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ORIGINAL ARTICLE

Novel waves structures for two nonlinear partial differential equations arising in the nonlinear optics via Sardar-subequation method

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Abstract In this paper, our aim is to construct the novel wave structures for two non-linear evolution equations which are rising in non-linear optics, mathematical biological models, fluid dynamics, waves theory, mechanics, quantum mechanics, and many more. We employed an efficient analytical technique namely, the Sardar-subequation method to build the wave solutions for the modified Benjamin-Bona-Mahony equation and the Coupled Klein–Gordon equations. We have built the numerous type of soliton wave structures of modified Benjamin-Bona-Mahony equation and the Coupled Klein–Gordon equations via the Sardar-subequation method. Acquired results reveal the dynamics behavior of waves structures including the bright, singular, dark, and periodic singular solitons solutions. To illustrate the behavior of these solutions some selected solutions are sketched in two-, and three-dimensional graphs. On the basis of these results, our technique is suitable, up-to-date, and powerful. The obtained solutions are very efficacious and influential in

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non-linear optics, mathematical biology, mechanics, fluid mechanics, plasma physics, and many more. This study will assist to predict some new hypothesis and theories in the field of mathematical physics.

1. Introduction

In recent times, the extensive contribution of non-linear partial differential equations (NLPDEs) in scientific domains have become more significant including, mathematical biology, chemical physics, optical fibers, mechanics, hydrodynamics, etc. and also used to define the complex characteristics in these fields [1–5]. Therefore, the analysis of these models is important and precarious to understand these existences and support to researchers using them to systematic developments and attaining the most momentous goal [6–8]. To facilitate, how complex things, perform in non-linear evolution equations (NLEEs), it is our requirement to inspect the wave structures for the NLEEs [9,10]. For this purpose, many distinct non-linear physical phenomena are used in domains of robust state physical, liquid mechanics and atoms, physical compound and optical fibers [11–15]. Therefore, searching for wave solutions of NLEEs is enormously crucial. Thus, to search such solutions many efficient techniques have been established by various researchers and scientists to develop the exact solutions of non-linear PDEs in various form, such as the tanh-coth method, the improved F-expansion method, the Frobenius integrable decomposition, the extended tanh-function method, the simple equation method, the Hirota bilinear method, the multiple simplest equation method, the Darboux transformation, the trial solution method, the Hirota technique, [16–22], new extended generalized Kudryashov scheme, rational $(G\prime/G)$-expansion and improved tanh scheme, improved auxiliary equation and enhanced rational $(G\prime/G)$-expansion methods, the fourth-order Runge–Kutta technique and so on [23–29]. Furthermore, in soliton theory, integrable equations are produced from matrix spectral problems and come in hierarchies. Using certain symmetric reductions on potentials, one can construct reduced integrable equations such as modified Korteweg-de Vries equations and non-linear Schrodinger equations. Riemann–Hilbert problems, expressed from the associated given matrix spectral problems, also deliver an influential method that permits us to solve integrable equations, mainly to construct soliton solutions [30–33]. In this work, our aim is to construct and analyze the novel soliton wave solutions via the Sardar-subequation method (SSM) [34] on the modified Benjamin-Bona-Mahony (BBM) equation and the Coupled Klein–Gordon (CKG) equations [35–38]. Our focus is on the construction of various waves solitons solutions for instance bright, singular, dark-bright and periodic-singular wave solitons. The under consideration method is measured more universal among mentioned above. Likewise, these outcomes support us to recognize the active performance of several physical configurations. Moreover, these results are positive, special, exact and may be cooperative for illuminating specific non-linear real phenomena in non-linear mathematical models.

This article is designed as: The BBM equation is defined in Section 2. The description of Sardar-subequation method and its application on BBM are given in Section 3. The CKG equation and the application of SSM are given in Section 4. Physical interpretation of the results is written in Section 5 while findings of this paper are presented in Section 6.

2. The Sardar-Subequation Method

Consider the PDE

$$W(P, P_t, P_x, P_{xx}, P_{xt}, P_{tt}, P_{tx}, \dots) = 0, \tag{1}$$

where $P = P(x, t)$ is an un-known function.

$$P(x, t) = V(\zeta), \;\; \zeta = x + \lambda t, \tag{2}$$

where λ is non-zero constant. By applying the above transformation, given below ODE is obtained

$$P(V, V', V'', \dots) = 0, \tag{3}$$

in which $V = V(\zeta), V' = \frac{dV}{d\zeta}, V\prime\prime = \frac{d^2V}{d\zeta^2}, \dots,$.

Consider (3) has the solution as follows

$$V(\zeta) = \sum_{j=0}^{M} \Lambda_j \Omega^j(\zeta), \tag{4}$$

where $\Lambda_j (0 \leqslant j \leqslant M)$ are constants.

$$(\Omega\prime(\zeta))^2 = G + F\Omega^2(\zeta) + \Omega^4(\zeta), \tag{5}$$

where G and F are real constants and (5) yields the sets of solutions as

Case 1:When $F > 0 and G = 0$, then

$$\Omega_1^{\pm}(\zeta) = \pm\sqrt{-Fpq}sech_{pq}(\sqrt{F}\zeta), \tag{6}$$

$$\Omega_2^{\pm}(\zeta) = \pm\sqrt{Fpq}csch_{pq}(\sqrt{F}\zeta), \tag{7}$$

where

$sech_{pq}(\zeta) = \frac{2}{pe^{\zeta} + qe^{-\zeta}}, \;\; csch_{pq}(\zeta) = \frac{2}{pe^{\zeta} - qe^{-\zeta}}$.

