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A Comprehensive Study on Pythagorean Fuzzy Normal Subgroups and Pythagorean Fuzzy Isomorphisms

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Abstract: The Pythagorean fuzzy set is an extension of the intuitionistic fuzzy set used to handle uncertain circumstances in various decisions making problems. Group theory is a mathematical technique for dealing with problems of symmetry. This study deals with Pythagorean fuzzy group theory. In this article, we characterize the notion of a Pythagorean fuzzy subgroup and examine various algebraic properties of this concept. An extensive study on Pythagorean fuzzy cosets of a Pythagorean fuzzy subgroup, Pythagorean fuzzy normal subgroups of a group and Pythagorean fuzzy normal subgroup of a Pythagorean fuzzy subgroup is performed. We define the notions of Pythagorean fuzzy homomorphism and isomorphism and generalize the notion of factor group of a classical group W relative to its normal subgroup S by defining a PFSG of W/S . At the end, the Pythagorean fuzzy version of fundamental theorems of isomorphisms is proved.

Keywords: Pythagorean fuzzy set; Pythagorean fuzzy subgroup; Pythagorean fuzzy normal subgroup; Pythagorean fuzzy isomorphism

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1. Introduction

Zadeh [1] is the pioneer of fuzzy set theory used to deal with uncertainty in real-life problems. A fuzzy subset A of a crisp set X is expressed by the function $\mu_A : X \rightarrow [0, 1]$ and can be written as $A = \{(x, \mu_A(x)) : x \in X\}$. The function μ_A is called the membership function of A . It can be easily observed that a fuzzy set is a generalization of a crisp set where the membership function of a set A is the characteristic function of A , which is equal to 0 if $x \notin A$ and 1 if $x \in A$. After the development of fuzzy set theory, numerous concepts and abstractions have been extended to cope with uncertainty and vagueness in an effective way [2–6]. With the passage of time, researchers noticed that the membership function alone is not enough to manage certain kinds of situations. Therefore, Atanassov [7] linked a negative membership with the positive membership function in a fuzzy set and put forward the notion of intuitionistic fuzzy set (IFS) A of a classical set X . This is in fact an object of the type $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X, \mu_A(x) + \nu_A(x) \leq 1\}$, where $\mu_X : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ are called positive and negative membership functions, respectively. The presence of the non-membership function along with that of the membership function in IFS ensure that it can deal with vague and ambiguous circumstances more efficiently, as compared to Zadeh's fuzzy sets, particularly in the decision-making field [8–12]. In spite of this, there are many circumstances where intuitionistic fuzzy set theory does

not work. For instance, if the membership and non-membership scores suggested by a decision-maker are 0.80 and 0.50, respectively, then $0.80 + 0.50 > 1$; therefore, such problems cannot be solved by IFSs. In order to obtain an appropriate solution under these circumstances, Yager [13] introduced the idea of Pythagorean fuzzy subset (PFS) A of X . This can be represented by $\{(x, \mu_A(x), \nu_A(x)) : x \in X, (\mu_A(x))^2 + (\nu_A(x))^2 \leq 1\}$. The concept of PFS was developed to describe vague and ambiguous situations in mathematical form and to provide a formal tool for dealing with imprecision in real-life problems [14–17]. To find out more about PFS, we recommend reading [18–21]. In [22,23], the importance of preference relations for decision makers to express their evaluation information is highlighted. A study regarding social network group decision making is presented in [24].

Group theory is one of the important branches of mathematics. The history of group theory is old and rich. Although the concept of group theory and its name are creations of the 19th century, its roots can be traced back to the Babylonian civilization in the 5th to 4th century BC, when mathematicians strived to find a general method for finding the roots of a linear equation. Although they did not have the notion of an equation, they developed an algorithmic scheme to solve problems, which, in our terminology, would yield a linear equation. In Galois' words, group theory is a "metaphysics of theory of equations". Around 1830, Galois used the term "group" to represent the sets of injective mappings over finite sets that can be combined together to design a set which obeys the closure property of composition. Just like other basic concepts in mathematics, the current definition of a group has evolved over a long evolutionary process. Although Heinrich Weber and Walther von Dyck defined the modern version of groups in 1882, it did not acquire widespread acknowledgment until the twentieth century. The theory of groups by this time had developed to a great extent into a systematic theory. Its scope started widening. In P. Hall's words: "It is not stretching too far to say that any type of algebra of any degree of importance is in some sense or the other related to group theory" [25]. Its linkage with other branches of mathematics started appearing. The relationship between group theory and physics became significantly strong and concrete when Weyl (1885–1955) published his famous book *Gruppentheorie und Quantenmechanik* in 1927 [26]. In the last two decades, many papers have been written to describe the applications of group theory in chemistry [27], physics [28], graph theory [29,30], cryptography [31–33], and differential equations [34].

Group theory is a mathematical approach for dealing with problems of symmetry. These sorts of problems appear repeatedly in all fields of science. The use of formal group theory has only recently been developed, but the concept of symmetry was widely used as early as 1000 years ago. The design of ornaments with symmetries, the observation of periodic patterns, and the regular appearance of the sun and other astronomical objects showed that symmetry is a useful concept. Pierre Curie first stated the importance of symmetry around 1870. Since then, group theory has become the principal tool for dealing with difficult problems in relativity theory, solid-state theory, atomic and nuclear spectroscopy, and the theory of elementary particles. Symmetry operations and symmetry elements are two basic and important concepts in group theory. In order to find more linkage between group theory and symmetry, we refer the readers to [35–38].

Rosenfeld [39] generalized the concept of classical subgroups and revealed the notion of fuzzy subgroups. Since the introduction of fuzzy subgroups, many studies have been performed to generalize the concepts of classical group theory in five different fuzzy environments. Das [40] defined the idea of a level subgroup of a fuzzy subgroup and studied its various algebraic characteristics. In [41], Liu introduced the concept of invariant subgroups and proved some fundamental properties of this notion. In 1984, Mukherjee and Bhattacharya proved a fuzzy version of the well-known Lagrange theorem [42]. In [43], the same authors proved several theorems that are analogs to some basic results of classical group theory. Wetherilt [44] introduced the notion of semidirect product of fuzzy subgroups and explored various related results. Some extensive studies on level fuzzy subgroups and normal fuzzy subgroups were conducted in [45–47]. The study on intuitionistic fuzzy

subgroups started with Biswas [48]. In [49], some group theoretic results related to cosets of a subgroup were studied in intuitionistic fuzzy environment. The notion of a direct product of intuitionistic fuzzy subgroups was defined in [50]. Altassan et al. [51] introduced the notion of ω -fuzzy subrings. Recently, Alharbi and Alghazzawi presented the concept of (ρ, η) complex fuzzy subgroups [52]. For more results about intuitionistic fuzzy subgroups, the readers are referred to [53–55]. The study on Pythagorean fuzzy group theory was initiated by Bhunia et al. in [56].

The above literature review highlights some achievements in the research on classical and intuitionistic fuzzy group theory. Moreover, some results related to Pythagorean fuzzy cosets of a group, Pythagorean fuzzy normal subgroups and Pythagorean fuzzy level subgroups have been proved, but there are still some open questions to be answered.

- (1) In classical group theory, if H is a subgroup of G and xH and yH are two cosets of H in G , then $xH = yH \Leftrightarrow x \in yH$ or $y \in xH$. Therefore, a question arises—under what condition are two Pythagorean fuzzy cosets the same?
- (2) In the existing literature, the characterization of the classical/intuitionistic fuzzy normal subgroups with regard to classical/intuitionistic fuzzy level subgroups is given. Since a Pythagorean fuzzy subgroup needs to not be an intuitionistic fuzzy subgroup, it is therefore important to know about the characterization of Pythagorean fuzzy normal subgroups with respect to Pythagorean fuzzy level subgroups.
- (3) Although the factor groups and three fundamental theorems of group isomorphisms are of great significance in group theory, in the existing literature, there is a lack of focus on these topics in a Pythagorean fuzzy environment. Hence, another question to address is: what is the Pythagorean fuzzy versions of all three fundamental theorems of group isomorphisms?

