i1.2\pi}$	$0.2e^{i\pi}$	$0.7e^{i1.2\pi}$	$e^{i2\pi}$
y_6	$0.6e^{i1.2\pi}$	$0.6e^{i1.4\pi}$	$0.5e^{i\pi}$	$0.4e^{i\pi}$	$e^{i2\pi}$
y_7	$0.8e^{i1.4\pi}$	$0.7e^{i\pi}$	$0.5e^{i1.2\pi}$	$0.7e^{i1.2\pi}$	$e^{i2\pi}$

Table 4 Upper approximations of CF information given in Table 2

	ρ_1	ρ_2	ρ_3	ρ_4	ρ
y_1	$0.9e^{i0.8\pi}$	$0.8e^{i1.6\pi}$	$0.3e^{i1.2\pi}$	$0.7e^{i1.4\pi}$	$e^{i2\pi}$
y_2	$0.7e^{i1.2\pi}$	$0.6e^{i1.4\pi}$	$0.8e^{i1.4\pi}$	$0.7e^{i1.6\pi}$	$e^{i2\pi}$
y_3	$0.7e^{i1.2\pi}$	$0.6e^{i1.4\pi}$	$0.8e^{i1.4\pi}$	$0.7e^{i1.6\pi}$	$e^{i2\pi}$
y_4	$0.7e^{i1.2\pi}$	$0.8e^{i1.8\pi}$	$0.6e^{i1.4\pi}$	$0.8e^{i1.6\pi}$	$e^{i2\pi}$
y_5	$0.9e^{i0.8\pi}$	$0.8e^{i1.6\pi}$	$0.3e^{i1.2\pi}$	$0.7e^{i1.4\pi}$	$e^{i2\pi}$
y_6	$0.7e^{i1.2\pi}$	$0.8e^{i1.6\pi}$	$0.6e^{i1.4\pi}$	$0.8e^{i1.6\pi}$	$e^{i2\pi}$
y_7	$0.8e^{i1.4\pi}$	$0.7e^{i\pi}$	$0.5e^{i1.2\pi}$	$0.7e^{i1.2\pi}$	$e^{i2\pi}$

Table 5 Distance measures of unknown material from known material

Xnown material	ρ_1	ρ_2	ρ_3	ρ_4
Distance measure	0.2054	0.3550	0.3773	0.2811
Rank	1	3	4	2

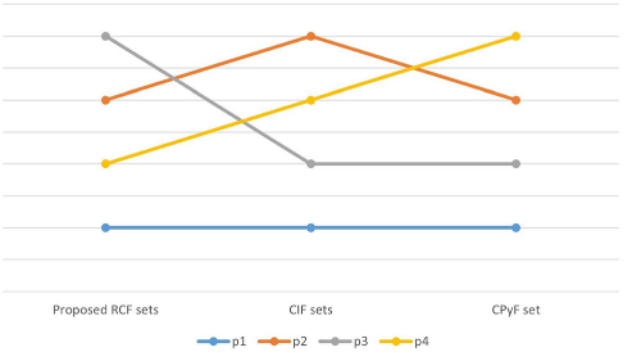

Fig. 1 Comparison of proposed RCF sets with existing approaches

2011; Akram and Bashir 2021) study uncertainty in data but the values of non-membership functions are based on membership functions or the choice of experts which make the results skewed and uncertain. RCF sets can be effectively used to discuss uncertainty from given data without additional membership functions, suppositions and adjustments. The proposed RCF sets also reduce the dependency of decision makers and give effective results. The subjective uncertainty in the data can be handled using upper and lower approximations without additional parameters.

2. The theory of RF sets (Dubois and Prade 1990) is a powerful mathematical approach to deal with uncertainty and approximations in decision making problems. This theory is an active domain of research not only in Mathematics but also real-world problems. The perspective of fuzziness with rough information can be effectively used to study various problems involving uncertainty, but the addition of phase value in RCF sets enables us to study the variations and fluctuations of objects from their attributes. This addition make RCF sets more stronger as compared to RF sets and related extensions.
3. Fuzzy soft sets (Ahmad and Kharal 2009) and their extensions are used to deal with different types of uncertainties based on parametrization tool. The existence of parameters in given information and may not be available in some cases. Rough models can discuss MCDM problems more objectively and rationally then CF sets and parametrization based models. Soft knowledge requires the addition of parameters or attribute. The change in the list of attributes can change the results. This phenomenon increases more uncertainty. Rough sets discuss the uncertainty in an interval rather a single value, that is, the range of uncertainty of is greater than any other approach. In case of less information, we can convert the data using rough sets or rough numbers and study the uncertainty with additional suppositions.
4. The integration of rough sets with other mathematical models is very useful to discuss multiple uncertainties with single mathematical approach. In this study, the hybridization of of CF sets with rough approximations is proposed to deal with two types of uncertainties. It can study interpersonal and intrapersonal vagueness efficiently to give more reliable and rational results. However, along with the merits, the presented model also has some shortcomings including increased calculation complexity and lack of multi-polar behaviour.

7 Conclusions and future directions

The integration of CF sets with rough information retain the characterization to describe uncertainty using upper and lower approximations of a complex-valued membership function. RCF sets incorporate the capability to study uncertainty as well as fluctuations and uneven properties of objects with respect to certain characteristics. In this paper, the notion of rough complex fuzzy sets is introduced by

Table 6 Identification of class on unknown materials with different approaches

Unknown material	Proposed RCF sets		CIF sets (Ullah et al. 2020)		CPyF set (Ullah et al. 2020)	
	Distance measure	Rank	Distance measure	Rank	Distance measure	Rank
ρ_1	0.2054	1	0.0284	1	0.270	1
ρ_2	0.3550	3	0.314	4	0.282	3
ρ_3	0.3773	4	0.294	2	0.275	2
ρ_4	0.2811	2	0.307	3	0.288	4

combining the concept of lower and upper rough approximations with complex fuzzy sets. Certain operation properties of rough complex fuzzy sets are described using s-norms and t-norms. Using the properties of various distance measures, the δ-approximation properties of rough complex fuzzy sets are discussed to study the affects of δ-equalities on various operations of rough complex fuzzy sets. The relations of certain variations of δ-approximations with RCF sets is studied with different algebraic operations and logical operators. The necessity and importance of rough complex fuzzy information is illustrated, with a numerical example, for the classification of unknown building materials. The presented study can expanded towards 1) Rough complex intuitionistic fuzzy sets; 2) Approximation properties of complex pythagorean fuzzy sets; 3) Rough complex fuzzy graphs.

Author Contributions Musavarah Sarwar, Muhammad Akram, Sundas Shahzadi conceptualized and designed the study, analysed the data, and wrote the manuscript.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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