Case 2:When $F < 0 and G = 0$, then

$$\Omega_3^{\pm}(\zeta) = \pm\sqrt{-Fpq}sec_{pq}(\sqrt{-F}\zeta), \tag{8}$$

$$\Omega_4^{\pm}(\zeta) = \pm\sqrt{-Fpq}csc_{pq}(\sqrt{-F}\zeta), \tag{9}$$

where

$sec_{pq}(\zeta) = \frac{2}{pe^{\imath\zeta} + qe^{-\imath\zeta}}, \;\; csc_{pq}(\zeta) = \frac{2\imath}{pe^{\imath\zeta} - qe^{-\imath\zeta}}$.

Case 3:When $F < 0 and G = \frac{a^2}{4b}$, then

$$\Omega_5^{\pm}(\zeta) = \pm\sqrt{-\frac{F}{2}}tanh_{pq}(\sqrt{-\frac{F}{2}}\zeta), \tag{10}$$

$$\Omega_6^{\pm}(\zeta) = \pm\sqrt{-\frac{F}{2}}coth_{pq}(\sqrt{-\frac{F}{2}}\zeta), \tag{11}$$

$$\Omega_7^{\pm}(\zeta) = \pm\sqrt{-\frac{F}{2}}\Big(tanh_{pq}(\sqrt{-2F}\zeta) \pm \imath\sqrt{pq}sech_{pq}(\sqrt{-2F}\zeta)\Big), \tag{12}$$

$$\Omega_8^{\pm}(\zeta) = \pm\sqrt{-\frac{F}{2}}\Big(coth_{pq}(\sqrt{-2F}\zeta) \pm \sqrt{pq}csch_{pq}(\sqrt{-2F}\zeta)\Big), \tag{13}$$

$$\Omega_9^{\pm}(\zeta) = \pm\sqrt{-\frac{F}{8}}\left(tanh_{pq}(\sqrt{-\frac{F}{8}}\zeta) + coth_{pq}(\sqrt{-\frac{F}{8}}\zeta)\right), \tag{14}$$

where

$tanh_{pq}(\zeta) = \frac{pe^{\zeta}-qe^{-\zeta}}{pe^{\zeta}+qe^{-\zeta}},\ coth_{pq}(\zeta) = \frac{pe^{\zeta}+qe^{-\zeta}}{pe^{\zeta}-qe^{-\zeta}}.$

Case 4:When $F > 0 and G = \frac{a^2}{4}$, then

$$\Omega_{10}^{\pm}(\zeta) = \pm\sqrt{\frac{F}{2}}tan_{pq}(\sqrt{\frac{F}{2}}\zeta), \tag{15}$$

$$\Omega_{11}^{\pm}(\zeta) = \pm\sqrt{\frac{F}{2}}cot_{pq}(\sqrt{\frac{F}{2}}\zeta), \tag{16}$$

$$\Omega_{12}^{\pm}(\zeta) = \pm\sqrt{\frac{F}{2}}\Big(tan_{pq}(\sqrt{2F}\zeta) \pm \sqrt{pq}sec_{pq}(\sqrt{2F}\zeta)\Big), \tag{17}$$

$$\Omega_{13}^{\pm}(\zeta) = \pm\sqrt{\frac{F}{2}}\Big(cot_{pq}(\sqrt{2F}\zeta) \pm \sqrt{pq}csc_{pq}(\sqrt{2F}\zeta)\Big), \tag{18}$$

$$\Omega_{14}^{\pm}(\zeta) = \pm\sqrt{\frac{F}{8}}\left(tan_{pq}(\sqrt{\frac{F}{8}}\zeta) + cot_{pq}(\sqrt{\frac{F}{8}}\zeta)\right), \tag{19}$$

where

$tan_{pq}(\zeta) = -\imath\frac{pe^{\imath\zeta}-qe^{-\imath\zeta}}{pe^{\imath\zeta}+qe^{-\imath\zeta}},\ cot_{pq}(\zeta) = \imath\frac{pe^{\imath\zeta}+qe^{-\imath\zeta}}{pe^{\imath\zeta}-qe^{-\imath\zeta}}.$

3. The Benjamin-Bona-Mahony Equation

The modified BBM equation is defined as

$$P_t + P_x + \mu P^2 P_x + P_{xxt} = 0, \tag{20}$$

where μ is a constant that will be found far ahead. Using the following wave transformation

$$P(x,t) = V(\zeta),\ \zeta = x + \lambda t,$$

converts (20) into the ODE, as follows

$$3(\lambda+1)V + \mu V^3 + 3\lambda V'' = 0. \tag{21}$$

3.1. Application of the SSM on BBM

In order to achieve the soliton wave structures of BBM, the SSM is applied on (21). By balance rule on terms of V'' and V^3 in (21), gives $M = 1$, so (4) changes to

$$V(\zeta) = \Lambda_0 + \Lambda_1\Omega(\zeta), \tag{22}$$

where Λ_0, Λ_1 are constants. Replacing the (22) along with (5) into (21), and gets the set of equations in $\Lambda_0, \Lambda_1, \mu, and \lambda$. On resolving the system of equations, we attains

$$\Lambda_0 = 0,\ \Lambda_1 = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}},$$

$$\lambda = \frac{1}{-1-F}. \tag{23}$$

Case 1: When $F > 0 and G = 0$, then

$$P_{1,1}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\Big(\pm\sqrt{-pqF}sech_{pq}(\sqrt{F}(x+\lambda t))\Big), \tag{24}$$

$$P_{1,2}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\Big(\pm\sqrt{pqF}csch_{pq}(\sqrt{F}(x+\lambda t))\Big). \tag{25}$$