The basic purpose of this study is to answer the above research questions and to fill a gap in the existing knowledge. The structure of the remainder of this article is as follows. Section 2 contains preliminary definitions and notions which are necessary to prove our main results. In Section 3, the concepts of Pythagorean fuzzy cosets of a PFSG and Pythagorean fuzzy normal subgroups of a group are described. We also prove various theorems associated with these notions. In Section 4, we define a Pythagorean fuzzy normal subgroup of a Pythagorean fuzzy subgroup and conduct a comprehensive study of this concept. The notions of Pythagorean fuzzy homomorphism and isomorphism are defined in Section 5. We also generalize the notion of a factor group of a classical group W relative to its normal subgroup S by defining a PFSG of W/S . Furthermore, the Pythagorean fuzzy version of fundamental theorems of isomorphisms are proved.

2. Preliminaries

This section contains some notions and concepts which are needed to prove our main theorems.

Definition 1. [56] Let W be a classical group, then an FS $J = \{(w, \mu_J(w)) : w \in W\}$ of W is termed as an FSG of W if the following conditions are satisfied:

- (i) $\mu_J(w_1w_2) \geq \min\{\mu_J(w_1), \mu_J(w_2)\}$ for all $w_1, w_2 \in W$.
- (ii) $\mu_J(w^{-1}) \geq \mu_J(w)$ for all $w \in W$.

Definition 2. [56] An IFS $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ of W is called an IFSG of W if the following assertions are true:

- (i) $\mu_J(w_1w_2) \geq \min\{\mu_J(w_1), \mu_J(w_2)\}$ and $\nu_J(w_1w_2) \leq \max\{\nu_J(w_1), \nu_J(w_2)\}$ for all $w_1, w_2 \in W$.
- (ii) $\mu_J(w^{-1}) \geq \mu_J(w)$ and $\nu_J(w^{-1}) \leq \nu_J(w)$ for all $w \in W$.

Definition 3. [56] A PFS $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ of W is called a PFSG of W if the following inequalities are satisfied:

- (i) $(\mu_J(w_1w_2))^2 \geq \min\{(\mu_J(w_1))^2, (\mu_J(w_2))^2\}$ and $(\nu_J(w_1w_2))^2 \leq \max\{(\nu_J(w_1))^2, (\nu_J(w_2))^2\}$ for all $w_1, w_2 \in W$.
- (ii) $(\mu_J(w^{-1}))^2 \geq (\mu_J(w))^2$ and $(\nu_J(w^{-1}))^2 \leq (\nu_J(w))^2$ for all $w \in W$.

Theorem 1. [56] A PFS J of W is a PFSG of W if and only if $(\mu_J(w_1w_2^{-1}))^2 \geq \min\{(\mu_J(w_1))^2, (\mu_J(w_2))^2\}$ and $(\nu_J(w_1w_2^{-1}))^2 \leq \max\{(\nu_J(w_1))^2, (\nu_J(w_2))^2\}$ for all $w_1, w_2 \in W$.

Theorem 2. [56] Let J and K be two PFSGs of W . Then, $J \cap K$ is a PFSG of W .

In [56], the notion of Pythagorean fuzzy level subset was introduced. Here, we reproduce the definition of a Pythagorean fuzzy level subset (PFLS) and some related results.

Definition 4. [56] Let $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ be a PFS of W . Then, the set $J_{(\theta, \tau)} = \{w \in W : (\mu_J(w))^2 \geq \theta, (\nu_J(w))^2 \leq \tau\}$ is known as the PFLS of the PFS J of W .

It is important to mention the following theorems which were proven in [56].

Theorem 3. [56] Let J and K be two PFSs of W and $J \subseteq K$, then $J_{(\theta, \tau)} \subseteq K_{(\theta, \tau)}$ for all $\theta, \tau \in [0, 1]$, such that $\theta^2 + \tau^2 \leq 1$.

Theorem 4. [56] A PFS J of W is a PFSG of W if and only if $J_{(\theta, \tau)}$ is a subgroup of W for all $\theta \in [0, (\mu_J(e))^2]$ and $\tau \in [(\nu_J(e))^2, 1]$.

Next, we rewrite the concept of Pythagorean fuzzy left cosets of a PFSG of a group defined in [56]

Definition 5. [56] Let $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ be a PFSG of W and $x \in W$. Then, the Pythagorean fuzzy left coset of J with respect to x is denoted by xJ and is defined as $xJ = \{(w, \mu_{xJ}(w), \nu_{xJ}(w)) : w \in W\}$, where $(\mu_{xJ}(w))^2 = (\mu_J(x^{-1}w))^2$ and $(\nu_{xJ}(w))^2 = (\nu_J(x^{-1}w))^2$. Similarly, $Jx = \{(w, \mu_{Jx}(w), \nu_{Jx}(w)) : w \in W\}$, where $(\mu_{Jx}(w))^2 = (\mu_J(wx^{-1}))^2$ and $(\nu_{Jx}(w))^2 = (\nu_J(wx^{-1}))^2$, is called the Pythagorean fuzzy right coset of J with respect to x .

The notion of Pythagorean fuzzy normal subgroup of W was defined in [56].

Next, we rewrite its definition.

Definition 6. [56] A PFSG $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ of W is called a Pythagorean fuzzy normal subgroup of W if $xJ = Jx$ for all $x \in W$.

Theorem 5. [56] Let $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ be a PFSG of W . Then J is a PFNS of W if and only if $(\mu_J(w_1w_2))^2 = (\mu_J(w_2w_1))^2$ and $(\nu_J(w_1w_2))^2 = (\nu_J(w_2w_1))^2$ for all $w_1, w_2 \in W$.

Theorem 6. [56] Let $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ be a PFSG of W . Then J is a PFNS of W if and only if $(\mu_J(wxw^{-1}))^2 = (\mu_J(x))^2$ and $(\nu_J(wxw^{-1}))^2 = (\nu_J(x))^2$ for all $x, w \in W$.

3. More on Pythagorean Fuzzy Cosets of a Pythagorean Fuzzy Subgroup and Pythagorean Fuzzy Normal Subgroups of a Group

In this section, we expand the concepts of Pythagorean fuzzy cosets of a PFSG and Pythagorean fuzzy normal subgroups of a group by proving many results related to these topics. The following example illustrates the notion of Pythagorean fuzzy left cosets of a PFSG of a group.

Example 1. Consider a well-known group

$$W = S_3 = \langle \alpha, \beta : \alpha^2 = \beta^3 = (\alpha\beta)^1 = 1 \rangle = \{e, \alpha, \beta, \beta^2, \alpha\beta, \alpha\beta^2\}$$

of symmetries of an equilateral triangle. Here, the generators α and β denote reflection along the diagonal line and anti-clockwise rotation through an angle 120° , respectively. In this way, we obtain total six symmetries, $e, \alpha, \beta, \beta^2, \alpha\beta$ and $\alpha\beta^2$, of an equilateral triangle. Let us define a PFSG of $W = S_3$, that is,

$$J = \left\{ (e, 0.90, 0.40), (\beta, 0.90, 0.40), (\beta^2, 0.90, 0.40), \right. \\ \left. (\alpha, 0.80, 0.50), (\alpha\beta, 0.80, 0.50), (\alpha\beta^2, 0.80, 0.50) \right\}.$$

Next, we find Pythagorean fuzzy left cosets J with respect to all $w \in W$.

(i) The Pythagorean fuzzy left coset of J with respect to $e \in W$ is

$$eJ = \left\{ (e, \mu_J(e^{-1}e), \nu_J(e^{-1}e)), (\beta, \mu_J(e^{-1}\beta), \nu_J(e^{-1}\beta)), \right. \\ \left. (\beta^2, \mu_J(e^{-1}\beta^2), \nu_J(e^{-1}\beta^2)), (\alpha, \mu_J(e^{-1}\alpha), \nu_J(e^{-1}\alpha)), \right. \\ \left. (\alpha\beta, \mu_J(e^{-1}\alpha\beta), \nu_J(e^{-1}\alpha\beta)), (\alpha\beta^2, \mu_J(e^{-1}\alpha\beta^2), \nu_J(e^{-1}\alpha\beta^2)) \right\} \\ = \left\{ (e, 0.90, 0.40), (\beta, 0.90, 0.40), (\beta^2, 0.90, 0.40), \right. \\ \left. (\alpha, 0.80, 0.50), (\alpha\beta, 0.80, 0.50), (\alpha\beta^2, 0.80, 0.50) \right\}.$$