Case 2: When $F < 0 and G = 0$, then

$$P_{1,3}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\Big(\pm\sqrt{-pqF}sec_{pq}(\sqrt{-F}(x+\lambda t))\Big), \tag{26}$$

$$P_{1,4}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\Big(\pm\sqrt{-pqF}csc_{pq}(\sqrt{-F}(x+\lambda t))\Big). \tag{27}$$

Case 3: When $F < 0 and G = \frac{a^2}{4b}$, then

$$P_{1,5}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\left(\pm\sqrt{-\frac{F}{2}}tanh_{pq}(\sqrt{-\frac{F}{2}}(x+\lambda t)\right), \tag{28}$$

$$P_{1,6}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\left(\pm\sqrt{-\frac{F}{2}}coth_{pq}(\sqrt{-\frac{F}{2}}(x+\lambda t)\right), \tag{29}$$

$$P_{1,7}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F+\mu}}\Big(\pm\sqrt{-\tfrac{F}{2}}\Big(tanh_{pq}(\sqrt{-2F}(x-\lambda t)) \pm\imath\sqrt{pq}sech_{pq}(\sqrt{-2F}(x+\lambda t))\Big)\Big), \tag{30}$$

$$P_{1,8}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F+\mu}}\Big(\pm\sqrt{-\tfrac{F}{2}}\Big(coth_{pq}(\sqrt{-2F}(x+\lambda t)) \pm\sqrt{pq}csch_{pq}(\sqrt{-2F}(x+\lambda t))\Big)\Big), \tag{31}$$

$$P_{1,9}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F+\mu}}\Big(\pm\sqrt{-\tfrac{F}{8}}\Big(tanh_{pq}(\sqrt{-\tfrac{A}{8}}(x+\lambda t)) +coth_{pq}(\sqrt{-\tfrac{F}{8}}(x+\lambda t))\Big)\Big). \tag{32}$$

Case 4: When $F > 0 and G = \frac{a^2}{4}$, then

$$P_{1,10}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\left(\pm\sqrt{\frac{F}{2}}tan_{pq}(\sqrt{\frac{F}{2}}(x+\lambda t))\right), \tag{33}$$

$$P_{1,11}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F + \mu}}\left(\pm\sqrt{\frac{F}{2}}cot_{pq}(\sqrt{\frac{F}{2}}(x+\lambda t))\right), \tag{34}$$

$$P_{1,12}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F+\mu}}\Big(\pm\sqrt{\tfrac{F}{2}}\Big(tan_{pq}(\sqrt{2A}(x+\lambda t)) \pm\sqrt{pq}sec_{pq}(\sqrt{2F}(x+\lambda t))\Big)\Big), \tag{35}$$

$$P_{1,13}^{\pm}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F+\mu}}\Big(\pm\sqrt{\tfrac{A}{2}}\Big(cot_{pq}(\sqrt{2F}(x+\lambda t)) \pm\sqrt{pq}csc_{pq}(\sqrt{2F}(x+\lambda t))\Big)\Big), \tag{36}$$

$$P^{\pm}_{1,14}(x,t) = \frac{\sqrt{6}}{\sqrt{\mu F+\mu}}\Big(\pm\sqrt{\tfrac{F}{8}}\Big(tan_{pq}(\sqrt{\tfrac{F}{8}}(x+\lambda t)) + cot_{pq}(\sqrt{\tfrac{F}{8}}(x+\lambda t))\Big)\Big). \tag{37}$$

4. The Coupled Klein–Gordon Equation

One other considered model in this paper, the CKG equation is defined as

$$\begin{cases} Q_{xx} - Q_{tt} - Q + 2Q^3 + 2QR = 0, \\ R_x - R_t - 4QQ_t = 0. \end{cases} \tag{38}$$

Using the following wave transformations

$Q(x,t) = V(\zeta),\ R(x,t) = U(\zeta),\ \zeta = x + \lambda t,$

converts (38) into the following ODE

$$(1-\lambda^2)V'' - V + \frac{2(1+\lambda)}{1-\lambda}V^3 = 0. \tag{39}$$

4.1. Application of the SSM on CKG

In order to achieve the soliton wave structures of CKG, the SSM is utilized on (39). Balancing the terms of V'' and V^3 in (39), yields $M = 1$, so (4) reduces to

$$V(\zeta) = \Lambda_0 + \Lambda_1\Omega(\zeta), \tag{40}$$

where Λ_0, Λ_1 are constants. Replacing the (40) along with (5) into (39) and gets the set of equations in $\Lambda_0, \Lambda_1, F, and \lambda$. On solving the set of equations, we yields

$$\Lambda_0 = 0,\ \Lambda_1 = \sqrt{2\lambda - \lambda^2 - 1},$$

$$F = \frac{1}{1-\lambda^2}. \tag{41}$$

Case 1: When $F > 0 and G = 0$, then

$$P^{\pm}_{2,1}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{\frac{-pq}{1-\lambda^2}}sech_{pq}\left(\sqrt{\frac{1}{1-\lambda^2}}(x+\lambda t)\right)\right), \tag{42}$$

$$P^{\pm}_{2,2}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{\frac{pq}{1-\lambda^2}}csch_{pq}\left(\sqrt{\frac{1}{1-\lambda^2}}(x+\lambda t)\right)\right). \tag{43}$$

Case 2: When $F < 0 and G = 0$, then

$$P^{\pm}_{2,3}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{\frac{-pq}{1-\lambda^2}}sec_{pq}\left(\sqrt{-\frac{1}{1-\lambda^2}}(x+\lambda t)\right)\right), \tag{44}$$

$$P^{\pm}_{2,4}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{\frac{-pq}{1-\lambda^2}}csc_{pq}\left(\sqrt{-\frac{1}{1-\lambda^2}}(x+\lambda t)\right)\right). \tag{45}$$

Case 3: When $F < 0 and G = \frac{a^2}{4b}$, then

$$P^{\pm}_{2,5}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{-\frac{1}{2(1-\lambda^2)}}tanh_{pq}\left(\sqrt{-\frac{1}{2(1-\lambda^2)}}(x+\lambda t)\right)\right), \tag{46}$$

$$P^{\pm}_{2,6}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{-\frac{1}{2(1-\lambda^2)}}coth_{pq}\left(\sqrt{-\frac{1}{2(1-\lambda^2)}}(x+\lambda t)\right)\right), \tag{47}$$

$$P^{\pm}_{2,7}(x,t) = \sqrt{2\lambda-\lambda^2-1}\Big(\pm\sqrt{-\tfrac{1}{2(1-\lambda^2)}}\Big(tanh_{pq}(\sqrt{-\tfrac{2}{1-\lambda^2}}(x+\lambda t)) \pm\imath\sqrt{pq}sech_{pq}(\sqrt{-\tfrac{2}{1-\lambda^2}}(x+\lambda t))\Big)\Big), \tag{48}$$

$$P^{\pm}_{2,8}(x,t) = \sqrt{2\lambda-\lambda^2-1}\Big(\pm\sqrt{-\tfrac{1}{2(1-\lambda^2)}}\Big(coth_{pq}(\sqrt{-\tfrac{2}{1-\lambda^2}}(x+\lambda t)) \pm\sqrt{pq}csch_{pq}(\sqrt{-\tfrac{2}{1-\lambda^2}}(x+\lambda t))\Big)\Big), \tag{49}$$

$$P^{\pm}_{2,9}(x,t) = \sqrt{2\lambda-\lambda^2-1}\Big(\pm\sqrt{-\tfrac{1}{8(1-\lambda^2)}}\Big(tanh_{pq}(\sqrt{-\tfrac{1}{8(1-\lambda^2)}}(x+\lambda t)) +coth_{pq}(\sqrt{-\tfrac{1}{8(1-\lambda^2)}}(x+\lambda t))\Big)\Big). \tag{50}$$

Case 4: When $F > 0 and G = \frac{a^2}{4}$, then

$$P^{\pm}_{2,10}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{\frac{1}{2(1-\lambda^2)}}tan_{pq}(\sqrt{\frac{1}{2(1-\lambda^2)}}(x+\lambda t))\right), \tag{51}$$

$$P^{\pm}_{2,11}(x,t) = \sqrt{2\lambda-\lambda^2-1}\left(\pm\sqrt{\frac{1}{2(1-\lambda^2)}}cot_{pq}(\sqrt{\frac{1}{2(1-\lambda^2)}}(x+\lambda t))\right), \tag{52}$$

$$P^{\pm}_{2,12}(x,t) = \sqrt{2\lambda-\lambda^2-1}\Big(\pm\sqrt{\tfrac{1}{2(1-\lambda^2)}}\Big(tan_{pq}(\sqrt{\tfrac{2}{1-\lambda^2}}(x+\lambda t)) \pm\sqrt{pq}sec_{pq}(\sqrt{\tfrac{2}{1-\lambda^2}}(x+\lambda t))\Big)\Big), \tag{53}$$

$$P^{\pm}_{2,13}(x,t) = \sqrt{2\lambda-\lambda^2-1}\Big(\pm\sqrt{\tfrac{1}{2(1-\lambda^2)}}\Big(cot_{pq}(\sqrt{\tfrac{2}{1-\lambda^2}}(x+\lambda t)) \pm\sqrt{pq}csc_{pq}(\sqrt{\tfrac{2}{1-\lambda^2}}(x+\lambda t))\Big)\Big), \tag{54}$$

$$P^{\pm}_{2,14}(x,t) = \sqrt{2\lambda-\lambda^2-1}\Big(\pm\sqrt{\tfrac{1}{8(1-\lambda^2)}}\Big(tan_{pq}(\sqrt{\tfrac{1}{8(1-\lambda^2)}}(x+\lambda t)) +cot_{pq}(\sqrt{\tfrac{1}{8(1-\lambda^2)}}(x+\lambda t))\Big)\Big). \tag{55}$$