(ii) The Pythagorean fuzzy left coset of J with respect to $\beta \in W$ is

$$\beta J = \left\{ (e, \mu_J(\beta^{-1}e), \nu_J(\beta^{-1}e)), (\beta, \mu_J(\beta^{-1}\beta), \nu_J(\beta^{-1}\beta)), \right. \\ \left. (\beta^2, \mu_J(\beta^{-1}\beta^2), \nu_J(\beta^{-1}\beta^2)), (\alpha, \mu_J(\beta^{-1}\alpha), \nu_J(\beta^{-1}\alpha)), \right. \\ \left. (\alpha\beta, \mu_J(\beta^{-1}\alpha\beta), \nu_J(\beta^{-1}\alpha\beta)), (\alpha\beta^2, \mu_J(\beta^{-1}\alpha\beta^2), \nu_J(\beta^{-1}\alpha\beta^2)) \right\} \\ = \left\{ (e, \mu_J(\beta^2), \nu_J(\beta^2)), (\beta, \mu_J(e), \nu_J(e)), (\beta^2, \mu_J(\beta), \nu_J(\beta)), \right. \\ \left. (\alpha, \mu_J(\alpha\beta), \nu_J(\alpha\beta)), (\alpha\beta, \mu_J(\alpha\beta^2), \nu_J(\alpha\beta^2)), (\alpha\beta^2, \mu_J(\alpha), \nu_J(\alpha)) \right\} \\ = \left\{ (e, 0.90, 0.40), (\beta, 0.90, 0.40), (\beta^2, 0.90, 0.40), \right. \\ \left. (\alpha, 0.80, 0.50), (\alpha\beta, 0.80, 0.50), (\alpha\beta^2, 0.80, 0.50) \right\}.$$

(iii) The Pythagorean fuzzy left coset of J with respect to $\beta^2 \in W$ is

$$\beta^2 J = \left\{ (e, \mu_J((\beta^2)^{-1}e), \nu_J((\beta^2)^{-1}e)), (\beta, \mu_J((\beta^2)^{-1}\beta), \nu_J((\beta^2)^{-1}\beta)), \right. \\ \left. (\beta^2, \mu_J((\beta^2)^{-1}\beta^2), \nu_J((\beta^2)^{-1}\beta^2)), (\alpha, \mu_J((\beta^2)^{-1}\alpha), \nu_J((\beta^2)^{-1}\alpha)), \right. \\ \left. (\alpha\beta, \mu_J((\beta^2)^{-1}\alpha\beta), \nu_J((\beta^2)^{-1}\alpha\beta)), (\alpha\beta^2, \mu_J((\beta^2)^{-1}\alpha\beta^2), \nu_J((\beta^2)^{-1}\alpha\beta^2)) \right\} \\ = \left\{ (e, \mu_J(\beta), \nu_J(\beta)), (\beta, \mu_J(\beta^2), \nu_J(\beta^2)), (\beta^2, \mu_J(e), \nu_J(e)), \right. \\ \left. (\alpha, \mu_J(\alpha\beta^2), \nu_J(\alpha\beta^2)), (\alpha\beta, \mu_J(\alpha), \nu_J(\alpha)), (\alpha\beta^2, \mu_J(\alpha\beta), \nu_J(\alpha\beta)) \right\} \\ = \left\{ (e, 0.90, 0.40), (\beta, 0.90, 0.40), (\beta^2, 0.90, 0.40), \right. \\ \left. (\alpha, 0.80, 0.50), (\alpha\beta, 0.80, 0.50), (\alpha\beta^2, 0.80, 0.50) \right\} = J.$$

(iv) The Pythagorean fuzzy left coset of J with respect to $\alpha \in W$ is

$$\begin{aligned}\alpha J &= \left\{ \begin{array}{l} (e, \mu_J(\alpha^{-1}e), \nu_J(\alpha^{-1}e)), (\beta, \mu_J(\alpha^{-1}\beta), \nu_J(\alpha^{-1}\beta)), \\ (\beta^2, \mu_J(\alpha^{-1}\beta^2), \nu_J(\alpha^{-1}\beta^2)), (\alpha, \mu_J(\alpha^{-1}\alpha), \nu_J(\alpha^{-1}\alpha)), \\ (\alpha\beta, \mu_J(\alpha^{-1}\alpha\beta), \nu_J(\alpha^{-1}\alpha\beta)), (\alpha\beta^2, \mu_J(\alpha^{-1}\alpha\beta^2), \nu_J(\alpha^{-1}\alpha\beta^2)) \end{array} \right\} \\ &= \left\{ \begin{array}{l} (e, \mu_J(\alpha), \nu_J(\alpha)), (\beta, \mu_J(\alpha\beta), \nu_J(\alpha\beta)), (\beta^2, \mu_J(\alpha\beta^2), \nu_J(\alpha\beta^2)), \\ (\alpha, \mu_J(e), \nu_J(e)), (\alpha\beta, \mu_J(\beta), \nu_J(\beta)), (\alpha\beta^2, \mu_J(\beta^2), \nu_J(\beta^2)) \end{array} \right\} \\ &= \left\{ (e, 0.0.80, 0.50), (\beta, 0.80, 0.50), (\beta^2, 0.80, 0.50), \right. \\ &\quad \left. (\alpha, 0.90, 0.40), (\alpha\beta, 0.90, 0.40), (\alpha\beta^2, 0.90, 0.40) \right\}.\end{aligned}$$

(v) The Pythagorean fuzzy left coset of J with respect to $\alpha\beta \in W$ is

$$\begin{aligned}\alpha\beta J &= \left\{ \begin{array}{l} (e, \mu_J((\alpha\beta)^{-1}e), \nu_J((\alpha\beta)^{-1}e)), (\beta, \mu_J((\alpha\beta)^{-1}\beta), \nu_J((\alpha\beta)^{-1}\beta)), \\ (\beta^2, \mu_J((\alpha\beta)^{-1}\beta^2), \nu_J((\alpha\beta)^{-1}\beta^2)), (\alpha, \mu_J((\alpha\beta)^{-1}\alpha), \nu_J((\alpha\beta)^{-1}\alpha)), \\ (\alpha\beta, \mu_J((\alpha\beta)^{-1}\alpha\beta), \nu_J((\alpha\beta)^{-1}\alpha\beta)), (\alpha\beta^2, \mu_J((\alpha\beta)^{-1}\alpha\beta^2), \nu_J((\alpha\beta)^{-1}\alpha\beta^2)) \end{array} \right\} \\ &= \left\{ \begin{array}{l} (e, \mu_J(\alpha\beta), \nu_J(\alpha\beta)), (\beta, \mu_J(\alpha\beta^2), \nu_J(\alpha\beta^2)), (\beta^2, \mu_J(\alpha), \nu_J(\alpha)), \\ (\alpha, \mu_J(\beta^2), \nu_J(\beta^2)), (\alpha\beta, \mu_J(e), \nu_J(e)), (\alpha\beta^2, \mu_J(\beta), \nu_J(\beta)) \end{array} \right\} \\ &= \left\{ (e, 0.0.80, 0.50), (\beta, 0.80, 0.50), (\beta^2, 0.80, 0.50), \right. \\ &\quad \left. (\alpha, 0.90, 0.40), (\alpha\beta, 0.90, 0.40), (\alpha\beta^2, 0.90, 0.40) \right\}.\end{aligned}$$

(vi) The Pythagorean fuzzy left coset of J with respect to $\alpha\beta^2 \in W$ is

$$\begin{aligned}\alpha\beta^2 J &= \left\{ \begin{array}{l} (e, \mu_J((\alpha\beta^2)^{-1}e), \nu_J((\alpha\beta^2)^{-1}e)), (\beta, \mu_J((\alpha\beta^2)^{-1}\beta), \nu_J((\alpha\beta^2)^{-1}\beta)), \\ (\beta^2, \mu_J((\alpha\beta^2)^{-1}\beta^2), \nu_J((\alpha\beta^2)^{-1}\beta^2)), (\alpha, \mu_J((\alpha\beta^2)^{-1}\alpha), \nu_J((\alpha\beta^2)^{-1}\alpha)), \\ (\alpha\beta, \mu_J((\alpha\beta^2)^{-1}\alpha\beta), \nu_J((\alpha\beta^2)^{-1}\alpha\beta)), \\ (\alpha\beta^2, \mu_J((\alpha\beta^2)^{-1}\alpha\beta^2), \nu_J((\alpha\beta^2)^{-1}\alpha\beta^2)) \end{array} \right\} \\ &= \left\{ \begin{array}{l} (e, \mu_J(\alpha\beta^2), \nu_J(\alpha\beta^2)), (\beta, \mu_J(\alpha), \nu_J(\alpha)), (\beta^2, \mu_J(\alpha\beta), \nu_J(\alpha\beta)), \\ (\alpha, \mu_J(\beta), \nu_J(\beta)), (\alpha\beta, \mu_J(\beta), \nu_J(\beta)), (\alpha\beta^2, \mu_J(e), \nu_J(e)) \end{array} \right\} \\ &= \left\{ (e, 0.0.80, 0.50), (\beta, 0.80, 0.50), (\beta^2, 0.80, 0.50), \right. \\ &\quad \left. (\alpha, 0.90, 0.40), (\alpha\beta, 0.90, 0.40), (\alpha\beta^2, 0.90, 0.40) \right\}.\end{aligned}$$

Thus, there are two distinct Pythagorean fuzzy left cosets of J with respect to all elements of W , namely $eJ = \beta J = \beta^2 J$ and $\alpha J = \alpha\beta J = \alpha\beta^2 J$.