5. Physical interpretation of the results

In this study, we have built the various type of soliton wave structures of BBM equation and CKG equations by using the SSM. It is exhibited that the use of wave transformations converted the PDEs into ODEs. By using the Mathematica software, we have satisfied all calculations in the methods and have verified that all consequent solutions fulfilled the core equations. To exhibit the plots of the attained results, we have employed the Maple software. In Fig. (1), the plot of the solution $|P^{\pm}_{1,1}(x,t)|$ in eq. (24), are the demonstrated with parameters $F = 1.5$, $\mu = 0.7$, $p = 1.3, q = 1.2$, and $c = 0.65$. In Fig. (a), we have plot 3-dim graph of $|P^{\pm}_{1,1}(x,t)|$ that gives bright soliton wave structure and Fig. (a-1), is the 2-dim graph of $|P^{\pm}_{1,1}(x,t)|$ with $t = 1$. In Fig. (2), the plot of the solution $|P^{\pm}_{1,5}(x,t)|$ in eq. (28), are the demonstrated with parameters $F = 1.4$, $\mu = 0.8$, $p = 1.2, q = 1.2$, and $c = 0.75$. In Fig. (b), we have plot 3-dim graph of $|P^{\pm}_{1,5}(x,t)|$ that gives dark soliton wave structure and Fig. (b-1), is the 2-dim graph of $|P^{\pm}_{1,5}(x,t)|$ with $t = 1$. In Fig. (3), the plot of the solution $|P^{\pm}_{1,7}(x,t)|$ in eq. (30), are the demonstrated with parameters $F = 1$, $\mu = 0.9$, $p = 1, q = 1$, and $c = 0.98$. In Fig. (c), we have plot 3-dim graph of $|P^{\pm}_{1,7}(x,t)|$ that is combo dark-bright soliton wave structure and Fig. (c-1), is the 2-dim graph of $|P^{\pm}_{1,7}(x,t)|$ with $t = 1$. In Fig. (4), the plot of the solution $|P^{\pm}_{1,14}(x,t)|$ in eq. (37), are the demonstrated with parameters $F = 1.2$, $\mu = 0.5$, $p = 1.8, q = 1.7$, and $c = 0.77$. In Fig. (d), we have plot 3-dim graph of $|P^{\pm}_{1,14}(x,t)|$ which is a combo periodic-singular wave structure and Fig. (d-1), is the 2-dim plot of $|P^{\pm}_{1,14}(x,t)|$ with $t = 1$. In Fig. (5), the plot of the solution $|P^{\pm}_{2,1}(x,t)|$ in eq. (42), are the demonstrated with parameters $p = 1.2, q = 1.2$, and $c = 0.75$. In Fig. (e), we have plot 3-dim graph of $|P^{\pm}_{2,1}(x,t)|$ which is a bright soliton wave structure and Fig. (e-1), is the 2-dim plot of $|P^{\pm}_{2,1}(x,t)|$ with $t = 1$. In Fig. (6), the plot of the solution $|P^{\pm}_{2,2}(x,t)|$ in eq. (43), are the demonstrated with parameters $p = 1.4, q = 1.3$, and $c = 0.7$. In Fig. (e), we have plot 3-dim graph of $|P^{\pm}_{2,2}(x,t)|$ which is a periodic soliton wave structure and Fig. (e-1), is the 2-dim graph of $|P^{\pm}_{2,2}(x,t)|$ with $t = 1$. In Fig. (7), the plot of the solution $|P^{\pm}_{2,5}(x,t)|$ in eq. (46), are the demonstrated with parameters $p = 1, q = 1$, and

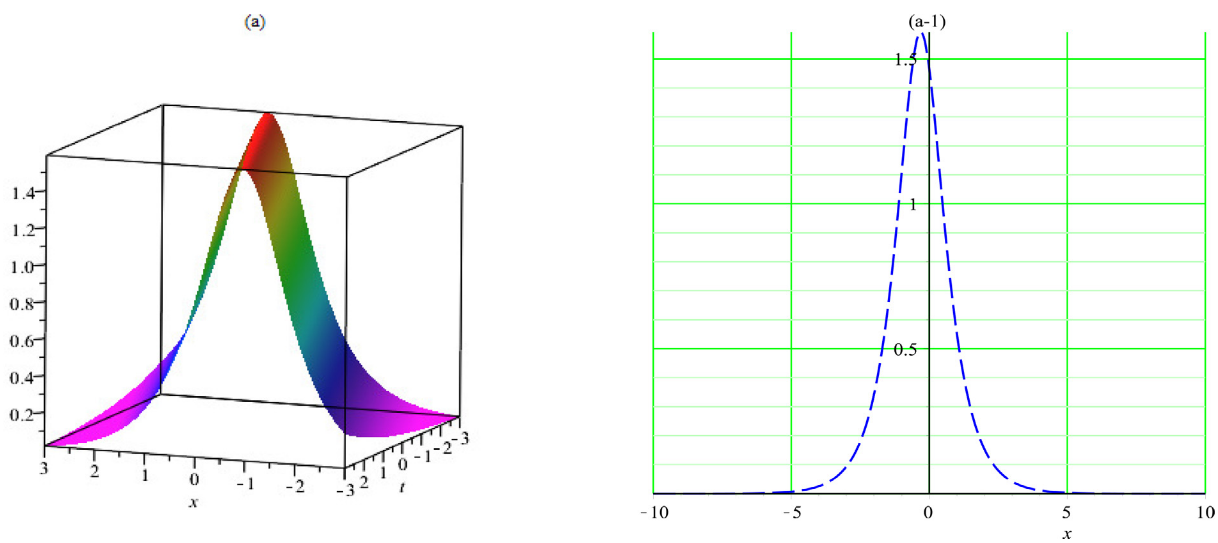

Fig. 1 (a) 3D plot of bright soliton wave structure of $|P^{\pm}_{1,1}(x,t)|$ and (a-1) 2D graph with $t = 1$.

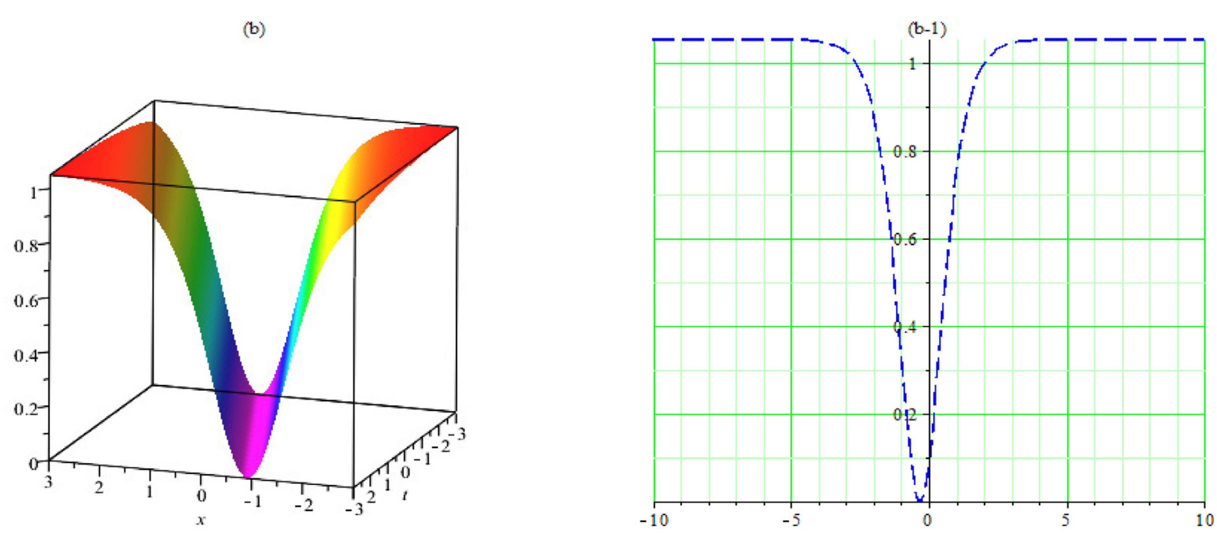

Fig. 2 (b) 3D plot of dark soliton wave structure of $|P^{\pm}_{1,5}(x,t)|$ and (b-1) 2D graph taking $t = 1$.

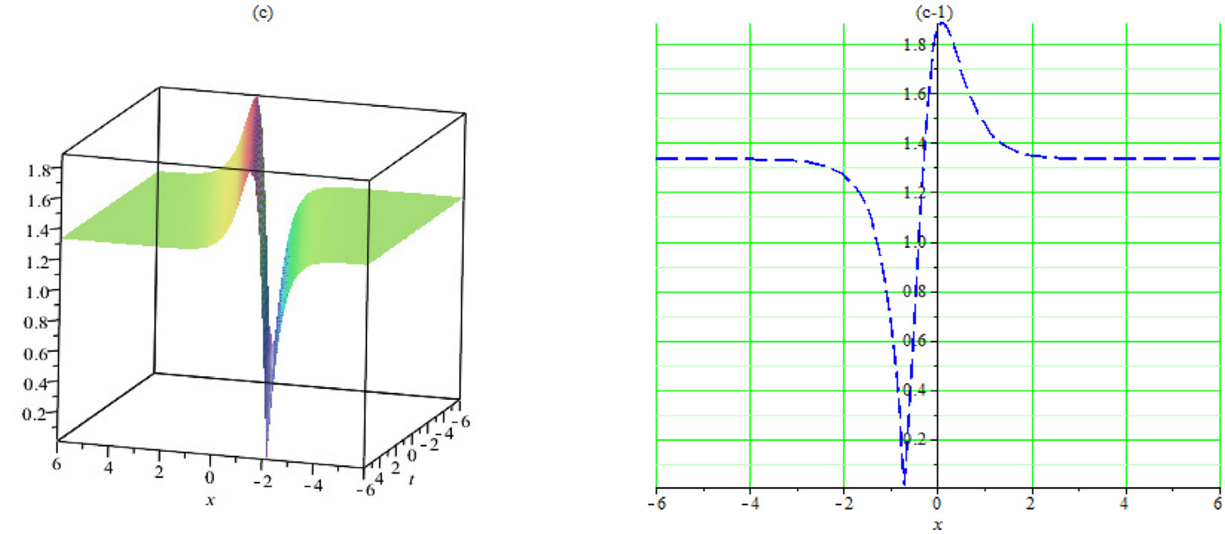

Fig. 3 (c) 3D plot of combo dark-bright soliton wave structure of $|P^{\pm}_{1,7}(x,t)|$ and (c-1) 2D plot taking $t = 1$.