Definition 7. Let $J = (w, \mu_J(w), \nu_J(w))$ be a PFSG of W . Then,

$$J_* = \left\{ w \in W : (\mu_J(w))^2 = (\mu_J(e))^2 \text{ and } (\nu_J(w))^2 = (\nu_J(e))^2 \right\}.$$

Definition 8. Let $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ be a PFSG of W . Then, the support set of J is denoted by J^* and is defined as:

$$J^* = \left\{ w \in W : (\mu_J(w))^2 > 0 \text{ and } (\nu_J(w))^2 < 1 \right\}.$$

The following theorem gives a necessary and sufficient condition for two Pythagorean fuzzy cosets to be the same.

Theorem 7. Let $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ be a PFSG of W . Then for all $x, y \in W$

- (i) $xJ = yJ \Leftrightarrow xJ_* = yJ_*$
- (ii) $Jx = Jy \Leftrightarrow J_*x = J_*y$.

Proof 1. As the proofs of the (i) and (ii) are similar, here we only prove (i).

Suppose that $xJ = yJ$, then $\mu_{xJ}(w) = \mu_{yJ}(w)$ and $\nu_{xJ}(w) = \nu_{yJ}(w)$ for all $w \in W$. Therefore, $\mu_J(x^{-1}w) = \mu_J(y^{-1}w)$ and $\nu_J(x^{-1}w) = \nu_J(y^{-1}w)$ for all $w \in W$. In particular, $y \in W$, then $\mu_J(x^{-1}y) = \mu_J(e)$ and $\nu_J(x^{-1}y) = \nu_J(e)$, which means that $x^{-1}y \in J_*$. So $x^{-1}yJ_* = J_*$ yields $yJ_* = xJ_*$.

Conversely, let $yJ_* = xJ_*$, then $x^{-1}y \in J_*$ and $y^{-1}x \in J_*$. For all $w \in W$, we have

$$\begin{aligned}\mu_J(x^{-1}w) &= \mu_J(x^{-1}yy^{-1}w) \\ &\geq \min\{\mu_J(x^{-1}y), \mu_J(y^{-1}w)\} \\ &= \min\{\mu_J(e), \mu_J(y^{-1}w)\} \\ &= \mu_J(y^{-1}w)\end{aligned}$$

Similarly, we can obtain $\mu_J(y^{-1}w) \geq \mu_J(x^{-1}w)$. Therefore $\mu_J(x^{-1}w) = \mu_J(y^{-1}w)$ for all $w \in W$.

In a similar fashion, we have $\nu_J(x^{-1}w) = \nu_J(y^{-1}w)$ for all $w \in W$. Thus, $\mu_{xJ}(w) = \mu_{yJ}(w)$ and $\nu_{xJ}(w) = \nu_{yJ}(w)$ for all $w \in W$, which further gives $xJ = yJ$. $\square$

Lemma 1. If $(\mu_J(wxw^{-1}))^2 \geq (\mu_J(x))^2$ and $(\nu_J(wxw^{-1}))^2 \leq (\nu_J(x))^2$ for all $x, w \in W$, then J is a PFNS of W .

Proof 2. Let $(\mu_J(wxw^{-1}))^2 \geq (\mu_J(x))^2$ and $(\nu_J(wxw^{-1}))^2 \leq (\nu_J(x))^2$ for all $x, w \in W$. Since $(\mu_J(wxw^{-1}))^2 \leq (\mu_J(w^{-1}(wxw^{-1})(w^{-1})^{-1}))^2$ (since $wxw^{-1}, w^{-1} \in W$)

$$= (\mu_J(x))^2$$

Similarly, $(\nu_J(wxw^{-1}))^2 \geq (\nu_J(x))^2$.

Thus, $(\mu_J(wxw^{-1}))^2 = (\mu_J(x))^2$ and $(\nu_J(wxw^{-1}))^2 = (\nu_J(x))^2$

This together with Theorem 6 implies that J is PFNS of W . $\square$

Theorem 8. Let $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ be a PFSG of W . Then, J is a PFNSG of W if and only if $J_{(\theta, \tau)} \leq W$ for all $\theta \in [0, (\mu_J(e))^2]$ and $\tau \in [(\nu_J(e))^2, 1]$.

Proof 3. In [56], it was proved that if J is a PFNS of W , $J_{(\theta, \tau)} \leq W$ for all $\theta \in [0, (\mu_J(e))^2]$ and $\tau \in [(\nu_J(e))^2, 1]$.

Conversely, suppose that $J_{(\theta, \tau)} \leq W$ for all $\theta \in [0, (\mu_J(e))^2]$ and $\tau \in [(\nu_J(e))^2, 1]$. According to Theorem 4, J is a PFSG of W . Let $x, w \in W$, such that $(\mu_J(x))^2 = a$ and $(\nu_J(x))^2 = b$. Then, $x \in J_{(a, b)}$, and therefore $wxw^{-1} \in J_{(a, b)}$. This further implies that $(\mu_J(wxw^{-1}))^2 \geq a = (\mu_J(x))^2$ and $(\nu_J(wxw^{-1}))^2 \leq b = (\nu_J(x))^2$. Then, according to Lemma 1, J is a PFNSG of W . $\square$

Lemma 2. If $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ is a PFSG of W , then both J^* and J_* are subgroups of W .

The proof can be obtained easily by using the previous notions.

Theorem 9. If $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ is a PFNSG of W , then

- (i) $J^* \leq W$.
- (ii) $J_* \leq W$.

Proof 4. Suppose that $J = \{(w, \mu_J(w), \nu_J(w)) : w \in W\}$ is a PFNSG of W .

Then

$$(\mu_J(x))^2 = (\mu_J(wxw^{-1}))^2 \text{ and } (\nu_J(x))^2 = (\nu_J(wxw^{-1}))^2 \text{ for all } x, w \in W.$$

(i) Let $w \in W$ and $x \in J^*$, then

$$(\mu_J(wxw^{-1}))^2 = (\mu_J(x))^2 \text{ (since } J \text{ is a PFNSG of } W)$$

> 0 (since $x \in J^*$) Similarly, we can obtain, $(\nu_J(wxw^{-1}))^2 < 1$ for all $w \in W$ and $x \in J^*$. This means that $wxw^{-1} \in J^*$ for all $w \in W$ and $x \in J^*$. Thus, $J^* \leq W$.

(ii) The proof is identical to that of (i). The upcoming example demonstrates that the converse of this theorem is invalid. $\square$

Example 2. Considering $S_3 = \{e, x, y, y^2, xy, xy^2\}$, let us define PFSG of S_3 as follows;

$$J = \left\{ (e, 1, 0), (x, 0.8, 0.7)(y, 0.7, 0.1), (y^2, 0.7, 0.1), (xy, 0.7, 0.1), (xy^2, 0.7, 0.1) \right\}.$$

Then, $J^* = S_3$ and $J_* = \{e\}$ are normal subgroups of S_3 , but J is not a PFNSG of S_3 because $\mu_J((y)(xy^2)) = \mu_J(xy) = 0.7 \neq 0.8 = \mu_J(x) = \mu_J((xy^2)(y))$.

4. Pythagorean Fuzzy Normal Subgroup of a Pythagorean Fuzzy Subgroup

In this section, we define a Pythagorean fuzzy normal subgroup of a Pythagorean fuzzy subgroup and present a comprehensive study of this notion.

Definition 9. Let J and K be two PFSGs of W such that $J \subseteq K$. Then, J is a PFNSG of K if

$$(\mu_J(wxw^{-1}))^2 \geq \min\{(\mu_J(x))^2, (\mu_K(w))^2\} \text{ and } (\nu_J(wxw^{-1}))^2 \leq \max\{(\nu_J(x))^2, (\nu_K(w))^2\}$$

for all $x, w \in W$.