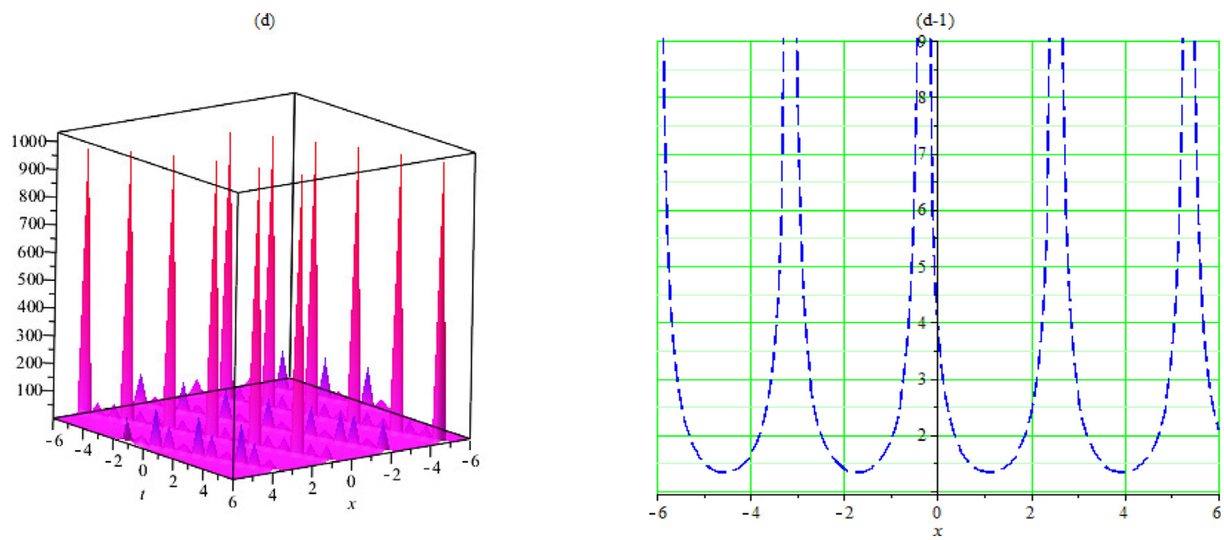

Fig. 4 (d) 3D plot of combo periodic-singular soliton wave structure of $|P^{\pm}_{1,14}(x,t)|$ and (d-1) 2D plot taking $t = 1$.

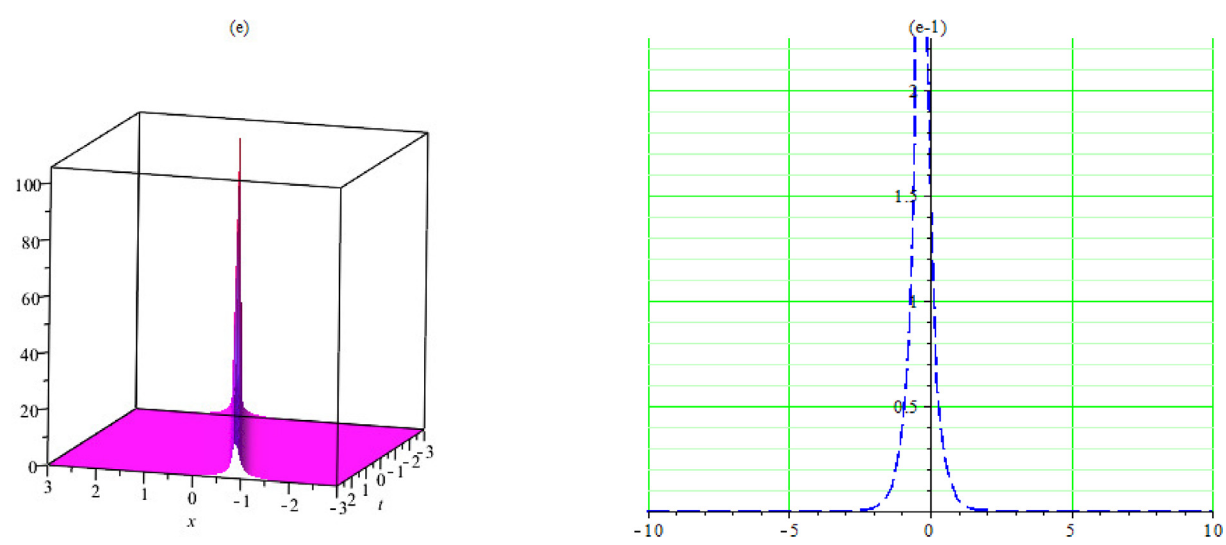

Fig. 5 (e) 3D plot of bright soliton wave structure of $|P^{\pm}_{2,1}(x,t)|$ and (e-1) 2D graph taking $t = 1$.