The following theorem describes the sufficient condition of normality between the supports of two PFSGs of W .

Theorem 10. Let J and K be two PFSGs of W such that J is PFNSG of K . Then $J^* \leq K^*$.

Proof 5. Let $x \in J^*$ and $w \in K^*$, then

$$(\mu_J(x))^2, (\mu_K(w))^2 > 0 \text{ and } (\nu_J(x))^2, (\nu_K(w))^2 < 1.$$

This means that $\min\{(\mu_J(x))^2, (\mu_K(w))^2\} > 0$ and $\max\{(\nu_J(x))^2, (\nu_K(w))^2\} < 1$.

Ultimately, we obtain

$$(\mu_J(wxw^{-1}))^2 \geq \min\{(\mu_J(x))^2, (\mu_K(w))^2\} > 0 \text{ and } (\nu_J(wxw^{-1}))^2 \leq \max\{(\nu_J(x))^2, (\nu_K(w))^2\} < 1$$

implying that $wxw^{-1} \in J^*$. Thus, $J^* \leq K^*$. $\square$

Theorem 11. Let J and K be two PFSGs of W such that $J \subseteq K$. Then, J is a PFNSG of K if and only if

$$(\mu_J(xw))^2 \geq \min\{(\mu_J(wx))^2, (\mu_K(x))^2\} \text{ and } (\nu_J(xw))^2 \leq \max\{(\nu_J(wx))^2, (\nu_K(x))^2\}$$

for all $w, x \in W$.

Proof 6. Let J be a PFNSG of K , then

$$\begin{aligned}(\mu_J(xw))^2 &= (\mu_J(xwxx^{-1}))^2 \geq \min\{(\mu_J(wx))^2, (\mu_K(x))^2\} \text{ and} \\ (\nu_J(xw))^2 &= (\nu_J(xwxx^{-1}))^2 \leq \max\{(\nu_J(wx))^2, (\nu_K(x))^2\}\end{aligned}$$

for all $w, x \in W$.

Conversely, suppose that

$$\begin{aligned}(\mu_J(xw))^2 &\geq \min\{(\mu_J(wx))^2, (\mu_K(x))^2\} \text{ and} \\ (\nu_J(xw))^2 &\leq \max\{(\nu_J(wx))^2, (\nu_K(x))^2\} \text{ for all } w, x \in W.\end{aligned}$$

Then

$$\begin{aligned}(\mu_J(wxw^{-1}))^2 &\geq \min\{(\mu_J(xw^{-1}w))^2, (\mu_K(w))^2\} \\ &= \min\{(\mu_J(x))^2, (\mu_K(w))^2\}\end{aligned}$$

and

$$\begin{aligned}(\nu_J(wxw^{-1}))^2 &\leq \max\{(\nu_J(xw^{-1}w))^2, (\nu_K(w))^2\} \\ &= \max\{(\nu_J(x))^2, (\nu_K(w))^2\}\end{aligned}$$

for all $w, x \in W$. Thus, J is a PFNSG of K . $\square$

Theorem 12. Let J and K be two PFSGs of W . Then, J is a PFNSG of K if and only if

$$J_{(\theta, \tau)} \trianglelefteq K_{(\theta, \tau)} \text{ for all } \theta \in [0, (\mu_J(e))^2] \text{ and } \tau \in [(\nu_J(e))^2, 1].$$

Proof 7. Let J be a PFNSG of J' and $\theta \in [0, \mu_J(e)]$ and $\tau \in [\nu_J(e), 1]$, then Theorem 3 and 4 together imply that $J_{(\theta, \tau)}$ is a subgroup of $K_{(\theta, \tau)}$. Suppose that $x \in J_{(\theta, \tau)}$ and $w \in K_{(\theta, \tau)}$, then $(\mu_J(x))^2, (\mu_K(w))^2 \geq \theta$ and $(\nu_J(x))^2, (\nu_K(w))^2 \leq \tau$.

Now

$$\begin{aligned}(\mu_J(wxw^{-1}))^2 &\geq \min\{(\mu_J(x))^2, (\mu_K(w))^2\} \geq \theta \\ \text{and } (\nu_J(wxw^{-1}))^2 &\leq \max\{(\nu_J(x))^2, (\nu_K(w))^2\} \leq \tau.\end{aligned}$$

This means that $wxw^{-1} \in J_{(\theta, \tau)}$. Thus, $J_{(\theta, \tau)} \trianglelefteq K_{(\theta, \tau)}$.

Conversely, suppose that $J_{(\theta, \tau)} \trianglelefteq K_{(\theta, \tau)}$ for all $\theta \in [0, \mu_J(e)]$ and $\tau \in [\nu_J(e), 1]$. Let $x, w \in W$, such that $(\mu_J(x))^2 = a$, $(\nu_J(x))^2 = b$ and $(\mu_K(w))^2 = c$, $(\nu_K(w))^2 = d$. Then, there are four cases:

- (i) $c \geq a$ and $d \geq b$
- (ii) $c \geq a$ and $d \leq b$
- (iii) $c \leq a$ and $d \geq b$
- (iv) $c \leq a$ and $d \leq b$

- (i) If $c \geq a$ and $d \geq b$, then $w \in K_{(a, d)}$ and $x \in J_{(a, d)}$. Then, by assumption, $wxw^{-1} \in J_{(a, d)}$, which gives

$$(\mu_J(wxw^{-1}))^2 \geq a = \min(a, c) = \min((\mu_J(x))^2, (\mu_K(w))^2)$$

and

$$(\nu_J(wxw^{-1}))^2 \leq d = \max(b, d) = \max((\nu_J(x))^2, (\nu_K(w))^2)$$

(ii) If $c \geq a$ and $d \leq b$, then $w \in K_{(a,b)}$ and $x \in J_{(a,b)}$.

Then, by assumption, $wxw^{-1} \in J_{(a,b)}$, which gives

$$\left(\mu_J(wxw^{-1})\right)^2 \geq a = \min(a, c) = \min\left((\mu_J(x))^2, (\mu_K(w))^2\right)$$

and

$$\left(\nu_J(wxw^{-1})\right)^2 \leq b = \max(b, d) = \max\left((\nu_J(x))^2, (\nu_K(w))^2\right)$$

Similarly, for (iii) and (iv), we obtain the same result.

Hence, J is a PFNSG of K . $\square$

Theorem 13. Suppose that J and K are two PFSGs of W . Then, $J^* \trianglelefteq K^*$ and $J_* \trianglelefteq K_*$ if J is a PFNSG of K .

Proof 8. Suppose that J is a PFNSG of K and $x \in J^*$ and $w \in K^*$, then

$$\left(\mu_J(wxw^{-1})\right)^2 \geq \min\left((\mu_J(x))^2, (\mu_K(w))^2\right) = (\mu_J(e))^2$$

and

$$\left(\nu_J(wxw^{-1})\right)^2 \leq \max\left((\nu_J(x))^2, (\nu_K(w))^2\right) = (\nu_J(e))^2$$

This means that $wxw^{-1} \in J^*$. Thus, $J^* \trianglelefteq K^*$.

A similar reasoning show that $J_* \trianglelefteq K_*$. $\square$

Theorem 14. Assume that J is a PFNSG and K is a PFSG of W . Then, $J \cap K$ is a PFNSG of K .

Proof 9. In view of Theorem 2, $J \cap K$ is a PFSG of W and $J \cap K \subseteq K$. Let $x, w \in W$, then

$$\begin{aligned} \left(\mu_{J \cap K}(wxw^{-1})\right)^2 &= \min\left\{\left(\mu_J(wxw^{-1})\right)^2, \left(\mu_K(wxw^{-1})\right)^2\right\} \\ &\geq \min\left\{\left(\mu_J(x)\right)^2, \min\left\{\left(\mu_K(wx)\right)^2, \left(\mu_K(w^{-1})\right)^2\right\}\right\} \\ &\geq \min\left\{\left(\mu_J(x)\right)^2, \min\left\{\min\left\{\left(\mu_K(w)\right)^2, \left(\mu_K(x)\right)^2\right\}, \left(\mu_K(w)\right)^2\right\}\right\} \\ &= \min\left\{\left(\mu_J(x)\right)^2, \min\left\{\left(\mu_K(w)\right)^2, \left(\mu_K(x)\right)^2\right\}\right\} \\ &= \min\left\{\min\left\{\left(\mu_J(x)\right)^2, \left(\mu_K(x)\right)^2\right\}, \left(\mu_K(w)\right)^2\right\} \\ &= \min\left\{\left(\mu_{J \cap K}(x)\right)^2, \left(\mu_K(w)\right)^2\right\} \end{aligned}$$

So, $\left(\mu_{J \cap K}(wxw^{-1})\right)^2 \geq \min\left\{\left(\mu_{J \cap K}(x)\right)^2, \left(\mu_K(w)\right)^2\right\}$ for all $x, w \in W$. Similarly, we can show that $\left(\nu_{J \cap K}(wxw^{-1})\right)^2 \leq \max\left\{\left(\nu_{J \cap K}(x)\right)^2, \left(\nu_K(w)\right)^2\right\}$ for all $x, w \in W$. Thus, $J \cap K$ is a PFNSG of K . $\square$

Theorem 15. Suppose that J, K and L are PFSGs of W such that J and K are PFNSGs of L . Then $J \cap K$ is PFSG of L .

Proof 10. Observe that $J \cap K$ is a PFSG of and $J \cap K \subseteq L$. Let $x, w \in W$, then

$$\begin{aligned} (\mu_{J \cap K}(wxw^{-1}))^2 &= \min\{(\mu_J(wxw^{-1}))^2, (\mu_K(wxw^{-1}))^2\} \\ &\geq \min\{\min\{(\mu_J(x))^2, (\mu_L(w))^2\}, \min\{(\mu_K(x))^2, (\mu_L(w))^2\}\} \\ &= \min\{\min\{(\mu_J(x))^2, (\mu_K(x))^2\}, (\mu_L(w))^2\} \\ &= \min\{(\mu_{J \cap K}(x))^2, (\mu_L(w))^2\} \end{aligned}$$

Similarly, we can obtain

$$(\nu_{J \cap K}(wxw^{-1}))^2 \leq \max\{(\nu_{J \cap K}(x))^2, (\nu_L(w))^2\}$$

Therefore, $J \cap K$ is a PFSG of L . $\square$

Theorem 16. Assume that $\alpha : W \rightarrow X$ is a group homomorphism and J and K are PFSGs of W . Then, $\alpha(J)$ is a PFNSG of $\alpha(K)$ if J is a PFNSG of K .

Proof 11. It is easy to prove that $\alpha(J)$ and $\alpha(K)$ are PFNSGs of X and $\alpha(J) \subseteq \alpha(K)$. Now

$$\begin{aligned} (\mu_{\alpha(J)}(wxw^{-1}))^2 &= \max\{(\mu_J(z))^2 : z \in W, \alpha(z) = wxw^{-1}\} \\ &= \max\{(\mu_J(kjk^{-1}))^2 : k, j \in W, \alpha(k) = w, \alpha(j) = x\} \\ &\geq \max\{\min\{(\mu_J(j))^2, (\mu_K(k))^2\} : k, j \in W, \alpha(k) = w, \alpha(j) = x\} \\ &= \min\{\max\{(\mu_J(j))^2 : j \in W, \alpha(j) = x\}, \max\{(\mu_K(k))^2 : k \in W, \alpha(k) = w\}\} \\ &= \min\{(\mu_{\alpha(J)}(x))^2, (\mu_{\alpha(K)}(w))^2\} \end{aligned}$$

In a similar manner, we

$$(\nu_{\alpha(J)}(wxw^{-1}))^2 \leq \max\{(\nu_{\alpha(J)}(x))^2, (\nu_{\alpha(K)}(w))^2\}$$

Thus, $\alpha(J)$ is a PFNSG of $\alpha(K)$. $\square$

Theorem 17. Assume that $\alpha : W \rightarrow X$ is a group homomorphism and J and K are PFSGs of X . Then $\alpha^{-1}(J)$ is a PFNSG of $\alpha^{-1}(K)$ if J is a PFNSG of K .

Proof 12. Clearly, $\alpha^{-1}(J)$ and $\alpha^{-1}(K)$ are PFNSGs of W and $\alpha^{-1}(J) \subseteq \alpha^{-1}(K)$. Now

$$\begin{aligned} (\mu_{\alpha^{-1}(J)}(wxw^{-1}))^2 &= (\mu_J(\alpha(wxw^{-1})))^2 \\ &= (\mu_J(\alpha(w), \alpha(x), (\alpha(w))^{-1}))^2 \\ &\geq \min\{(\mu_J(\alpha(x)))^2, (\mu_K(\alpha(w)))^2\} \\ &= \min\{(\mu_{\alpha^{-1}(J)}(x))^2, (\mu_{\alpha^{-1}(K)}(w))^2\} \end{aligned}$$

Similarly

$$(\nu_{\alpha^{-1}(J)}(wxw^{-1}))^2 \leq \max\{(\nu_{\alpha^{-1}(J)}(x))^2, (\nu_{\alpha^{-1}(K)}(w))^2\}$$

Therefore, $\alpha^{-1}(J)$ is a PFNSG of $\alpha^{-1}(K)$. $\square$

5. Fundamental Theorems of Pythagorean Fuzzy Isomorphism

In this section, we define the notions of Pythagorean fuzzy homomorphism and isomorphism. We generalize the concept of a factor group of a classical group W relative to its normal subgroup S by defining a PFSG of $\frac{W}{S}$. Furthermore, we prove a Pythagorean fuzzy version of the fundamental theorems of isomorphisms.

Definition 10. Let J and K be two PFSGs of W and X , respectively. Then a homomorphism $\alpha : W \rightarrow X$ is called

- (i) weak Pythagorean fuzzy homomorphism of J_1 to J_2 if $\alpha(J_1) \subseteq J_2$. In such a case, we write $J_1 \sim J_2$.
- (ii) Pythagorean fuzzy homomorphism of J_1 to J_2 if $\alpha(J_1) = J_2$. In such a case, we write $J_1 \approx J_2$.

Definition 11. Let J_1 and J_2 be two PFSGs of W_1 and W_2 , respectively. Then an isomorphism $\alpha : W_1 \rightarrow W_2$ is called

- (i) weak Pythagorean fuzzy isomorphism of J_1 to J_2 if $\alpha(J_1) \subseteq J_2$. In such a case, we write $J_1 \simeq J_2$.
- (ii) Pythagorean fuzzy isomorphism of J_1 to J_2 if $\alpha(J_1) = J_2$. In such a case, we write $J_1 \cong J_2$.

The segmentation of color images is a challenge which is in constant development, as they provide more information than a gray-scale image. In particular, the analysis of biomedical homomorphic images is especially helpful for many purposes. The concept of Pythagorean fuzzy homomorphism is effectively applied in the framework of color image segmentation, by means of which one can obtain a good performance to achieve the segmentation of the objects of interest.

Definition 12. Let J be a PFSG of W and $S \trianglelefteq W$. Define a PFS J' of $\frac{W}{S}$ in the following way:

- (i) $(\mu_{J'}(wS))^2 = \max\{(\mu_J(t))^2 : t \in wS\}$
- (ii) $(\nu_{J'}(wS))^2 = \min\{(\nu_J(t))^2 : t \in wS\}$ for all $wS \in \frac{W}{S}$.

Theorem 18. J' is a PFSG of $\frac{W}{S}$.

Proof 13. Let $wS \in \frac{W}{S}$, then

$$\begin{aligned} (\mu_{J'}(wS)^{-1})^2 &= (\mu_{J'}(w^{-1}S))^2 \\ &= \max\{(\mu_J(t))^2 : t \in w^{-1}S\} \\ &= \max\{(\mu_J(t^{-1}))^2 : t^{-1} \in wS\} \text{ (since } S \text{ is normal in } W) \\ &= (\mu_{J'}(wS))^2 \end{aligned}$$

In a similar way, we obtain $(\nu_{J'}(wS)^{-1})^2 = (\nu_{J'}(wS))^2$.