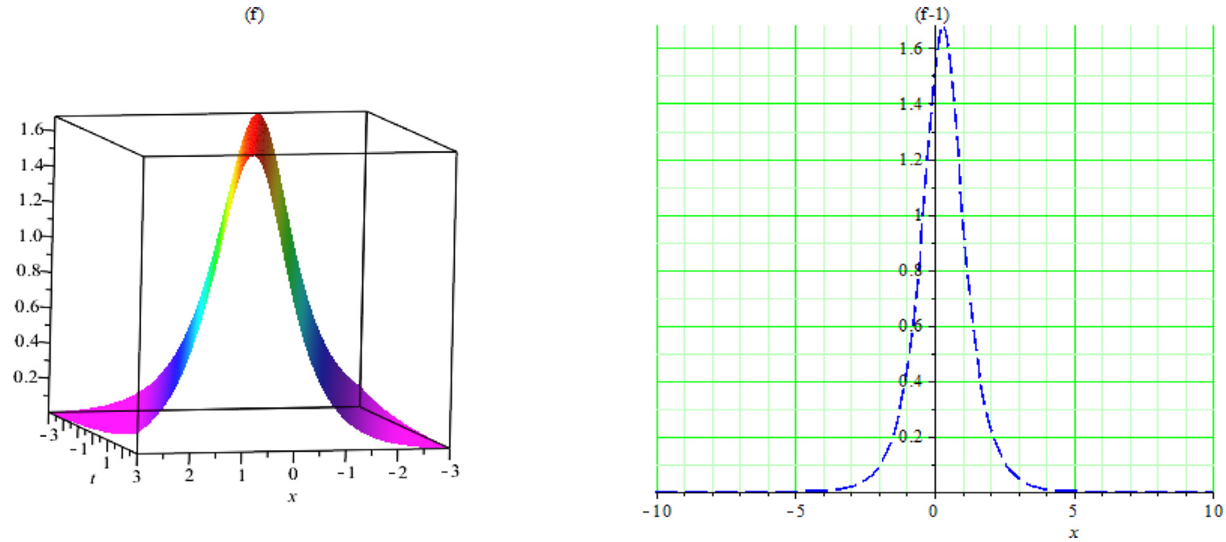

Fig. 6 (f) 3D plot of singular soliton wave structure of $|P^{\pm}_{2,2}(x,t)|$ and (f-1) 2D graph taking $t = 1$.

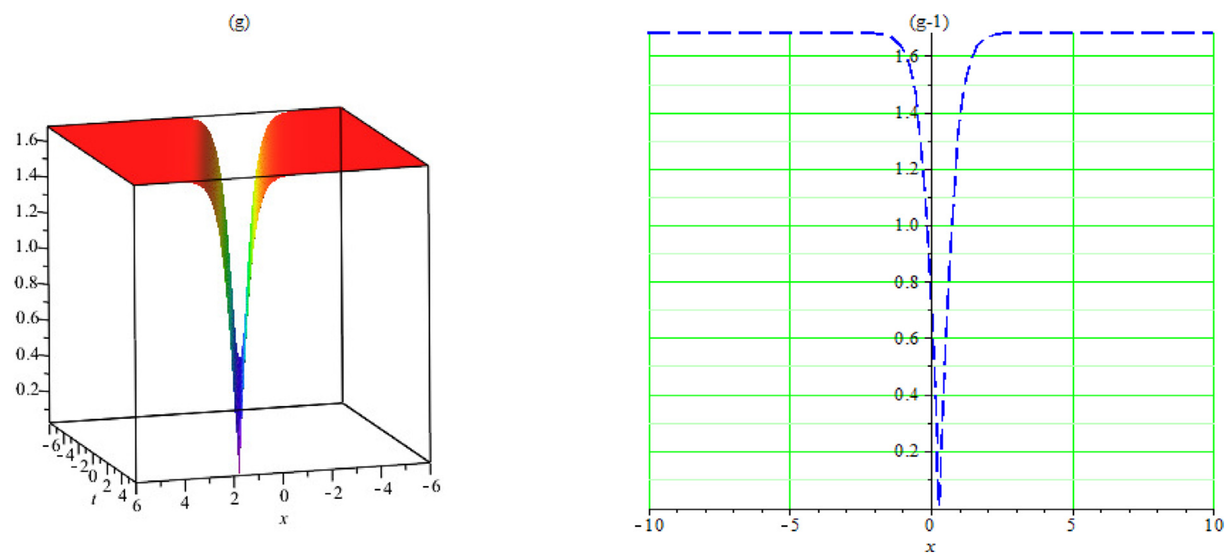

Fig. 7 (g) 3D plot of dark soliton wave structure of $|P^{\pm}_{2,5}(x,t)|$ and (g-1) 2D graph taking $t = 1$.

$c = 0.85$. In Fig. (g), we have plot 3-dim graph of $|P^{\pm}_{2,5}(x,t)|$ which is a bright soliton wave structure and Fig. (g-1), is the 2-dim graph of $|P^{\pm}_{2,5}(x,t)|$ with $t = 1$.

6. Conclusion

In this paper, we have successfully applied the SSM on two different non-linear models, namely, BBM and CKG equations. The acquired results are in the forms of trigonometric and hyperbolic functions solutions which are very efficacious and influential in non-linear optics, mathematical biology, mechanics, fluid mechanics, plasma physics. The obtained results show the bright, singular, dark, and periodic singular solitons wave behaviors. Selecting the suitable values of parameters and to realize the physical meaning of the solutions, 2-dim and 3-dim plots have been represented. By the results and figures, it may be observed that all solutions stimulate their estimated physical behavior. These solutions produce different soliton wave behaviors to the considered models, physically. From outcomes, we can say that these solitons present wave behavior, physically, of the considered models. Also, this study will help to predict some new hypothesis and theories in the field of mathematical physics. Moreover, we compare our results with that have been newly published [39,40] to show the paper's novelty and its scientific contributions.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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