Next, let $w_1S, w_2S \in W/S$, then

$$\begin{aligned} (\mu_{J'}(w_1S.w_2S))^2 &= (\mu_{J'}(w_1w_2S))^2 \\ &= \max\{(\mu_J(t))^2 : t \in w_1w_2S = w_1S.w_2S\} \\ &= \max\{(\mu_J(xy))^2 : x \in w_1S, y \in w_2S\} \\ &\geq \max\{\min((\mu_J(x))^2, (\mu_J(y))^2) : x \in w_1S, y \in w_2S\} \\ &= \min(\max\{(\mu_J(x))^2 : x \in w_1S\}, \max\{(\mu_J(y))^2 : y \in w_2S\}) \\ &= \min((\mu_{J'}(w_1S))^2, (\mu_{J'}(w_2S))^2) \end{aligned}$$

Similarly, it can be proved that

$$(\nu_{J'}(w_1S.w_2S))^2 \leq \max((\nu_{J'}(w_1S))^2, (\nu_{J'}(w_2S))^2).$$

Thus, J' is a PFSG of W/S . $\square$

Remark 1. The PFSG J' is called a factor Pythagorean fuzzy subgroup of J relative to S .

Suppose that J and K are PFSGs of W such that J is a PFNSG of K . Then, according to Theorem 13, $J^* \trianglelefteq K^*$, and therefore the quotient group J^*/K^* exists.

Let us define a PFS J'' of J^* in the following way:

$$\begin{aligned} \mu_{J''}(w) &= \mu_J(w) \text{ for all } w \in J^* \\ \nu_{J''}(w) &= \nu_J(w) \text{ for all } w \in J^* \end{aligned}$$

Obviously, J'' is a PFSG of J^* . Now, by Definition 12 and Theorem 18, Pythagorean fuzzy quotient group of J'' with respect to K^* exists. For the sake of convenience, we denote it by J/K .

Clearly, J/K is a PFSG of J^*/K^* and

$$\begin{aligned} \text{(i)} \quad (\mu_{J/K}(wK^*))^2 &= \max\{(\mu_{J''}(t))^2 : t \in wK^*\} \\ \text{(ii)} \quad (\nu_{J/K}(wK^*))^2 &= \min\{(\nu_{J''}(t))^2 : t \in wK^*\}. \end{aligned}$$

Theorem 19. Let J and K be PFSGs of W and J be a PFNSG of K . Then

$$J'' \cong J/K.$$

Proof 14. Observe that J'' is a PFSG of J^* and of J/K is a PFSG of J^*/K^* . Let $\alpha : J^* \rightarrow J^*/K^*$ be defined by $\alpha(w) = wK^*$. Then, obviously, α is a homomorphism. Now

$$\begin{aligned} (\mu_{\alpha(J'')}(wK^*))^2 &= \max\{(\mu_{J''}(t))^2 : t \in J^*, \alpha(t) = wK^*\} \\ &= \max\{(\mu_J(u))^2 : u \in wK^*\} \\ &= (\mu_{J/K}(wK^*))^2 \end{aligned}$$

By using the same arguments, we show that

$$(\nu_{\alpha(J'')}(wK^*))^2 = (\nu_{J/K}(wK^*))^2$$

So, we conclude that $\alpha(J'') = J/K$, which further implies that $J'' \cong J/K$. $\square$

Lemma 3. Let J be a PFSG of W and $\alpha : W \rightarrow X$ be a group homomorphism. Then, $(\alpha(J))^* = \alpha(J^*)$.

Proof 15. Let $x \in \alpha(J^*)$, then $\alpha(w) = x$ for some $w \in J^*$.

Consider $(\mu_{\alpha(J)}(x))^2 = \sup\{(\mu_J(z) : z \in \alpha^{-1}(x))^2\} \geq (\mu_J(w))^2 > 0$ and

$$(\nu_{\alpha(J)}(x))^2 = \inf\{(\nu_J(z) : z \in \alpha^{-1}(x))^2\} \leq (\nu_J(w))^2 < 1.$$

Therefore, $x \in (\alpha(J))^*$. Thus, $\alpha(J^*) \subseteq (\alpha(J))^*$.

Let $x \in (\alpha(J))^*$, then there exists $w \in W$, such that $\alpha(w) = x$ and $\mu_J(w) = \mu_{\alpha(J)}(x)$. Since $(\mu_{\alpha(J)}(x))^2 > 0$, so $(\mu_J(w))^2 > 0$, which implies that $(\nu_J(w))^2 < 1$. This gives $w \in J^*$ and therefore $x = \alpha(w) \in \alpha(J^*)$. Thus, $(\alpha(J))^* \subseteq \alpha(J^*)$. $\square$

Theorem 20. First Fundamental Theorem of Pythagorean Fuzzy Isomorphism. Assume that W and S are crisp groups and J and L are PFSGs of W and S , respectively. If $J \cong L$, then there exists a PFNSG K of J , such that $J/K \cong L''$.

Proof 16. Suppose that $J \cong L$, then there exists an epimorphism $\alpha : W \rightarrow S$, such that $\alpha(J) = L$. Define a PFSG K of W

$$\mu_K(w) = \begin{cases} \mu_J(w), & \text{if } w \in \ker \alpha \\ 0, & \text{otherwise} \end{cases} \text{ and } \nu_K(w) = \begin{cases} \nu_J(w), & \text{if } w \in \ker \alpha \\ 1, & \text{otherwise} \end{cases}.$$

Clearly, K is a PFSG of W and $K \subseteq J$.

Suppose that $x \in W$, then there are the two following possibilities;

(i) If $x \in \ker \alpha$, then $wxw^{-1} \in \ker \alpha$ for all $w \in W$. Furthermore,

$$\begin{aligned} (\mu_K(wxw^{-1}))^2 &= (\mu_J(wxw^{-1}))^2 \geq \min\{(\mu_J(x))^2, (\mu_J(w))^2\} \\ &= \min\{(\mu_K(x))^2, (\mu_J(w))^2\} \end{aligned}$$

and

$$\begin{aligned} (\nu_K(wxw^{-1}))^2 &= (\nu_J(wxw^{-1}))^2 \leq \max\{(\nu_J(x))^2, (\nu_J(w))^2\} \\ &= \max\{(\nu_K(x))^2, (\nu_J(w))^2\} \end{aligned}$$

(ii) If $x \in W / \ker \alpha$, then $\mu_K(x) = 0$ and $\nu_K(x) = 1$. Furthermore,

$$\begin{aligned} (\mu_K(wxw^{-1}))^2 &\geq \min\{(\mu_K(x))^2, (\mu_J(w))^2\} \\ \text{and } (\nu_K(wxw^{-1}))^2 &\leq \max\{(\nu_K(x))^2, (\nu_J(w))^2\}. \end{aligned}$$

Thus, in both cases, K is a PFNSG of J . In addition, $J \cong L$ yields $\alpha(J) = L$, so $(\alpha(J))^* = L^*$. The application of Lemma 3 gives $\alpha(J^*) = L^*$. Let $\beta = \alpha|_{J^*}$, then β is a homomorphism from J^* to L^* such that $\text{Ker } \beta = K^*$. According to the first isomorphism theorem, there exists an isomorphism $\psi : J^*/K^* \rightarrow L^*$ defined by $\psi(wK^*) = \beta(w) = \alpha(w)$ for all $wK^* \in J^*/K^*$. Now, for all $k \in L^*$:

$$\begin{aligned}
(\mu_{\psi(J/K)}(k))^2 &= \max\{(\mu_{J/K}(wK^*))^2 : w \in J^*, \psi(wK^*) = k\} \\
&= \max\left\{\max\{(\mu_J(t))^2 : t \in wK^*\} : w \in J^*, \beta(w) = k\right\} \\
&= \max\{(\mu_J(t))^2 : t \in J^*, \beta(t) = k\} \\
&= \max\{(\mu_J(t))^2 : t \in J^*, \beta(t) = k\} \\
&= \max\{(\mu_J(t))^2 : t \in W, \alpha(t) = k\} \\
&= (\mu_{\alpha(J)}(k))^2 \\
&= (\mu_L(k))^2 \\
&= (\mu_{L''}(k))^2 \text{ (since } k \in L^*)
\end{aligned}$$

In a similar way, we can obtain:

$$(\nu_{\psi(J/K)}(k))^2 = (\nu_{L''}(k))^2$$

Thus, $J/K \cong L''$. $\square$

Definition 13. Let J and K be two PFSGs of W , then the product of J and K is denoted by $J \circ K$ and is defined as:

- (i) $\mu_{J \circ K}(w) = \max\{\min(\mu_J(u), \mu_K(v)) : u, v \in W, uv = w\}$
- (ii) $\nu_{J \circ K}(w) = \min\{\max(\nu_J(u), \nu_K(v)) : u, v \in W, uv = w\}$ for all $w \in W$.

Lemma 4. Let J and K be two PFSGs of W . Then

- (i) $(J \cap K)^* = J^* \cap K^*$.
- (ii) $(J \circ K)^* = J^* K^*$

Proof 17.

- (i) Suppose that $w \in W$, then

$$\begin{aligned}
w \in (J_1 \cap J_2)^* &\Leftrightarrow (\mu_{J_1 \cap J_2}(w))^2 > 0 \text{ and } (\nu_{J_1 \cap J_2}(w))^2 < 1 \\
&\Leftrightarrow \min((\mu_{J_1}(w))^2, (\mu_{J_2}(w))^2) > 0 \text{ and } \min((\nu_{J_1}(w))^2, (\nu_{J_2}(w))^2) < 1 \\
&\Leftrightarrow (\mu_{J_1}(w))^2, (\mu_{J_2}(w))^2 > 0 \text{ and } (\nu_{J_1}(w))^2, (\nu_{J_2}(w))^2 < 1 \\
&\Leftrightarrow w \in J_1^*, J_2^* \Leftrightarrow w \in J_1^* \cap J_2^*.
\end{aligned}$$

Thus, $(J_1 \cap J_2)^* = J_1^* \cap J_2^*$.

- (ii) Suppose that $w \in W$, then

$$\begin{aligned}
w \in (J \circ K)^* &\Leftrightarrow (\mu_{J \circ K}(w))^2 > 0 \text{ and } (\nu_{J \circ K}(w))^2 < 1 \\
&\Leftrightarrow \max\{\min(\mu_J(u), \mu_K(v))^2 : u, v \in W, uv = w\} > 0 \\
&\text{and } \nu_{J \circ K}(w) = \min\{\max(\nu_J(u), \nu_K(v)) : u, v \in W, uv = w\} < 1 \\
&\Leftrightarrow \mu_J(u) > 0, \mu_K(v) > 0 \text{ and } \nu_J(u), \nu_K(v) < 1
\end{aligned}$$

where, $u, v \in W$ such that $uv = w \Leftrightarrow u \in J^*$ and $v \in K^*$, where $u, v \in W$, such that $uv = w \Leftrightarrow w = uv \in J^* K^*$. Thus, $(J \circ K)^* = J^* K^*$. $\square$

Theorem 21. Second Fundamental Theorem of Pythagorean Fuzzy Isomorphism.

Let J be a PFNSG of W and K be a PFSG of W . Then, $K/J \cap K \simeq (J \circ K)/J$.

Proof 18. By means of Theorem 9, we have $J^* \trianglelefteq W$. In addition, from Lemma 2, we obtain that K^* is a subgroup of W . Therefore, according to the second isomorphism theorem of classical group theory, there exists an isomorphism $\psi : K^* / J^* \cap K^* \rightarrow J^* K^* / J^*$, defined by

$$\psi(k(J^* \cap K^*)) = kJ^* \text{ for all } k \in K^*.$$

According to Lemma 4,

$$J^* \cap K^* = (J \cap K)^* \text{ and } J^* K^* = (JK)^*, \text{ respectively, so } K^* / J^* \cap K^* \cong J^* K^* / J^*.$$

Now

$$\begin{aligned} \left(\mu_{\psi(K/J \cap K)}(kJ^*) \right)^2 &= \left(\mu_{K/J \cap K}(k(J \cap K)^*) \right)^2 \text{ } \psi \text{ is one-one} \\ &= \max \left\{ (\mu_K(t))^2 : t \in k(J \cap K)^* \right\} \\ &\leq \max \left\{ ((\mu_J \circ \mu_K)(t))^2 : t \in k(J^* \cap K^*) \right\} \\ &\leq \max \left\{ ((\mu_J \circ \mu_K)(t))^2 : t \in kJ^* \right\} \\ &= ((\mu_J \circ \mu_K) / \mu_J)(kJ^*)^2 \end{aligned}$$

The same reasoning leads us to $\left(\nu_{\psi(K/J \cap K)}(kJ^*) \right)^2 \geq (((\nu_J \circ \nu_K) / \nu_J)(kJ^*))^2$. Therefore, $\psi(K/J \cap K) \subseteq (J \circ K) / J$, which further implies that $K/J \cap K \simeq (J \circ K) / J$. $\square$

Theorem 22. Third Fundamental Theorem of Pythagorean Fuzzy Isomorphism.

Let J, K and L be PFSGs of W . If J and K are PFNSGs of and $J \subseteq K$, then $(L/J)/(K/J) \simeq L/K$.

Proof 19. According to Theorem 9, $J^* \trianglelefteq L^*$ and $K^* \trianglelefteq L^*$. Since $J \subseteq K$, therefore $J^* \trianglelefteq K^*$. Then, the third isomorphism theorem of classical group theory yields an isomorphism $\psi : (L^* / J^*) / (K^* / J^*) \rightarrow L^* / K^*$, defined by $\psi(lJ^*(K^* / J^*)) = lK^*$ for all $l \in L^*$.

Next, we will prove that if $zJ^* \in lJ^*(K^* / J^*)$, then $z \in lK^*$. Let $zJ^* \in lJ^*(K^* / J^*)$, then $lJ^*(K^* / J^*) = zJ^*(K^* / J^*)$ (since $x \in gH$ implies $gH = xH$). So, $\psi(lJ^*(K^* / J^*)) = \psi(zJ^*(K^* / J^*))$, which further implies that $lK^* = zK^*$. Since $z \in zK^*$, thus, $z \in lK^*$. Now

$$\begin{aligned} \left(\mu_{\psi((L/J)/(K/J))}(lK^*) \right)^2 &= \left(\mu_{(L/J)/(K/J)}(lJ^*(K^* / J^*)) \right)^2 \\ &= \max \left\{ (\mu_{L/J}(l'J^*))^2 : l' \in L^*, l'J^* \in lJ^*(K^* / J^*) \right\} \\ &= \max \left\{ \max \left\{ (\mu_L(t))^2 : t \in l'J^* \right\} : l' \in L^*, l'J^* \in lJ^*(K^* / J^*) \right\} \\ &= \max \left\{ (\mu_L(z))^2 : z \in L^*, zJ^* \in lJ^*(K^* / J^*) \right\} \end{aligned}$$

We know that $zJ^* \in lJ^*(K^* / J^*)$, and thus $lJ^*(K^* / J^*) = zJ^*(K^* / J^*)$ (since $x \in gH$ implies $gH = xH$). So, $\psi(lJ^*(K^* / J^*)) = \psi(zJ^*(K^* / J^*))$, which further implies that $lK^* = zK^*$. Since $z \in zK^*$, thus, $z \in lK^*$. Therefore,

$$\begin{aligned} \left(\mu_{\psi((L/J)/(K/J))}(lK^*) \right)^2 &= \max \left\{ (\mu_L(z))^2 : z \in L^*, z \in lK^* \right\} \\ &= (\mu_{L/K}(lK^*))^2 \end{aligned}$$

By the same method, we obtain:

$$\left(\nu_{\psi((L/J)/(K/J))}(lK^*) \right)^2 = (\nu_{L/K}(lK^*))^2$$

Therefore, $\psi((L/J)/(K/J)) = L/K$, thus $(L/J)/(K/J) \simeq L/K$. $\square$

6. Conclusions

The main objective of this paper is to present the notion of group theory under the features of Pythagorean fuzzy sets. The Pythagorean fuzzy set is one of the extensions of the fuzzy set used to deal with uncertainties in the data. Keeping the features of this set in mind, we extend the study of the Pythagorean fuzzy cosets of a Pythagorean fuzzy subgroup. In addition, we examine the three fundamental theorems of group isomorphisms under a Pythagorean fuzzy environment. To address this, firstly, we define certain notions of group theory such as cosets of a subgroup, normal subgroups and factor groups in a Pythagorean fuzzy format and prove some results relevant to them. The results proved in this paper will help to generalize many other pure group theoretic notions such as the order of a group and its elements, Lagrange's theorem, Cayley's theorem, nilpotent and solvable groups, and group actions in a Pythagorean fuzzy environment. In the literature, group morphisms related to classical and intuitionistic fuzzy sets are used in pattern recognition problems and image encryption. Since PFS is one of the prime generalizations of IFS, the application of the concepts of Pythagorean fuzzy morphisms in pattern recognition problems and image encryption will thus produce better results as compared to classical and intuitionistic fuzzy morphisms.

In future work, we will extend the present concepts to different fields of the homomorphism and coset graphs of symmetric groups and under different extensions of the fuzzy sets. Apart from this, we will apply the results of this study to different applications such as image encryption, cryptography and image quality assessment, etc.

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